(1) Let \( A \subset \{ H_2, H_4, H_6 \} \subset S \)

\( B = \{ H_2, H_3, H_5, T_2, T_3, T_5 \} \)

\( \text{Therefore, } C = \{ T_1, T_3, T_5 \} \)

(2) Let \( A \subset \{ H_2, H_3, H_4, H_6 \} \subset \{ T_2, T_3, T_5 \} \)

a) \( \text{Then } B \cup A = \{ H_2, H_3, H_4, H_6, T_2, T_3, T_5 \} \)

b) \( B \cap C = \{ T_3, T_5 \} \)

c) \( B \cap (A \cap C) = \{ H_2, H_3, H_5, T_2, T_3, T_5 \} \cap \{ H_1, H_3, H_5, T_1, T_2, T_3, T_4, T_5, T_6 \} \)

(3) \( A, B, C \) are mutually exclusive and

\[ A \cap C = \emptyset \]

mutually exclusive if \( A, C \) are
3) Let $S = \{ a_1, a_2, a_3, a_4 \}$, and let $P$ be the probability function on $S$.

1) Find $P(a_1)$ if $P(a_2) = \frac{1}{3}$, $P(a_3) = \frac{1}{6}$, $P(a_4) = \frac{1}{9}$

ii) Find $P(a_1)$ and $P(a_2)$ if $P(a_3) = P(a_4) = \frac{1}{4}$ and $P(a_1) = 2P(a_2)$.

iii) Find $P(a_1)$ if $P(\{a_2, a_3\}) = \frac{2}{3}$, $P(\{a_2, a_4\}) = \frac{1}{2}$ and $P(a_2) = \frac{1}{3}$.

The Solution

1) $P(a_1) + P(a_2) + P(a_3) + P(a_4) = 1$

Given: $P(a_1) + \frac{1}{3} + \frac{1}{6} + \frac{1}{9} = 1 \Rightarrow P(a_1) = 1 - \frac{1}{3} + \frac{1}{6} + \frac{1}{9} = \frac{7}{18}$

ii) Let $P(a_2) = x \Rightarrow P(a_1) = 2x$

Sample (points)

$P(a_1) + P(a_2) + P(a_3) + P(a_4) = 1$

$2x + x + \frac{1}{4} + \frac{1}{4} = 1 \Rightarrow 3x = 1 - \frac{1}{2} = \frac{1}{2}$

$x = \frac{1}{6}$

$P(a_2) = \frac{1}{6}$

$P(a_1) = \frac{1}{3}$

iii) $P(\{a_2, a_3\}) = P(a_2) + P(a_3)$

$\frac{2}{3} = \frac{1}{3} + P(a_3) \Rightarrow P(a_3) = \frac{1}{3}$

$P(a_4) = P(\{a_2, a_3\}) - P(a_2) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

$P(a_1) = P(a_1) + \frac{1}{3} + \frac{1}{3} + \frac{1}{6} = 1 \Rightarrow P(a_1) = 1 - \frac{5}{6} = \frac{1}{6}$
Let $A$ be the event represent the number divides 3, so $A = \{3, 6, 9, 12, 15, 18\}$.

Let event $B$ contains odd numbers, $B = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$.

Since the two events not disjoint $A \cap B = \{3, 9, 15\} \neq \emptyset$.

So $P(A \cup B) = \frac{3}{20}$

$P(B) = \frac{10}{20}$

$P(A) = \frac{6}{20}$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{6}{20} + \frac{10}{20} - \frac{3}{20} = \frac{13}{20}$$

Problems:

1. A coin is weighted so that heads is twice as likely to appear as tails. Find $P(T)$ and $P(H)$.

2. Two men, $m_1$ and $m_2$, and three women, $w_1, w_2, w_3$, are in a chess tournament. Those of the same sex have equal probabilities of winning, but each man is twice as likely to
Win as any woman.

i) Find the probability that a woman wins the tournament.

ii) If \( m_1 \) and \( w_1 \) are married, find the probability that one of them wins the tournament.

2. Let the number of married men be 8.

Let the number of married women be 6.

Then, the total number of matches is 12.

So, the probability is \( \frac{12}{12} \).

Now, let us consider the probability of winning a match.
Conditional Probability and Independence:

Conditional probability: Let $E$ be an arbitrary event in a sample space $S$ with $P(E) > 0$, the conditional probability of $A$ given $E$, written $P(A|E)$, is defined as:

$$P(A|E) = \frac{P(A \cap E)}{P(E)}$$

or

$$P(A|E) = \frac{|A \cap E|}{|E|}$$

$|E|$ denotes the number of elements in an event $E$.

**Example:** Let a pair of fair dice be tossed. If the sum is 6, find the probability that one of the dice is a 2. In other words, if

$$E = \{ \text{sum is 6} \} = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$$

and

$$A = \{ \text{a 2 appears on at least one die} \}$$

Find $P(A|E)$.

**Solution:** Now $E$ consists of five elements and two of them, $(2,4)$ and $(4,2)$, belong to $A$. Therefore,

$$E \cap A = \{(2,4), (4,2)\}$$

Then

$$P(A|E) = \frac{2}{5}$$
Multiplication Law

\[ P(A) = P(A|k_1)P(k_1) + P(A|k_2)P(k_2) + P(A|k_3)P(k_3) \]

Let's denote the number of ways to choose a certain card as follows:

- For cards in hand $k_1$, there are 3 cards.
- For cards in hand $k_2$, there are 5 cards.
- For cards in hand $k_3$, there are 3 cards.

So, the probability of drawing a card is:

- $P(A|k_1) = \frac{3}{3+4+5} = \frac{3}{12} = \frac{1}{4}$
- $P(A|k_2) = \frac{5}{5+10+5} = \frac{5}{20} = \frac{1}{4}$
- $P(A|k_3) = \frac{3}{3+2+1} = \frac{3}{6} = \frac{1}{2}$

Thus, the overall probability is:

\[ P(k_1) = P(k_2) = P(k_3) = \frac{1}{3} \]

\[ P(A) = \frac{1}{4} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{3} \]
Independent Events

Independent Events:

If events A and B are independent, then $P(AB) = P(A)P(B)$.

Example:

Let $A = \{1, 2\}$ and $B = \{1, 3\}$.

- $P(A) = \frac{2}{4} = \frac{1}{2}$
- $P(B) = \frac{1}{4}$
- $P(AB) = \frac{1}{4}

Then $P(B|A) = \frac{P(AB)}{P(A)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$.

$p(B) = \frac{1}{4}$.
\[ P(B|A) = \frac{2}{3} \quad \text{and} \quad P(B) = \frac{3}{4} \neq \frac{2}{3} \]

\[ P(A) = \frac{1}{2}, \quad P(B) = \frac{1}{2} \quad \rightarrow \quad P(AB) = \frac{1}{4} \]

\[ P(B|A) = \frac{1}{2} \quad \text{or} \quad P(A|B) = \frac{1}{2} \]

\[ B \subset A \quad \text{imply} \]