Vector:

A vector is a matrix that has only one row – then we call the matrix a **row vector** – or only one column – then we call it a **column vector**.

**A row vector** is of the form: \( a = [a_1 \ a_2 \ ... \ a_n] \)

**A column vector** is of the form:

\[
\begin{bmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_m
\end{bmatrix}
\]

A quantity such as force, displacement, or velocity is called a vector and is represented by a directed line segment

A **vector** in the plane is directed line segment. The directed line segment \( \overrightarrow{AB} \) has **initial point** \( A \) and **terminal point** \( B \); its **length** is denoted by \( |\overrightarrow{AB}| \). Two vectors are **equal** if they have the same length and direction.

**Component form**

If \( \mathbf{v} \) is a **two dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point \( (v_1,v_2) \) , then the **Component form** of \( \mathbf{v} \) is:

\[ \mathbf{v} = (v_1,v_2) \]

If \( \mathbf{v} \) is a **three dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point \( (v_1,v_2,v_3) \), then the **Component form** of \( \mathbf{v} \) is:

\[ \mathbf{v} = (v_1,v_2,v_3) \]
The numbers \( v_1, v_2 \) and \( v_3 \) are called the components of \( v \).

Given the points \( P(x_1, y_1, z_1) \) and \( Q(x_2, y_2, z_2) \), the standard position vector \( v = (v_1, v_2, v_3) \) equal to \( \overrightarrow{PQ} \) is

\[
v = (x_2 - x_1, y_2 - y_1, z_2 - z_1)
\]

The magnitude or length of the vector \( v = \overrightarrow{PQ} \) is the nonnegative number

\[
|v| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}
\]

The only vector with length 0 is the zero vector \( 0 = (0, 0) \) or \( 0 = (0, 0, 0) \). This vector is also the only vector with no specific direction.

**Ex.:** Find a) component form and b) length of the vector with initial point \( P(-3, 4, 1) \) and terminal point \( Q(-5, 2, 2) \)

**Solution:**

a) \( v = (-5 + 3, 2 - 4, 2 - 1) \)

The component form of \( \overrightarrow{PQ} \) is \( v = (-2, -2, 1) \)

b) The length or magnitude of \( v = \overrightarrow{PQ} \) is \( |v| = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3 \)

**Vector Addition and Multiplication of a vector by a scalar**

Let \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \) be vectors with \( k \) a scalar.

**Addition:**

\[
u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3)
\]
Scalar multiplication: $ku = (ku_1, ku_2, ku_3)$

If the length of $ku$ is the absolute value of the scalar $k$ times the length of $u$. The vector $(-1)u = -u$ has the same length as $u$ but points in the opposite direction.

If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$, $u - v = (u_1 - v_1, u_2 - v_2, u_3 - v_3)$

Note that $(u - v) + v = u$ and the difference $u - v$ as the sum $u + (-v)$

Ex.: Let $u = (-1, 3, 1)$ and $v = (4, 7, 0)$, find

a) $2u + 3v$  

b) $u - v$  

c) $\frac{1}{2}u$

Solution:

a) $2u + 3v = (-2, 6, 2) + (12, 21, 0) = (10, 27, 2)$

b) $u - v = (-5, -4, 1)$

c) $\frac{1}{2}u = \left(\frac{-1}{2}, \frac{3}{2}, \frac{1}{2}\right) = \frac{1}{2}\sqrt{11}$

Properties of vector operations:

Let $u$, $v$ and $w$ be vectors and $a$ and $b$ be scalars.

