Chapter one

Functions
R: The set of all real numbers.
(a,b) or \( a < x < b \)
[a,b] or \( a \leq x \leq b \)

Cartesian coordinates or xy-plane
(x,y) is appoint in xy-plane

Function:
A function f from asset D={x\(\in\)R} to asset R is a rule that assigns a single element of R to each element in D, denoted by y=f(x). The set D is called domain of f and the set R={y\(\in\)R, y=f(x)} is called the range of f.

Axiom for domain:

1- we does not take the values of x which introduce division by zero.
2- We does not take the values of x which introduce a root anegative number.
Example:
Find the domain and the range of y=f(x)= \( \sqrt{x-2} \)

Solution:
x-2\(\geq\)0
x\(\geq\)2
D={x\(\in\)R, x\(\geq\)2} and R={y\(\in\)R, y\(\geq\)0} or R=R^+

Definition:
If f(x) and g(x) are two functions then f(x)+ g(x)
F(x).g(x) and \( \frac{f(x)}{g(x)} \), g(x)\(\neq\)0 are also functions.

Definition: The composition of f(x) and g(x) is denoted by f0g or g0f and is defined by : (g0f)(x)=g[f(x)]

Example
Write f0g if f(x)=\( \sqrt{x} \) and g(x)=\( \frac{x}{2} \)

Solution
(f0g)(x)=f[g(x)]=f(\( \frac{x}{2} \))= \( \sqrt{\frac{x}{2}} \)
Some functions
Trigonometric functions:

\[ \sin \theta = \frac{y}{r} \]
\[ \cos \theta = \frac{x}{r} \]
\[ \tan \theta = \frac{y}{x} = \frac{\sin \theta}{\cos \theta} \]
\[ \cot \theta = \frac{x}{y} = \frac{\cos \theta}{\sin \theta} \]
\[ \sec \theta = \frac{r}{x} = \frac{1}{\cos \theta} \]
\[ \csc \theta = \frac{r}{y} = \frac{1}{\sin \theta} \]

1. \( y = f(x) = \sin x, D = \mathbb{R}, R = [y / y \in \mathbb{R}, -1 \leq y \leq 1] \)
2. \( y = f(x) = \cos x, D = \mathbb{R}, R = [y / y \in \mathbb{R}, -1 \leq y \leq 1] \)
3. \( y = f(x) = \tan x, D = \{x / x \in \mathbb{R}, x \neq -\frac{n\pi}{2}, n, \text{ is odd number}\} \)
**Absolute value function:**

Let a be any real number, we define

\[ |a| = \begin{cases} 
  a & a \geq 0 \\
  -a & a < 0 
\end{cases} \]

Properties:

1. \( |a+b| \leq |a| + |b| \)
2. \( |a| \leq r \quad -r \leq a \leq r \)

**Example:**

Find the value of x satisfy \( \left| \frac{3x+1}{2} \right| < 1 \)

**Solution:**

\[-1 < \frac{3x+1}{2} < 1\]
\[-2 < 3x + 1 < 2\]
\[-3 < 3x < 1\]
\[-1 < x < \frac{1}{3}\]

**Step function**

\( Y = f(x) = [x], \quad D = \mathbb{R}, \quad R = \mathbb{R}, \quad [x] = \) the greatest integer less than or equal to x

**Example**

\( [1.9] = 1, [2] = 2, [0.5] = 0, [-0.5] = -1, [-2.7] = -3 \)
Limits

1. The limit of \( f(x) \) as \( x \) approaches \( a \) from the right is the number \( L \) if
   \[
   \lim_{x \to a^+} f(x) = L.
   \]

2. The limit of \( f(x) \) as \( x \) approaches \( a \) from the left is the number \( L \) if
   \[
   \lim_{x \to a^-} f(x) = L.
   \]

3. The number \( L \) is the limit of \( f(x) \) as \( x \) approaches \( a \), denoted by
   \[
   \lim_{x \to a} f(x) = L
   \] if and only if
   \[
   \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x) = L.
   \]

Theorem 1:
If \( \lim_{x \to a} f(x) = L_1 \) and \( \lim_{x \to a} g(x) = L_2 \), then

1. \( \lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L_1 + L_2 \)
2. \( \lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L_1 L_2 \)
3. \( \lim_{x \to a} k f(x) = k \lim_{x \to a} f(x) = kL_1 \)
4. \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L_1}{L_2} \) if \( L_2 \neq 0 \)

Theorem 2
1. \( \lim_{x \to a} (b_0 x^n + b_1 x^{n-1} + \ldots + b_n) = b_0 a^n + b_1 a^{n-1} + \ldots + b_n \)
2. \( \lim_{x \to a} \sin(x) = 1 \)
3. If \( g(x) \leq f(x) \leq h(x) \), are three functions such that
   \( \lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L \) then \( L \) is the limit of \( f(x) \) as \( x \) approaches \( a \).

Example:

\[
\lim_{x \to 3} \frac{x^2 + 2x + 4}{x + 2} = \frac{19}{5}
\]

Infinity as a limit:

1. \( \lim_{x \to 0^+} \frac{1}{x} = \infty \)
2. \( \lim_{x \to 0^-} \frac{1}{x} = -\infty \)
3. \( \lim_{x \to \infty} \frac{1}{x} = 0 \)
4. \( \lim_{x \to -\infty} \frac{1}{x} = 0 \)
Example: evaluate

\[
\lim_{{x \to \infty}} \frac{x}{2x + 4} = 0 = \frac{1}{2}
\]

\[
x = \cos \theta
\]

\[
y = \sin \theta
\]

1. \(\cos^2 \theta + \sin^2 \theta = 1\)
2. \(1 + \tan^2 \theta = \sec^2 \theta, \quad 1 + \cot^2 \theta = \csc^2 \theta\)
3. \(\sin(\theta_1 + \theta_2) \sin \theta_1 \cos \theta_2 + \cos \theta_1 + \sin \theta_2\)
4. \(\cos(\theta_1 + \theta_2) \cos \theta_1 \cos \theta_2 - \sin \theta_1 + \sin \theta_2\)
5. \(\sin^2(2\theta) = 2 \sin \theta \cos \theta, \quad \cos(2\theta) = \cos^2 \theta - \sin^2 \theta\)
6. \(\cos \theta = \frac{1 + \cos(2\theta)}{2}, \quad \sin \theta = \frac{1 - \cos(2\theta)}{2}\)

Example: evaluate

\[
\lim_{{x \to 0}} \frac{1 - \cos(2x)}{2x}
\]

\[
\lim_{{x \to 0}} \frac{1 - \cos(2x)}{2x} = \lim_{{x \to 0}} \frac{\sin^2 x}{2x} = \lim_{{x \to 0}} \frac{\sin x}{x} \cdot \lim_{{x \to 0}} \sin x = 1.0 = 0
\]

Continuity

A function \(y = f(x)\) is said to be continuous at a point \(a\) if

1. \(f(a)\) exists
2. \(\lim_{{x \to a}} f(x)\) exists
3. \(\lim_{{x \to a}} f(x) = f(a)\)

Example: test the continuity of

\(f(x) = \frac{1}{x - 2}\) \quad \(a = 2\)

Solution

\(F(x)\) does not exist of \(x = 2\) then \(f\) is not cont.

Derivatives:

The derivative of a function \(y = f(x)\) is denoted by \(f'(x)\) and defined by:

\[
f'(x) = \lim_{{\Delta x \to 0}} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

if the limit exists.