1) $u + v = v + u$  
2) $(u + v) + w = u + (v + w)$

3) $u + 0 = u$  
4) $u + (-u) = 0$

5) $0u = 0$  
6) $1u = u$
7) \( a(bu) = (ab)u \)
8) \( a(u + v) = au + av \)
9) \( (a+b)u = au + bu \)

**Unit vectors**

A vector \( v \) of length 1 is called **unit vector**. The standard unit vectors are:

\[
i = (1,0,0) \quad , \quad j = (0,1,0) \quad , \quad k = (0,0,1)
\]

\[
v = (v_1, v_2, v_3) = (v_1,0,0) + (0,v_2,0) + (0,0,v_3)
\]

\[
= v_1(1,0,0) + v_2(0,1,0) + v_3(0,0,1)
\]

\[
= v_1i + v_2j + v_3k
\]

We call the scalar (or number) \( v_1 \) the **i-component** of the vector \( v \), \( v_2 \) the **j-component** of the vector \( v \), and \( v_3 \) the **k-component**. In component form, \( P_1(x_1,y_1,z_1) \) and \( P_2(x_2,y_2,z_2) \) is

\[
\overrightarrow{P_1P_2} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k
\]

If \( v \neq 0 \), then

1) \( u = \frac{v}{|v|} \) is a unit vector in the direction of \( v \), called **the direction** of the nonzero vector \( v \).

2) The equation \( v = \frac{v}{|v|} |v| \) expresses \( v \) in terms of its **length** and **direction**.

Ex.:

Find a unit vector \( u \) in the direction of the vector \( P_1(1,0,1) \) and \( P_2(3,2,0) \).

**Solution**

\[
\overrightarrow{P_1P_2} = (3-1)i + (2 - 0)j + (0 - 1)k = 2i + 2j - k
\]

\[
\overrightarrow{P_1P_2} = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{9} = 3
\]
\[ u = \frac{P_1 P_2}{|P_1 P_2|} = \frac{2i + 2j - k}{3} = \frac{2}{3} i + \frac{2}{3} j - \frac{1}{3} k \]

The unit vector \( u \) is the direction of \( P_1 P_2 \).

**Midpoint of a line segment**

The midpoint \( M \) of a line segment joining points \( P_1(x_1, y_1, z_1) \) and \( P_2(x_2, y_2, z_2) \) is the point

\[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)
\]

\[
\overrightarrow{OM} = \overrightarrow{OP_1} + \frac{1}{2} (\overrightarrow{P_1 P_2})
\]

\[
= \overrightarrow{OP_1} + \frac{1}{2} (\overrightarrow{P_2} - \overrightarrow{P_1})
\]

\[
= \frac{1}{2} (\overrightarrow{OP_1} + \overrightarrow{OP_2})
\]

\[
= \frac{1}{2} \left( \left( x_1, y_1, z_1 \right) + \left( x_2, y_2, z_2 \right) \right)
\]

\[
= \frac{1}{2} \left( x_1 + x_2, y_1 + y_2, z_1 + z_2 \right)
\]

Ex.:

The midpoint of the segment joining \( P_1(3, -2, 0) \) and \( P_2(7, 4, 4) \) is

\[
\left( \frac{3 + 7}{2}, \frac{-2 + 4}{2}, \frac{0 + 4}{2} \right) = (5, 1, 2)
\]

**Product of vectors**

\( u \& \ v \) are vectors,

There are two kinds of multiplication of two vectors:

1. The scalar product (dot product) \( u \cdot v \). The result is a *scalar*.
2. The vector product (cross product) \( u \times v \). The result is a *vector*.

1) **The dot product**

In this section, we show how to calculate easily the angle between two vectors directly from their components. The dot product is also called *inner* or *scalar* products because the product results in scalar, not a vector.
Def.: The dot product \( u \cdot v \) ( \( u \) dot \( v \)) of vectors \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \) is:

\[
 u \cdot v = u_1v_1 + u_2v_2 + u_3v_3
\]

Note:

\[
\begin{align*}
    i \cdot i &= 1, \\
    j \cdot j &= 1, \\
    k \cdot k &= 1, \\
    j \cdot k &= 0,
\end{align*}
\]