Also \(f'(x)\) is called the slope of the curve \(y = f(x)\).
Laws of derivatives:
the derivative of a constant is zero.
1. \( (x^n)' = nx^{n-1} \)
2. \( (cf(x))' = cf(x) \)
3. \( (f(x)g(x))' = f(x)' + g(x)' \)
4. \( (f(x)g(x))' = f(x)g(x) + f(x)g(x) \)
5. \( [(f(x))^n]' = n(f(x))^{n-1} f(x)' \)
6. \( \left( \frac{f(x)}{g(x)} \right)' = \frac{g(x)f(x)' - f(x)g(x)'}{g(x)^2} \)

Example: Find \( y' \)

\[
y = f(x) = x\sqrt{x^2 - 2}
\]

Solution

\[
y' = x[1/2(x^2 - 2)^{-1/2}(2x)] + \sqrt{x^2 - 2}
\]

Higher Derivatives
Implicit differentiation
Chain rule
Derivative of trigonometric functions:
1. \( \sin u' = \cos u \frac{du}{dx} \)
2. \( \cos u' = -\sin u \frac{du}{dx} \)
3. \( \tan u' = \sec^2 u \frac{du}{dx} \)
4. \( \cot u' = -\csc^2 u \frac{du}{dx} \)
5. \( \sec u' = \sec u \tan u \frac{du}{dx} \)
6. \( \csc u' = -\csc u \cot u \frac{du}{dx} \)

Example: find \( y' \)

\[
y = f(x) = \sin 3x
\]

Solution

\[
y' = 3\cos(3x)
\]

Inverse function
1. A function \( f(x) \) is said to be one to one if \( f(x_1) = f(x_2) \) implies that \( x_1 = x_2 \). Geometrically \( f(x) \) is one to one if we draw a line parallel to x-axis it will cut or intersect curve of \( f(x) \) in one point only.
2. A function \( f(x) \) is said to be onto if for every \( b \) on y-axis there exists a point \( a \) on x-axis such that \( f(a) = b \)
3. if \( f(x) \) is one to one and onto then there exists \( f(x)^{-1} \) called the inverse of \( f(x) \) such that \( fof^{-1} = f^{-1}of = x \)
Example: find the inverse of $y=f(x)=\frac{1}{4}x+3$
Solution:
$f^{-1}=4x-12$

Application of Derivatives:
Increasing and Decreasing function:

Suppose that $f(x)$ has derivative at every point $x$ in an interval $I$ then:
1- $f$ increase on $I$ if $f'(x)>0$ for all $x$ in $I$
2- $f$ decrease on $I$ if $f'(x)<0$ for all $x$ in $I$

Maximum and Minimum:

Let $y=f(x)$ be a function such that $y'$ and $y''$ exist first find critical points by solving $y'=0$. let $a$ be one of these critical points

Test1
1- $f(a)$ is maximum if $f(a-1)>0$ and $f(a+1)<0$
2- $f(a)$ is minimum if $f(a-1)<0$ and $f(a+1)>0$

Test2
1- $f(a)$ is maximum if $f''(a)<0$
2- $f(a)$ is maximum if $f''(a)>0$

Concavity and points of inflection

$F(a)$ is a point of inflection to $y=f(x)$ if $f''(a)=0$

1- the graph of $f(x)$ is concave down on any interval where $y<0$
2- the graph of $f(x)$ is concave up on any interval where $y>0$

Example:
sketch the curve $y=\frac{1}{6}(x^3-6x^2+9x+6)$
solution
$y'=\frac{1}{6}(3x^2-12x+9)=\frac{1}{2}(x^2-4x+3)=\frac{1}{2}(x-1)(x-3)$

$y'=0$ x=1 and x=3 are critical points
$f(1)=\frac{5}{3}$ is maximum
$f(3)=1$ is minimum
$y''=\frac{1}{2}(2x-4)=x-2$
$y''=0$, $x=2$ f(2)=4/3 is point of inflection
L'Hopital Rule:

if \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \) or \( \frac{\infty}{\infty} \), then

\[ \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \]

Example: find

\[ \lim_{x \to 0} \frac{x - \sin x}{x^3} = \lim_{x \to 0} \frac{1 - \cos x}{3x^2} = \lim_{x \to 0} \frac{\sin x}{6x} = \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6} \]

Even and odd function:

1-A function \( f(x) \) is said to be an even function if \( f(-x) = f(x) \), geometrically the graph of \( f(x) \) must be symmetric about the y-axis.
2-A function \( f(x) \) is said to be an odd function if \( f(-x) = -f(x) \), geometrically the graph of \( f(x) \) must be symmetric about the origin.

Example
1- \( x^2 \), \( \cos x \) even function
2- \( x \), \( x^3 \), \( \sin x \) odd function
Chapter two
Integration
Laws of integration
1. \( \int dx = x + c \)
2. \( \int kf(x)dx = k \int f(x)dx \)
3. \( \int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx \)
4. \( \int x^n \, dx = \frac{x^{n+1}}{n+1} + c \quad \text{if } n \neq -1 \)
5. \( \int (f(x))^n \, f'(x) \, dx = \frac{(f(x)^{n+1})}{n+1} + c \quad \text{if } n \neq -1 \)

Example: Evaluate
\[
\int \sqrt{x} \, dx = \int x^{3/2} \, dx = \frac{x^{5/2}}{5/2} + c
\]

Integration of Trigonometric functions
1. - \( \int \cos u \, du = \sin u + c \)
2. - \( \int \sin u \, du = -\cos u + c \)
3. - \( \int \sec u \, du = \tan u + c \)
4. - \( \int \csc^2 u \, du = -\cot u + c \)
5. - \( \int \sec u \tan u \, du = \sec u + c \)
6. - \( \int \csc u \cot u \, du = -\csc u + c \)
Example: Evaluate
\[
\int \tan x \sec x \, dx
\]
Solution
\[
\frac{1}{2} \tan x + c
\]

Define Integral:
The area under the curve \( y=f(x) \) from \( x=a \) to \( x=b \) is defined by:
\[
A = \int_a^b f(x) \, dx
\]
Properties::
1. \( \int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx \quad \text{k is constant} \)
2. \( \int_a^b f(x) \, dx \geq 0 \)
3- \( \int_{a}^{b} f(x)dx = -\int_{b}^{a} f(x)dx \)

4- \( \int_{a}^{b} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{b} f(x)dx \) \( c \) is appoint in \([a,b]\)

5- if \( f \) is continuous on \([a,b]\) and \( f \) is any antderivatives of \( f \) on \([a,b]\) then

\( \int_{a}^{b} f(x)dx = f(x)\Big|_{a}^{b} = f(a) - f(b) \)

**Remark**

\( A = \int_{c}^{d} f(y)dy \)

**Example**

Find the area \( y=f(x)=\sin x \) from \( x=0 \) to \( x=\pi \)

**Solution**

\( A = \int_{0}^{\pi} \sin xdx = [-\cos x]\Big|_{0}^{\pi} = -\cos \pi + \cos 0 = 2 \)

**Application of Define integral:**

**Area between two curves:**

\( A = \int_{a}^{b} [f_1(x) - f_2(x)]dx \)

\( A = \int_{c}^{d} [f_1(y) - f_2(y)]dy \)

**Example**

Find the area bounded by \( y=2-x^2 \) and \( y=-x \)

**Solution**

\( 2-x^2=-x \)
\( x^2-x-2=0=(x-2)(x+1)=0 \)
\( x=-1, y=1 \) or \( x=2, y=2 \).

The intersetion points are \((-1,1)\) and \((2,-2)\)

\( A = \int_{-1}^{2} (2-x^2 + x)dx = 9/2 \)

**Volume of Rotation**

1- Disk Method

\( V = \int_{a}^{b} \pi [f(x)]^2dx \)

\( F(x) \) is rotated about x-axis

\( V = \int_{c}^{d} \pi [f(y)]^2dy \)

\( F(y) \) is rotated about y-axis
Example find the volume of the solid generated by rotating the function $y = \sqrt{x}$ about x-axis from $x=0$ to $x=4$

Solution

$V = \int_{a}^{b} \pi [f(x)]^2 \, dx = \int_{0}^{4} \pi (\sqrt{x})^2 \, dx = 8\pi$

2-- $V = \int \pi [f_2(x)^2 - f_1(x)^2] \, dx$

The area is rotated about x-axis

$V = \int \pi [f_2(y)^2 - f_1(y)^2] \, dy$

The area is rotated about y-axis

Example

The region bounded by $x=y$ is rotated about y-axis to generate a solid. Find its volume

Solution

$y^2 = y$

$y^2 - y = 0 \rightarrow y(y - 1) = 0 \rightarrow y = 0 \text{ or } y = 1$

$V = \int_{0}^{1} \pi (y^2 - y^4) \, dy = \pi \left[ \frac{y^3}{3} - \frac{y^5}{5} \right]_{0}^{1} = \pi \left[ \frac{1}{3} - \frac{1}{5} \right] = \frac{2\pi}{15}$

Chapter three

Transcendental Functions

Inverse of Trigonometric Functions

$y = f(x) = \sin x, \, D = \{ x / -1 \leq x \leq 1 \}, \, R = \{ y / -\pi/2 \leq y \leq \pi \}$

$y = f(x) = \cos x, \, D = \{ x / -1 \leq x \leq 1 \}, \, R = \{ y / 0 \leq y \leq \pi \}$

$y = f(x) = \tan x, \, D = \mathbb{R}, \, r = \{ y / -\pi/2 \leq y \leq \pi/2 \}$

$y = f(x) = \cot x, \, D = \mathbb{R}, \, R = \{ y / 0 \leq y \leq \pi \}$

$y = f(x) = \sec x, \, D = \{ x / x \geq -1, x \leq -1 \}, \, R = \{ y / y \neq n(\pi/2) \}$

$y = f(x) = \csc x, \, D = \{ x / x \geq -1, x \leq -1 \}, \, R = \{ y / 0 \leq y \leq \pi/2 \}$
Properties
\[-1\]
\[\sin(-x) = -\sin x\]
\[-1\]
\[\cos(-x) = \pi - \cos x\]
\[-1\]
\[\sin x + \cos x = \pi / 2\]
\[-1\]
\[\cot x = \pi / 2 - \tan x\]
\[-1\]
\[\sec x = \cos(1 / x)\]
\[-1\]
\[\csc x = \sin (1 / x)\]
\[-1\]
\[\csc(-x) = \pi - \sec x\]

Derivatives
\[-1\]
\(\sin u = \frac{du}{dx}\)
\[-1\]
\(\cos u = -\frac{du}{dx}\)
\[-1\]
\(\tan u = \frac{du}{dx}\)
\[-1\]
\(\cot u = -\frac{du}{dx}\)
\[-1\]
\(\sec u = \frac{du}{dx}\)
\[-1\]
\(\csc u = -\frac{du}{dx}\)

Example: find \(y'\)
\[y = \cos(2x)\]

Solution
\[y' = -\frac{2}{\sqrt{1-(2x)^2}}\]

Integration
\[\int \frac{du}{\sqrt{1-u^2}} = \sin u + c\]
\[\int \frac{du}{1+u^2} = \tan u + c\]
\[\int \frac{du}{u\sqrt{u^2-1}} = \sec|u| + c\]
Example

\[ \int \frac{x^2}{1 + x^2} \, dx = \frac{1}{3} \tan^{-1} x + c \]

Natural Logarithm:

\[ y = \ln x = \int \frac{dt}{t} \]

Properties:

1. \( \ln 1 = 0 \)
2. \( \ln(ax) = \ln a + \ln x \)
3. \( \ln(x/a) = \ln x - \ln a \)
4. \( \ln x^a = a \ln x \)
5. \( \ln 1/x = -\ln x \)

Derivative: \( \frac{du}{dx} = \int \frac{\ln u}{u} \)

Integration: \( \int \frac{du}{u} = \ln|u| + c \)

Examples:

1. \( y = \ln(6x^2 + 2x + 1) \) \( y' = \frac{12x + 2}{6x^2 + 2x + 1} \)

2. \( \int \frac{x \, dx}{1 + x^2} = \frac{1}{2} \int \frac{2 \, dx}{1 + x^2} = \frac{1}{2} \ln|1 + x^2| + c \)

The Exponential function

\( y = f(x) = e^x, D = \mathbb{R}, R = \mathbb{R}^+ \)

It is the inverse of \( y = \ln x \)

Properties

1. \( \lim_{x \to \infty} e^x = \infty \)
2. \( \lim_{x \to -\infty} e^x = 0 \)
3. \( e^a \cdot e^x = e^{a+x} \)
4. \( e^{-x} = \frac{1}{e^x} \)
5. \( \ln e^x = x \)
6. \( e^{\ln x} = x \)

Derivative: \( (e^u)' = e^u \frac{du}{dx} \)

Integration: \( \int e^u \, du = e^u + c \)
The Function: $a^u$

\[ y = f(x) = a^x, D = \Re, R = \Re^+, a > 0 \]

Derivative: \( (a^u)' = a^u \ln a \frac{du}{dx} \)

Integration: \[ \int a^u \, du = \frac{a^u}{\ln a} + c \]

The function \( y = \log_a u \)

\[ y = f(x) = \log_a(x) \] is the inverse of \( y = a^x \)

Properties:

1. \( \log_a x = \frac{\ln x}{\ln a} \)
2. \( \log_a a^x = x \)
3. \( \log_a a = x \)

Derivative: \( \left( \log_a u \right)' = \frac{du}{dx} \frac{1}{u \ln a} \)

Chapter four

Integration Methods

1. Integration by parts
\[ \int u \, dv = uv - \int v \, du \]

Example
\[ \int \ln x \, dx \]

Solution
\[ u = \ln x, dv = dx \]
\[ du = \frac{1}{x}, v = x \]
\[ \int \ln x \, dx = x \ln x - \int x \frac{1}{x} \, dx = x \ln x - x + c \]

2. Partial fraction
\[ \sqrt{a^2 + u^2}, \sqrt{a^2 - u^2}, \frac{u^2}{a^2 - u^2}, \frac{u^2}{a^2 + u^2} \]

3. Integral involves

put \( u = a \sin \theta \) then use \( 1 - \sin^2 \theta = \cos^2 \theta a^2 - u^2 \),

put \( u = a \tan \theta \) then use \( 1 + \tan^2 \theta = \sec^2 \theta a^2 + u^2 \),

put \( u = a \sec \theta \) then use \( 1 + \tan^2 \theta = \sec^2 \theta u^2 - a^2 \).
4-powers of Trigonometric functions

**Sin x or cos x of odd power**

\[
\int \cos x^2 \, dx = \int \cos x^2 \cos x \, dx = \int (1 - \sin x^2) \cos x \, dx \\
= \int \cos x \, dx - \int \sin x^2 \cos x \, dx \\
= \sin x - \sin x^3 / 3 + c
\]

**Sin x or cos x of even power**

\[
\int \cos x^2 \, dx = \int [1 + \cos 2x] / 2 \, dx = 1/2 x + 1/4 \sin 2x + c
\]

\[
\sin ax \cos bx, \sin ax \sin bx, \cos ax \cos bx
\]

- Sinax sinbx=1/2[cos(a-b)x-cos(a+b)x] 
- Sinax cosbx=1/2[sin(a-b)x+sin(a+b)x] 
- Cosax cosbx=1/2[cos(a-b)x+cos(a+b)x] 

Example

\[
\int \sin 5x \cos 3x \, dx = 1/2 \int \sin 2x + \sin 8x \, dx = -1/4 \cos 2x - 1/16 \cos 8x + c
\]

6- a- tanx, cotx , secx and cscx of even power 
Use tan^2 x=sec^2 x-1 
- b- tanx or cotx of odd power 
use cot^2 x=csc^2 x-1 
- c- secx or cscx of odd power 
use integration by part (udv)

**Chapter five**

**Matrices:**

When a system of equations has more than two equations, it is difficult to discuss them without using matrices and vectors.