Ex.:

a) 
\[
(3,5) \cdot (-1,2) = 3(-1) + 5(2) = 7 \quad \text{scalar}
\]
\[
(3i + 5j) \cdot (-i + 2j) = 7
\]

b) 
\[
(1,-3,4) \cdot (1,5,2) = 1 - 15 + 8 = -6 \quad \text{scalar}
\]
\[
(i - 3j + 4k) \cdot (i + 5j + 2k) = -6
\]

Angle between two vectors

The angle \( \theta \) between two nonzero vectors \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \) is given by

\[
\cos \theta = \frac{u \cdot v}{\| u \| \| v \|}
\]

\[
\theta = \cos^{-1}\left( \frac{u \cdot v}{\| u \| \| v \|} \right)
\]

where \( \theta \) \( (0 \leq \theta \leq \pi) \)

Ex.: Find the angle between two vectors in space

\[
\begin{align*}
    \vec{u} &= 2\hat{i} - \hat{j} + 2\hat{k}, \\
    \vec{v} &= \hat{i} - 2\hat{j} + 2\hat{k}
\end{align*}
\]

\[
\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\| u \| \| v \|} = \frac{2 + 4}{\sqrt{4+1+4} \cdot \sqrt{1+4+4}}
\]

\[
\cos \theta = \frac{8}{9} \quad \Rightarrow \quad \theta = \cos^{-1}\left( \frac{8}{9} \right)
\]
Ex.:
Find the angle $\theta$ in the triangle ABC determined by the vertices

$A = (0,0), B(3,5)$ and $C(5,2)$

$\overrightarrow{CA} = (-5,-2)$ and $\overrightarrow{CB} = (-2,3)$

$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-5)(-2) + (-2)(3) = 4$

$|\overrightarrow{CA}| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$

$|\overrightarrow{CB}| = \sqrt{(-3)^2 + (3)^2} = \sqrt{13}$

$\theta = \cos^{-1} \left( \frac{4}{\sqrt{29} \cdot \sqrt{13}} \right)$

**Orthogonal vectors**

Vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ are **orthogonal** (or **perpendicular**) if and only if $u \cdot v = 0$

Ex.:

a) $u = (3,-2)$ and $v = (4,6)$ are orthogonal because $u \cdot v = 0$

b) $u = 3i - 2j + k$ and $v = 2j + 4k$ are orthogonal because $u \cdot v = 0$

c) $0$ is orthogonal to every vector $u$ since

$0 \cdot u = (0,0,0) \cdot (u_1, u_2, u_3)$

$= 0$

**Properties of the Dot product**

If $u$, $v$ and $w$ are any vectors and $c$ is a scalar, then

1) $u \cdot v = v \cdot u$

2) $(cu) \cdot v = u \cdot (cv) = c(u \cdot v)$

3) $u \cdot (v + w) = u \cdot v + u \cdot w$

4) $u \cdot u = |u|^2$
5) \( 0 \cdot u = 0 \)

**Vector projection**

Vector projection of \( u \) onto \( v \)

\[
\text{proj}_v u = \left( \frac{u \cdot v}{|v|^2} \right) v \quad \ldots \quad (1)
\]

\( \text{proj}_v u \) (**The vector projection of** \( u \) **onto** \( v \)**)

Scalar component of \( u \) in the direction of \( v \)

\[
|u| \cos \theta = \frac{u \cdot v}{|v|} = \frac{u}{|v|} \cdot \frac{v}{|v|} \quad \ldots \quad (2)
\]

**Ex.:**

Find the vector projection of \( u = 6i + 3j + 2k \) onto \( v = i - 2j - 2k \) and the scalar component of \( u \) in the direction of \( v \).