The size of the matrix is described by the number of its row and columns. A matrix of \( n \) rows and \( m \) columns is said to be \( n \times m \) matrix.

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1m} \\
a_{21} & a_{22} & \cdots & a_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nm}
\end{bmatrix} = [a_{ij}] \quad i = 1,2,\ldots,n \quad , \quad j = 1,2,\ldots,m.
\]

**Types of matrices:**

**Square matrix:** It is a matrix whose number of rows are equal to the number of columns \( (n = m) \). For example:
\[
A = \begin{bmatrix}
1 & 5 \\
2 & 4
\end{bmatrix}_{2 \times 2}, \quad B = \begin{bmatrix}
1 & 3 & 0 \\
3 & 2 & 1 \\
1 & 8 & 0
\end{bmatrix}_{3 \times 3}
\]

**Diagonal matrix:** It is a square matrix which all its elements are zero except the elements on the main diagonal. For example:

\[
A = \begin{bmatrix}
4 & 0 & 0 \\
0 & 9 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

**Identity matrix:** It is a diagonal matrix whose elements on the main diagonal are equal to 1, and it is denoted by \( I_n \). For example:

\[
I_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad I_2 = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

**Transpose matrix:** Transpose of \( A \) is denoted by \( A^T \), means that write the rows of \( A \) as columns in \( A^t \). For example:

\[
A = \begin{bmatrix}
3 & 7 & 1 \\
-2 & 1 & -3
\end{bmatrix}_{2 \times 3}, \quad A^T = \begin{bmatrix}
3 & -2 \\
7 & 1 \\
-3 & \end{bmatrix}_{3 \times 2}
\]

**Matrix addition and multiplication**

If \( A = [a_{ij}] \) and \( B = [b_{ij}] \) and both \( A \) & \( B \) are \( n \times m \) matrices, then

\[
A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]
\]

Ex.1:

\[
\begin{bmatrix}
1 & -1 \\
0 & 2
\end{bmatrix} + \begin{bmatrix}
2 & 3 \\
4 & 5
\end{bmatrix} = \begin{bmatrix}
3 & 2 \\
4 & 7
\end{bmatrix}
\]

For any scalar (number) \( c \), we can multiply \( A \) by \( c \) as follows:

\[
cA = c[a_{ij}] = [ca_{ij}]
\]

Ex.2:

\[
\begin{bmatrix}
3 & -1 \\
0 & 2
\end{bmatrix} = \begin{bmatrix}
3 & -3 \\
0 & 6
\end{bmatrix}
\]

A matrix with only one column, \( n \times 1 \) in size, is called a **column vector**, and one of only one row, \( 1 \times m \) in size, is called a **row vector**.
Matrices multiplication

Let A be an \( n \times k \) matrix and B be a \( k \times m \) matrix then \( C=AB \) is an \( n \times m \) matrix, where the element in the \( i^{th} \) row and \( j^{th} \) column of \( AB \) is the sum

\[
c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{im}b_{mj} = \sum_{k=1}^{n} a_{ik}b_{kj}, \quad i = 1,2,\ldots,m \text{ and } j = 1,2,\ldots,p.
\]

Ex.3

Suppose \( A = \begin{bmatrix} 3 & 7 & 1 \\ -2 & 1 & -3 \end{bmatrix} \) \(_{2 \times 3}\), \( B = \begin{bmatrix} 5 & -2 \\ 0 & 3 \\ 1 & -1 \end{bmatrix} \) \(_{3 \times 2}\) then

\[
AB = \begin{bmatrix} 16 & 14 \\ -13 & 10 \end{bmatrix} \quad BA = \begin{bmatrix} -6 & 3 & -9 \\ 5 & 6 & 4 \end{bmatrix}
\]

Determinants

With each square matrix \( A \) we associate a number \( \det(A) \) or \( |a_{ij}| \) called the determinant of \( A \), calculated from the entries of \( A \) as follows:

For \( n=1 \), \( \det(a)=a \),

For \( n=2 \), \( \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21} \)

Minors

To each element of a \( 3 \times 3 \) matrix there corresponds a \( 2 \times 2 \) matrix that is obtained by deleting the row and column of that element. The determinant of the \( 2 \times 2 \) matrix is called the minor of that element.

For a matrix of dimension \( 3 \times 3 \), we define

\[
\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}
\]

where \( \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \) is the minor of \( a_{11} \), \( \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \) is the minor of \( a_{12} \),

and \( \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \) is the minor of \( a_{13} \).
Ex.4: Find the determinant of each matrix

a) \[
\begin{vmatrix}
1 & 3 \\
-2 & 5
\end{vmatrix}
\]
\[= 1(5) - 3(-2) = 5 + 6 = 11\]

b) \[
\begin{vmatrix}
2 & 4 \\
6 & 12
\end{vmatrix}
\]
\[= 2(12) - 4(6) = 0\]

Ex.5: Find the determinant of A where:
\[
A = \begin{bmatrix}
1 & 3 & -5 \\
-2 & 4 & 6 \\
0 & -7 & 9
\end{bmatrix}
\]
Sol.: By choosing the first column we get
\[
\det(A) = \begin{vmatrix}
1 & 3 & -5 \\
-2 & 4 & 6 \\
0 & -7 & 9
\end{vmatrix} = 1 \cdot \begin{vmatrix} 4 & 6 \\ -7 & 9 \end{vmatrix} - (-2) \cdot \begin{vmatrix} 3 & -5 \\ -7 & 9 \end{vmatrix} + 0 \cdot \begin{vmatrix} 3 & -5 \\ 4 & 6 \end{vmatrix}
\]
\[= 1 \cdot [36 - (-42)] + 2 \cdot (27 - 35)
\[= 78 - 16 = 62\]

Ex.6: Evaluate the determinant of A if:
\[
A = \begin{bmatrix}
1 & 3 & -5 \\
-2 & 4 & 6 \\
0 & -7 & 9
\end{bmatrix}
\]
Solution:
By choosing the second row we get
\[
\det(A) = \begin{vmatrix}
1 & 3 & -5 \\
-2 & 4 & 6 \\
0 & -7 & 9
\end{vmatrix} = (-2) \cdot \begin{vmatrix} 3 & -5 \\ -7 & 9 \end{vmatrix} + 4 \cdot \begin{vmatrix} 1 & -5 \\ 0 & 9 \end{vmatrix} - 6 \cdot \begin{vmatrix} 1 & 3 \\ 0 & -7 \end{vmatrix}
\]
\[= 2(27 - 35) + 4(9 - 0) - 6(-7 - 0)
\[= -16 + 36 + 42 = 62\]
Note that 62 is the same value that was obtained for this determinant in Example above.

Note:
If a matrix A is triangular (either upper or lower), its determinant is just the product of the diagonal elements:
Solving a system of linear equations

Let A be a matrix, X a column vector, B a column vector then the system of linear equations is denoted by AX=B.

The augmented matrix

The solution to a system of linear equations such as
\[ \begin{align*}
  x - 2y &= -5 \\
  3x + y &= 6
\end{align*} \]

Depends on the coefficients of \( x \) and \( y \) and the constants on the right-hand side of the equation. The matrix of coefficients for this system is the \( 2 \times 2 \) matrix
\[
\begin{bmatrix}
  1 & -2 \\
  3 & 1
\end{bmatrix}
\]

If we insert the constants from the right-hand side of the system into the matrix of coefficients, we get the \( 2 \times 3 \) matrix
\[
\begin{bmatrix}
  1 & -2 & -5 \\
  3 & 1 & 6
\end{bmatrix}
\]

We use a vertical line between the coefficients and the constants to represent the equal signs. This matrix is the augmented matrix of the system also it can be written as:
\[
\begin{bmatrix}
  1 & -2 & | & x \\
  3 & 1 & | & y
\end{bmatrix}
= \begin{bmatrix}
  -5 \\
  6
\end{bmatrix}
\]

Note:

Two systems of linear equations are equivalent if they have the same solution set. Two augmented matrices are equivalent if the systems they represent are equivalent.

Ex.1:

Write the augmented matrix for each system of equations.
\[ x + y - z = 5 \]
a) \[ 2x + z = 3 \]
\[ 2x - y + 4z = 0 \]
\[
\begin{bmatrix}
  1 & 1 & -1 & | & 5 \\
  2 & 0 & 1 & | & 3 \\
  2 & -1 & 4 & | & 0
\end{bmatrix}
\]
\[ x + y = 1 \]
b) \[ y + z = 6 \]
\[ z = 0 \]
\[
\begin{bmatrix}
  1 & 1 & 0 & | & 1 \\
  0 & 1 & 1 & | & 6 \\
  0 & 0 & 1 & | & -5
\end{bmatrix}
\]
We’ll take two methods to solve the system \( AX = B \)

1) **Cramer's rule**

The solution to the system

\[
\begin{align*}
  a_1x + b_1y &= c_1 \\
  a_2x + b_2y &= c_2
\end{align*}
\]

Is given by \( x = \frac{D_x}{D} \) and \( y = \frac{D_y}{D} \) where

\[
D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \quad \text{and} \quad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}
\]

Provided that \( D \neq 0 \)

**Notes:**
1. Cramer’s rule works on systems that have **exactly one** solution.
2. Cramer’s rule gives us a precise formula for finding the solution to an independent system.
3. Note that \( D \) is the determinant made up of the original coefficients of \( x \) and \( y \). \( D \) is used in the denominator for both \( x \) and \( y \). \( D_x \) is obtained by replacing the first (or \( x \) ) column of \( D \) by the constants \( c_1 \) and \( c_2 \). \( D_y \) is found by replacing the second (or \( y \) ) column of \( D \) by the constants \( c_1 \) and \( c_2 \).

**Ex.1:** Use Cramer's rule to solve the system:

\[
\begin{align*}
  3x - 2y &= 4 \\
  2x + y &= -3
\end{align*}
\]

**Sol.**

First find the determinants \( D, D_x, \) and \( D_y \):

\[
D = \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = 3 - (-4) = 7
\]

\[
D_x = \begin{vmatrix} 4 & -2 \\ -3 & 1 \end{vmatrix} = 4 - 6 = -2, \quad D_y = \begin{vmatrix} 3 & 4 \\ 2 & -3 \end{vmatrix} = -9 - 8 = -17
\]

By Cramer's rule, we have

\[
x = \frac{D_x}{D} = \frac{-2}{7} \quad \text{and} \quad y = \frac{D_y}{D} = \frac{-17}{7}
\]

Check in the original equations. The solution set is \( \left\{ \left( -\frac{2}{7}, -\frac{17}{7} \right) \right\} \).
Ex.2: Solve the system:
\[
\begin{align*}
2x + 3y &= 9 \\
-2x - 3y &= 5
\end{align*}
\]
Sol.: Cramer's rule does not work because
\[
D = \begin{vmatrix} 2 & 3 \\ -2 & -3 \end{vmatrix} = -6 - (-6) = 0
\]
Because Cramer's rule fails to solve the system, we apply the addition method:
\[
\begin{align*}
2x + 3y &= 9 \\
-2x - 3y &= 5 \\
\hline
0 &= 14
\end{align*}
\]
Because this last statement is false, the solution set is empty. The original equations are inconsistent.

Ex.3: Solve the system:
\[
\begin{align*}
3x - 5y &= 7 \\
6x - 10y &= 14
\end{align*}
\]
Sol.: Cramer's rule does not apply because
\[
D = \begin{vmatrix} 3 & -5 \\ 6 & -10 \end{vmatrix} = -30 - (-30) = 0
\]
Multiply Eq.(1) by -2 and add it to Eq.(2)
\[
\begin{align*}
-6x + 10y &= -14 \\
6x - 10y &= 14 \\
\hline
0 &= 0
\end{align*}
\]
Because the last statement is an identity, the equations are dependent. The solution set is \(\{(x, y) | 3x - 5y = 7\}\).

Ex.4: Use Cramer's rule to solve the system:
\[
\begin{align*}
2x - 3(y + 1) &= -3 \\
2y &= 3x - 5
\end{align*}
\]
Sol.: First write the equations in standard form, \(Ax + By = C\)
\[
\begin{align*}
2x - 3y &= 0 \\
-3x + 2y &= -5
\end{align*}
\]
Find \(D, D_x,\) and \(D_y:\)
\[
D = \begin{vmatrix} 2 & -3 \\ -3 & 2 \end{vmatrix} = 4 - 9 = -5
\]
\[
D_x = \begin{vmatrix} 0 & -3 \\ -5 & 2 \end{vmatrix} = 0 - 15 = -15
, \quad D_y = \begin{vmatrix} 2 & 0 \\ -3 & -5 \end{vmatrix} = -10 - 0 = -10
\]
Using Cramer's rule, we get
\[
\begin{align*}
x &= \frac{D_x}{D} = \frac{-15}{-5} = 3 \\
y &= \frac{D_y}{D} = \frac{-10}{-5} = 2
\end{align*}
\]
Because (3,2) satisfies both of the original equations, the solution set is \{(3,2)\}.

2) The Gaussian Elimination method

When we solve a single equation, we write simpler and simpler equivalent equations to get an equation whose solution is obvious. In the Gaussian elimination method we write simpler and simpler equivalent augmented matrices until we get an augmented matrix in which the solution to the corresponding system is obvious.

Because each row of an augmented matrix represents an equation, we can perform the row operations on the augmented matrix.

Elementary Row Operation:
2. Interchange two rows \((R_i \leftrightarrow R_j)\).
3. Multiply any row by a constant different from zero \((R_i \leftrightarrow kR_i)\).
4. Add a constant multiply of any row to another row \((R_i \leftrightarrow R_i + kR_j)\).

Ex.1: Use Gaussian elimination method to solve the system (two equations in two variables):

\[
\begin{align*}
2x + y &= 11 \\
x - 3y &= 1
\end{align*}
\]

Sol.:

Start with the augmented matrix:

\[
\begin{bmatrix}
1 & -3 & 11 \\
2 & 1 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -3 & 11 \\
0 & 7 & -21
\end{bmatrix}
\]

\[R_2' = -2R_1 + R_2\]

\[
\begin{bmatrix}
1 & -3 & 11 \\
0 & 1 & -3
\end{bmatrix}
\]

\[R_2' = \frac{1}{7}R_2\]

\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & -3
\end{bmatrix}
\]

\[R_1' = 3R_2 + R_1\]

This augmented matrix represents the system \(x = 2\) and \(y = -3\). So the solution set to the system is \(\{(2,-3)\}\).

Ex.2: Use Gaussian elimination method to solve the system (three equations in three variables):

\[
\begin{align*}
2x - y + z &= -3 \\
x + y - z &= 6 \\
3x - y - z &= 4
\end{align*}
\]

Sol.:

\[
\begin{bmatrix}
2 & -1 & 1 & | & -3 \\
1 & 1 & -1 & | & 6 \\
3 & -1 & -1 & | & 4
\end{bmatrix}
\]
This augmented matrix represents the system \( x = 1, \ y = 2 \) and \( z = -3 \). So the solution set to the system is \( \{ (1, 2, -3) \} \).

**Ex. 3:** Solve the system
\[
\begin{align*}
x - y &= 1 \\
-3x + 3y &= 4
\end{align*}
\]

**Sol.:**
\[
\begin{bmatrix}
1 & -1 & 1 \\
-3 & 3 & 4
\end{bmatrix} \rightarrow R_2' = 3R_1 + R_2 \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 7 \end{bmatrix}
\]

\( R_2 \) corresponds to the equation \( 0 = 7 \). So the equations are **inconsistent**, and there is no solution to the system.