**Solution:**

We find \( \text{proj}_v u \) from eq.(1):

\[
\text{proj}_v u = \left( \frac{u \cdot v}{|v|^2} \right) v = \frac{u \cdot v}{v \cdot v} \cdot v = \frac{6 - 6 - 4}{1 + 4 + 4} (i - 2j - 2k) = \frac{-4}{9} (i - 2j - 2k) = \frac{-4}{9} i + \frac{8}{9} j + \frac{8}{9} k
\]

We find the scalar component of \( u \) in the direction of \( v \) from eq.(2):

\[
|u| \cos \theta = u \cdot \frac{v}{|v|} = 6i + 3j + 2k \cdot \left( \frac{1}{3} i - \frac{2}{3} j - \frac{2}{3} k \right) = 2 \cdot 2 - \frac{4}{3} = \frac{-4}{3}
\]

**Problems:**

1) Let \( u = (3,-2) \) and \( v = (-2,5) \). Find the **a)** component form and **b)** magnitude (length) of the vector.

1. \(-2u + 5v\)
2. \(- \frac{3}{5} u + \frac{4}{5} v\)

2) Find the component form of the vector:

a. The vector \( \overrightarrow{PQ} \) where \( P = (1,3) \) and \( Q(2,-1) \).

b. The vector \( \overrightarrow{OP} \) where \( O \) is the origin and \( P \) is the midpoint of segment \( RS \), where \( R = (2,-1) \) and \( S = (-4,3) \).

c. The vector from the point \( A = (2,3) \) to the origin.
d. The sum of $\overrightarrow{AB}$ and $\overrightarrow{CD}$, where

$A = (1,-1)$, $B = (2,0)$, $C = (-1,3)$ and $D = (-2,2)$

3) Let $v$, $u$ and $w$ as in the figure: find a) $u + v$, b) $u + v + w$, c) $u - v$ and
d) $u - w$

4) Express each vector as a product of its length and direction

a. $2i + j - 2k$

b. $5k$

c. $\frac{3}{5}i + \frac{4}{5}k$

d. $\frac{1}{\sqrt{6}}i - \frac{1}{\sqrt{6}}j - \frac{1}{\sqrt{6}}k$

5) Find the vectors whose lengths and directions are given. Try to do the calculation without writing:

<table>
<thead>
<tr>
<th>Length</th>
<th>Direction</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. 2</td>
<td>i</td>
</tr>
<tr>
<td>b. $\sqrt{3}$</td>
<td>-k</td>
</tr>
<tr>
<td>c. $\frac{1}{2}$</td>
<td>$\frac{3}{5}j + \frac{4}{5}k$</td>
</tr>
<tr>
<td>d. 7</td>
<td>$\frac{6}{7}i - \frac{2}{7}j + \frac{3}{7}k$</td>
</tr>
</tbody>
</table>

6) Find a) the direction of $\overrightarrow{P_1P_2}$ and b) the midpoint of line segment $P_1P_2$.

a. $P_1(-1,1,5)$ and $P_2(2,5,0)$

b. $P_1(0,0,0)$ and $P_2(2,-2,-2)$

7) Find $v \cdot u$, $|v|$, $|u|$, the cosine of the angle between $v$ and $u$, the scalar component of $u$ in the direction of $v$ and the vector $proj_v u$.

a) $v = 2i - 4j + \sqrt{5}k$, $u = -2i + 4j - \sqrt{5}k$
b) \( v = \left( \frac{3}{5} \right)i + \left( \frac{4}{5} \right)k \), \( u = 5i + 12j \)

c) \( v = -i + j \), \( u = \sqrt{2}i + \sqrt{3}j + 2k \)

d) \( v = 5i + j \), \( u = 2i + \sqrt{17}j \)

e) \( v = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right) \), \( u = \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}} \right) \)

8) Find the angles between the vectors:

a) \( u = 2i - 2j + k \), \( v = 3i + 4k \)

b) \( u = \sqrt{3}i - 7j \), \( v = \sqrt{3}i + j - 2k \)

c) \( u = i + \sqrt{2}j - \sqrt{2}k \), \( v = -i + j + k \)

9) Find the measures of the angles between the diagonals of the rectangle whose vertices are \( A = (1,0) \), \( B(0,3) \), \( C(3,4) \) and \( D(4,1) \)

References:

2- Calculus (Haward Anton).
3- Advanced Mathematics for Engineering Studies (أ.رياض أحمد عزت)