**Ex. 4:** Solve the system
\[
\begin{align*}
3x + y &= 1 \\
6x + 2y &= 2
\end{align*}
\]

**Sol.:**
\[
\begin{bmatrix}
3 & 1 & 1 \\
6 & 2 & 2
\end{bmatrix} \rightarrow R_2' = 2R_1 + R_2 \begin{bmatrix} 3 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\]

In the \( R_2 \) of the augmented matrix we have the equation \( 0 = 0 \). So the equations are **dependent**. For ordered pair that satisfies the first equation satisfies both equations. The solution set is \( \{ (x, y) | 3x + y = 1 \} \)

**Exercises:** Solve the following systems:

1) \[
\begin{align*}
x + y &= 3 \\
-3x + y &= -1
\end{align*}
\]

2) \[
\begin{align*}
x + 2y &= 1 \\
3x + 6y &= 3
\end{align*}
\]

3) \[
\begin{align*}
x + y - z &= 1 \\
x - y + 2z &= 2
\end{align*}
\]
**Matrix Inverse**

The matrix A has an inverse denoted by $A^{-1}$ if $|A| \neq 0$ where $A.A^{-1} = I$. We'll take two methods to find $A^{-1}$ where A is an $n \times n$ matrix.

2) **By Cofactor Method** (Using determinant of the matrix)

The cofactor of the element $a_{ij}$ of the matrix $A = (a_{ij})$ is defined by $c_{ij} = (-1)^{i+j} A_{ij}$ where $A_{ij}$ is the determinant of the matrix that remains when the row $i$ and the column $j$ are deleted.

To find the inverse of a matrix whose determinant is not zero

1- construct the matrix of cofactors of $A$, $\text{cof} (A) = c_{ij}$

2- Construct the transposed matrix of cofactors called the adjoin of $A = \text{adj} (A) = (\text{cof} (A))^T$

3- then $A^{-1} = \frac{1}{\text{det} (A)} \text{adj} A$

4- to check your answer $A.A^{-1} = I$ or $A^{-1}.A = I$

**Ex.**: Use determinant to find $A^{-1}$ where $A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} \text{adj}(A)$

$|A| = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 8 - 1 = 7$

$\text{Cof}(A) = \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}$

$C_{11} = (-1)^{1+1} |4| = 4$

$C_{12} = (-1)^{1+2} |1| = -1$

$C_{21} = (-1)^{2+1} |1| = -1$

$C_{22} = (-1)^{2+2} |2| = 2$

$\text{Adj}(A) = \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}^T = \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}$

$\therefore A^{-1} = \frac{1}{7} \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{4}{7} & \frac{-1}{7} \\ \frac{-1}{7} & \frac{2}{7} \end{pmatrix}$

**Ex2**: Find $A^{-1}$ if $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & -2 \\ 4 & 0 & 2 \end{bmatrix}$
Solution

\[
\begin{vmatrix}
2 & -1 & 3 \\
1 & 0 & -2 \\
4 & 0 & 2 \\
\end{vmatrix} = 2 + 8 = 10
\]

\[\text{cof}(A) = \begin{pmatrix} 0 & -10 & 0 \\ 2 & -8 & -4 \\ 2 & 7 & 1 \end{pmatrix} \]

\[\text{c}_{11} = 0, \quad \text{c}_{12} = (-1)10 = -10, \quad \text{c}_{13} = 0,\]

\[\text{c}_{21} = -(2) = 2, \quad \text{c}_{22} = -8, \quad \text{c}_{23} = -4,\]

\[\text{c}_{31} = 2, \quad \text{c}_{32} = (1)(-7) = 7, \quad \text{c}_{33} = 1\]

\[\text{Adj}(A) = \text{cof}^t = \begin{pmatrix} 0 & 2 & 2 \\ -10 & -8 & 7 \\ 0 & -4 & 1 \end{pmatrix} \]

\[\therefore A^{-1} = -1 \begin{pmatrix} 0 & \frac{1}{5} & \frac{1}{5} \\ -4 & \frac{7}{5} & \frac{10}{5} \\ 0 & -\frac{2}{5} & \frac{1}{10} \end{pmatrix} \]

Problems:

1) write the augment matrix to the following systems then find the solution:

\[x - y + z = 1\]

a) \[2x - 2y + 2z = 2\]

\[-3x + 3y - 3z = -3\]

\[x + y - z = 2\]

b) \[2x - y + z = 1\]

\[3x + 3y - 3z = 8\]

\[2x_2 = 1\]

\[x_3 - 2x_3 = 2\]

c) \[x_1 - 2x_4 = -10\]

\[x_1 + x_4 = 5\]

2) Find the inverse of each following matrix

\[
\begin{pmatrix}
2 & 1 & -1 \\
3 & 5 & 2 \\
5 & -2 & 4 \\
\end{pmatrix}
\]

a)

\[
\begin{pmatrix}
1 & 1 & -3 \\
2 & -1 & 1 \\
1 & 2 & -1 \\
\end{pmatrix}
\]
Chapter six

Hyperbolic Function

\[ \text{Coshu} = \frac{e^u + e^{-u}}{2} \]  
(hyperbolic cosine of \( u \))

\[ \text{Sinhu} = \frac{e^u - e^{-u}}{2} \]  
(hyperbolic sine of \( u \))

\[ \text{tanh} = \frac{\text{Cosh}u}{\text{Sinhu}} = \frac{e^u + e^{-u}}{e^u - e^{-u}} \]

\[ \text{sechu} = \frac{1}{\text{Cosh}u} = \frac{2}{e^u + e^{-u}} \]

\[ \text{cothu} = \frac{\text{Cosh}u}{\text{Sinhu}} = \frac{e^u + e^{-u}}{e^u} \]

\[ \text{csc hu} = \frac{1}{\text{Sinhu}} = \frac{2}{e^u - e^{-u}} \]

Properties

1. \( \cosh \theta - \sinh \theta = 1 \)
2. \( 1 - \tanh \theta = \sec h \theta \), \( \coth \theta - 1 = \csc h \theta \)
3. \( \sinh(\theta_1 + \theta_2) = \sinh \theta_1 \cosh \theta_2 + \cosh \theta_1 \sinh \theta_2 \)
4. \( \cosh(\theta_1 + \theta_2) \cosh \theta_1 \cosh \theta_2 + \sinh \theta_1 \sinh \theta_2 \)
5. \( \sinh(2\theta) = 2\sinh \theta \cosh \theta \)  
   \( \cosh(2\theta) = \cosh^2 \theta + \sinh^2 \theta \)
6. \( \cosh \theta = \frac{1 + \cosh(2\theta)}{2} \)  
   \( \sinh \theta = \frac{\cosh(2\theta) - 1}{2} \)
7. \( \cosh \theta - \sinh \theta = e^{-\theta} \), \( \cosh \theta + \sinh \theta = e^\theta \)

Derivatives

1. \( (\sinh u)' = \cosh u \frac{du}{dx} \)
2. \( (\cosh u)' = \sinh u \frac{du}{dx} \)
3. \( (\tanh)' = \sec h^2 u \frac{du}{dx} \)
4. \( (\coth u)' = -\csc u^2 \frac{du}{dx} \)
5. \( (\sec h u)' = -\sec h \cdot \tanh u \frac{du}{dx} \)
6. \( (\csc hu)' = -\csc hu \cdot \coth u \frac{du}{dx} \)

Example

\( y = \sinh(3x) \)
\( y' = 3\cosh(3x) \)
Integrations:

1. \( \int \sinh u \, du = \cosh u + c \)
2. \( \int \cosh u \, du = \sinh u + c \)
3. \( \int \text{sech}^2 u \, du = \tanh u + c \)
4. \( \int \text{csch}^2 u \, du = -\coth u + c \)
5. \( \int \text{sech} u \, \tanh u \, du = -\text{sec} h u + c \)

Example

\[ \int \sin(3x) \, dx = \frac{1}{3} \cosh(3x) + c \]

The inverse hyperbolic Functions

1. \( y = f(x) = \sinh^{-1} x, D = \mathbb{R}, R = \mathbb{R} \)
2. \( y = f(x) = \cosh^{-1} x, D = \{ x / x \in \mathbb{R}, x \geq 0 \}, R = \mathbb{R} \)
3. \( y = f(x) = \tanh^{-1} x, D = [-1,1], R = \mathbb{R} \)
4. \( y = f(x) = \coth^{-1} x, D = (x \in \mathbb{R}, x > 1, x < 1), R = \mathbb{R} \setminus \{0\} \)
5. \( y = f(x) = \text{sech}^{-1} x, D = [0,1], R = \mathbb{R} \)
6. \( y = f(x) = \text{csch}^{-1} x, D = \mathbb{R} \setminus \{0\}, R = \mathbb{R} \setminus \{0\} \)

Derivatives

\[ (\sinh u)^{-1}' = \frac{du}{dx} \frac{1}{\sqrt{1 + u^2}} \]

\[ (\cosh u)^{-1}' = \frac{du}{dx} \frac{1}{\sqrt{u^2 - 1}} \]

\[ (\tanh u)^{-1}' = \frac{du}{dx} \frac{1}{1 - u^2} \]

\[ (\coth u)^{-1}' = \frac{du}{dx} \frac{1}{1 - u^2} \]

\[ (\text{sech} u)^{-1}' = \frac{du}{dx} \frac{1}{|1 - u^2|} \]

\[ (\text{csch} u)^{-1}' = \frac{du}{dx} \frac{1}{|1 + u^2|} \]

Example

\[ y = \sinh(2x)^{-1} = y' = \frac{2}{\sqrt{1 + (2x)^2}} \]
Integrations:
1- \[ \int \frac{du}{\sqrt{1 + u^2}} = \sinh^{-1} u + c \]
2- \[ \int \frac{du}{\sqrt{u^2 - 1}} = \cosh^{-1} u + c \]
3- \[ \int \frac{du}{1 - u^2} = \begin{cases} \tanh^{-1} u + c, & |u| < 1 \\ \coth^{-1} u + c, & |u| > 1 \end{cases} \]
4- \[ \int \frac{du}{u\sqrt{1 - u^2}} = -\sec^{-1}|u| + c \]
5- \[ \int \frac{du}{u\sqrt{1 + u^2}} = -\csc^{-1}|u| + c \]

Properties
1- \[ \text{sec}^{-1} x = \cosh^{-1} \left( \frac{1}{x} \right) \]
2- \[ \text{csch}^{-1} x = \sinh^{-1} \left( \frac{1}{x} \right) \]
3- \[ \text{coth}^{-1} x = \tanh^{-1} \left( \frac{1}{x} \right) \]

Chapter seven

Complex Numbers
The general form of a complex number is \( z = a + bi \) Where a and b are real numbers , \( \sqrt{-1} \). The number a is called real part denoted by \( \text{Re}(z) = a \), and \( \text{Im}(z) = b \) is called the imaginary part.

**Algebra of complex numbers:**

Let \( z_1 = a + bi \) and \( z_2 = c + di \) be two complex numbers

1- Equality: \( z_1 = z_2 \rightarrow a = c \) and \( b = d \)

2- Addition: \( z_1 + z_2 = (a + c) + (b + d)i \)

3- Multiplication: \( z_1 . z_2 = (a + bi)(c + di) = \) \( (ac - bd) + (ad + bc)i \)

4- Complex conjugate: \( \overline{z} = (a + bi) = a - bi \)

5- Division: \( \frac{z_1}{z_2} = \frac{\overline{z_2}}{z_2} \cdot \frac{z_1}{z_2} \)

6- Length of \( z \) is \( |z| = \sqrt{a^2 + b^2} \)
Polar form

\[ a = r \cos \theta, b = r \sin \theta \text{ then} \]

\[ z = a + bi = r \cos \theta + ir \sin \theta \text{ or } z = r(\cos \theta + i \sin \theta) \text{ is called the polar form.} \]

The angle \( \theta \) is called argument of \( z \) and written \( \arg(z) = \theta \)

Exponential form:

We know that \( e^{i\theta} = \cos \theta + i \sin \theta \), \( z = re^{i\theta} \) is called exponential form.

Theorem:

\[ z_1 = \eta_1(\cos \theta_1 + i \sin \theta_1) \]
\[ z_2 = \eta_2(\cos \theta_2 + i \sin \theta_2) \]

\[ 1 - z_1 \cdot z_2 = \eta_1 \eta_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \]

\[ 2 - \frac{z_1}{z_2} = \frac{\eta_1}{\eta_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)] \]

**Demoisvers Theorem:**

\((\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)\)

Powers: let \( z = r(\cos \theta + i \sin \theta) \), then

\( Z^n = r^n[\cos(n\theta) + i \sin(n\theta)] \)

Roots: let \( z = r(\cos \theta + i \sin \theta) \), then

\( W_k = \sqrt[n]{z} = \sqrt[n]{r[\cos(\theta + \frac{2k\pi}{n}) + i \sin(\theta + \frac{2k\pi}{n})]} \)

\( K = 0, 1, 2, \ldots, n-1 \)

**Chapter eight**

**Vector:**

A vector is a matrix that has only one row – then we call the matrix a *row vector*

– or only one column – then we call it a *column vector*.

A *row vector* is of the form: \([a_1 \ a_2 \ \ldots \ a_n]\)

A *column vector* is of the form:
A quantity such as force, displacement, or velocity is called a vector and is represented by a directed line segment

A vector in the plane is directed line segment. The directed line segment $\overline{AB}$ has initial point $A$ and terminal point $B$; its length is denoted by $|\overline{AB}|$. Two vectors are equal if they have the same length and direction.

**Component form**

If $v$ is a two dimensional vector in the plane equal to the vector with initial point at the origin and terminal point $(v_1, v_2)$, then the Component form of $v$ is:

$$v = (v_1, v_2)$$

If $v$ is a three dimensional vector in the plane equal to the vector with initial point at the origin and terminal point $(v_1, v_2, v_3)$, then the Component form of $v$ is:

$$v = (v_1, v_2, v_3)$$
The numbers \( v_1, v_2 \) and \( v_3 \) are called the components of \( v \).

Given the points \( P(x_1, y_1, z_1) \) and \( Q(x_2, y_2, z_2) \), the standard position vector \( v = (v_1, v_2, v_3) \) equal to \( PQ \) is

\[
v = (x_2 - x_1, y_2 - y_1, z_2 - z_1)
\]

The **magnitude** or **length** of the vector \( v = PQ \) is the nonnegative number

\[
|v| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}
\]

The only vector with length 0 is the zero vector \( 0 = (0,0) \) or \( 0 = (0,0,0) \). This vector is also the only vector with no specific direction.

**Ex.:** Find a) component form and b) length of the vector with initial point \( P(-3, 4, 1) \) and terminal point \( Q(-5, 2, 2) \)

**Solution:**

a) \( v = (-5 + 3, 2 - 4, 2 - 1) \)

The component form of \( PQ \) is \( v = (-2, -2, 1) \)

b) The length or magnitude of \( v = PQ \) is \( |v| = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3 \)

**Vector Addition and Multiplication of a vector by a scalar**

Let \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \) be vectors with \( k \) a scalar.

**Addition:**

\[
u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3)
\]
**Scalar multiplication:** $ku = (ku_1, ku_2, ku_3)$

If the length of $ku$ is the absolute value of the scalar $k$ times the length of $u$. The vector $(-1)u = -u$ has the same length as $u$ but points in the opposite direction.

If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$, $u - v = (u_1 - v_1, u_2 - v_2, u_3 - v_3)$

Note that $(u - v) + v = u$ and the difference $u - v$ as the sum $u + (-v)$

**Ex.:**

Let $u = (-1, 3, 1)$ and $v = (4, 7, 0)$, find

a) $2u + 3v$    b) $u - v$    c) $\left| \frac{1}{2}u \right|$

**Solution:**

a) $2u + 3v = (-2, 6, 2) + (12, 21, 0) = (10, 27, 2)$

b) $u - v = (-5, -4, 1)$

c) $\left| \frac{1}{2}u \right| = \left| \left( \frac{-1}{2}, \frac{3}{2}, \frac{1}{2} \right) \right| = \frac{1}{2} \sqrt{11}$

**Properties of vector operations:**

Let $u$, $v$ and $w$ be vectors and $a$ and $b$ be scalars.

1) $u + v = v + u$  
2) $(u + v) + w = u + (v + w)$

3) $u + 0 = u$  
4) $u + (-u) = 0$

5) $0u = 0$  
6) $1u = u$
7) \( a(bu) = (ab)u \)

8) \( a(u + v) = au + av \)

9) \( (a + b)u = au + bu \)

**Unit vectors**

A vector \( v \) of length 1 is called **unit vector**. The standard unit vectors are:

\[ i = (1,0,0) \quad , \quad j = (0,1,0) \quad , \quad k = (0,0,1) \]

\[ v = (v_1, v_2, v_3) = (v_1,0,0) + (0,v_2,0) + (0,0,v_3) \]

\[ = v_1(1,0,0) + v_2(0,1,0) + v_3(0,0,1) \]

\[ = v_1i + v_2j + v_3k \]

We call the scalar (or number) \( v_1 \) the **i-component** of the vector \( v \) , \( v_2 \) the **j-component** of the vector \( v \), and \( v_3 \) the **k-component**. In component form, \( P_1(x_1,y_1,z_1) \) and \( P_2(x_2,y_2,z_2) \) is

\[ \overrightarrow{P_1P_2} = (x_2-x_1)i + (y_2-y_1)j + (z_2-z_1)k \]

If \( v \neq 0 \), then

1) \( \vec{u} = \frac{v}{|v|} \) is a unit vector in the direction of \( v \), called **the direction** of the nonzero vector \( v \).

2) The equation \( v = \frac{v}{|v|} |v| \) expresses \( v \) in terms of its **length** and **direction**.

**Ex.**

Find a unit vector \( u \) in the direction of the vector \( P_1(1,0,1) \) and \( P_2(3,2,0) \).

**Solution**

\[ \overrightarrow{P_1P_2} = (3-1)i + (2-0)j + (0-1)k = 2i + 2j - k \]

\[ \overrightarrow{P_1P_2} = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{9} = 3 \]
The unit vector \( u \) is the direction of \( \overrightarrow{P_1P_2} \).

**Product of vectors**

u & v are vectors,

There are two kinds of multiplication of two vectors:

1. The scalar product (dot product) u\cdot v. The result is a **scalar**.
2. The vector product (cross product) u\times v. The result is a **vector**.

**1) The dot product**

In this section, we show how to calculate easily the angle between two vectors directly from their components. The dot product is also called **inner** or **scalar** products because the product results in scalar, not a vector.

**Def.:** The dot product \( u \cdot v \) (u dot v) of vectors \( u = (u_1,u_2,u_3) \) and \( v = (v_1,v_2,v_3) \) is:

\[
u \cdot v = u_1v_1 + u_2v_2 + u_3v_3\]

**Note:**

\[
\begin{align*}
  i \cdot i &= 1, \\
  j \cdot j &= 1, \\
  k \cdot k &= 1, \\
  i \cdot j &= 0, \\
  j \cdot k &= 0, \\
  k \cdot j &= 0.
\end{align*}
\]

**Ex.:**

a) \((3,5)\cdot(-1,2) = 3(-1) + 5(2) = 7\) scalar
\((3i + 5j)\cdot(-i + 2j) = 7\)

b) \((1,-3,4)\cdot(1,5,2) = 1 - 15 + 8 = -6\) scalar
\((i - 3j + 4k)\cdot(i + 5j + 2k) = -6\)
Angle between two vectors

The angle $\theta$ between two nonzero vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ is given by

$$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{\| \vec{u} \| \| \vec{v} \|}\right)$$

where $\theta$ $(0 \leq \theta \leq \pi)$

**Ex.:** Find the angle between two vectors in space

\[ \vec{u} = 2\vec{i} - \vec{j} + 2\vec{k}, \quad \vec{v} = \vec{i} - 2\vec{j} + 2\vec{k} \]

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\| \vec{u} \| \| \vec{v} \|} = \frac{2 + 2 + 4}{\sqrt{4 + 1 + 4} \cdot \sqrt{1 + 4 + 4}}$$

$$\cos \theta = \frac{8}{9} \implies \theta = \cos^{-1} \frac{8}{9}$$

**Ex.:**

Find the angle $\theta$ in the triangle ABC determined by the vertices $A = (0,0)$, $B(3,5)$, and $C(5,2)$
\[ \overrightarrow{CA} = (-5,-2) \quad \text{and} \quad \overrightarrow{CB} = (-2,3) \]
\[ \overrightarrow{CA} \cdot \overrightarrow{CB} = (-5)(-2) + (-2)(3) = 4 \]
\[ |\overrightarrow{CA}| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29} \]
\[ |\overrightarrow{CB}| = \sqrt{(-3)^2 + (3)^2} = \sqrt{13} \]
\[ \theta = \cos^{-1}\left( \frac{4}{\sqrt{29} \cdot \sqrt{13}} \right) \]

**Orthogonal vectors**

Vectors \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \) are **orthogonal** (or **perpendicular**) if and only if \( u \cdot v = 0 \)

**Ex.**

a) \( u = (3,-2) \) and \( v = (4,6) \) are orthogonal because \( u \cdot v = 0 \)

b) \( u = 3i - 2j + k \) and \( v = 2j + 4k \) are orthogonal because \( u \cdot v = 0 \)

c) \( 0 \) is orthogonal to every vector \( u \) since
\[ 0 \cdot u = (0,0,0) \cdot (u_1, u_2, u_3) \]
\[ = 0 \]

**Properties of the Dot product**

If \( u, v \) and \( w \) are any vectors and \( c \) is a scalar, then

1) \( u \cdot v = v \cdot u \)
2) \( (cu) \cdot v = u \cdot (cv) = c(u \cdot v) \)
3) \( u \cdot (v + w) = u \cdot v + u \cdot w \)
4) \( u \cdot u = |u|^2 \)
5) \( 0 \cdot u = 0 \)

**Vector projection**

Vector projection of \( u \) onto \( v \)
\[ \text{proj}_v u = \left( \frac{u \cdot v}{|v|^2} \right) v \quad \ldots \quad (1) \]

\( \text{proj}_v u \) (**The vector projection of \( u \) onto \( v \)**)
Scalar component of $u$ in the direction of $v$

\[ |u| \cos \theta = \frac{u \cdot v}{|v|} = \frac{u \cdot v}{|v|} \quad \ldots \ldots (2) \]

Ex.: Find the vector projection of $u = 6i + 3j + 2k$ onto $v = i - 2j - 2k$ and the scalar component of $u$ in the direction of $v$.

Solution:

We find $\text{proj}_v u$ from eq.(1):

\[ \text{proj}_v u = \left( \frac{u \cdot v}{|v|^2} \right)v = \frac{u \cdot v}{v \cdot v}v = \frac{6 - 6 - 4}{1 + 4 + 4} (i - 2j - 2k) = \frac{-4}{9} (i - 2j - 2k) = \frac{-4}{9}i + \frac{8}{9}j + \frac{8}{9}k \]

We find the scalar component of $u$ in the direction of $v$ from eq.(2):

\[ |u| \cos \theta = \frac{u \cdot v}{|v|} = 6i + 3j + 2k \cdot \left( \frac{1}{3}i - \frac{2}{3}j - \frac{2}{3}k \right) = 2 \cdot -2 \cdot \frac{4}{3} = \frac{-4}{3} \]