Def. 1: If \( y = f(x) \) is a continuous function, then we define the derivative of \( f \) as admit as:

\[
\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

Example:

Find the derivative of the function \( y = x^2 \)

\[
y = x^2 = f(x) \quad \Rightarrow \quad y + \Delta y = (x + \Delta x)^2 = f(x + \Delta x)
\]

\[
\Delta y = x^2 + 2x \Delta x + \Delta x^2 - f(x) = x^2 + 2x \Delta x + \Delta x^2
\]

\[
\frac{\Delta y}{\Delta x} = 2x + \Delta x
\]

\[
\Rightarrow \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} 2x + \Delta x = 2x
\]

Derivative of algebraic functions:

1. If \( y_1 = f(x) \),

\[
\frac{d}{dx}(f(x) + g(x)) = \frac{df(x)}{dx} + \frac{dg(x)}{dx}
\]

2. \( \frac{d}{dx}(f(x) \cdot g(x)) = f(x) \frac{dg(x)}{dx} + g(x) \frac{df(x)}{dx} \)

3. \( \frac{d}{dx}\left( \frac{f(x)}{g(x)} \right) = \frac{g(x) \frac{df(x)}{dx} - f(x) \frac{dg(x)}{dx}}{[g(x)]^2} \)

4. \( \frac{d}{dx}[k \cdot f(x)] = k \frac{df(x)}{dx} \quad \text{Known as constant} \)
$5 - \frac{d}{dx}\left[f(x)^n\right] = n f(x)^{n-1} \frac{df(x)}{dx}$

$6. \frac{d}{dx}(x^n) = nx^{n-1}$

**Chain rule.** Suppose that $h = g \circ f$ is the composition of the differentiable functions $g = g(u)$ and $x = f(t)$. Then $h$ is a differentiable function whose derivative at each value of $t$ is

$$h(t) = g(f(t)) \Rightarrow \frac{dh}{dt} = \frac{dy}{du} \cdot \frac{dt}{dx} \cdot \frac{du}{dt}$$

**Ex.** Find $\frac{dy}{dt}$ at $t = -1$. If $y = x^3 + 5x - 4 = g(x)$ and $x = t^2 + t = f(t)$

**Solution**

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \cdot \frac{dt}{dt} = (3x^2 + 5)(2t + 1)$$

$$= (-5)(-1) = 5$$

**L'Hôpital.** Suppose that $f(x) = g(x) = 0$ and $g'(x)$ exist, and that $g(x) \neq 0$. Then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

**Ex.** Evaluate

$$\lim_{x \to \infty} \frac{x^2 + 1}{x^4 - 3} = \frac{\infty}{\infty}$$

$$= \lim_{x \to \infty} \frac{2x}{3x^2} = \frac{0}{\infty} = 0$$
Def. (2) Higher order derivative: If \( y = f(x) \) a continuous function, then the first derivative of \( f \) is \( \dot{y} = \frac{dy}{dx} = \frac{df}{dx} = f'(x) \). The second order derivative is \( \ddot{y} = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) = f''(x) \). ... the \( n \)th order derivative is \( \frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = f^{(n)}(x) \).

Ex. If \( y = 4x^3 + 2x + 1 \), find \( \frac{d^2y}{dx^2} \).

Solution. \( y = 12x^2 + 2 \Rightarrow \ddot{y} = 24x \)

Derivative of a parametric function:

If \( y = f(t) \), \( x = g(t) \), then
\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \Rightarrow \dot{y} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dt} / \frac{dx}{dt}
\]

Ex. Find \( \frac{d^2y}{dx^2} \), if \( x = t - t^2 \) and \( y = t^2 + 3 \).

Solution:
\[
\dot{y} = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 - 3t^2}{1 - 2t}
\]
\[
\ddot{y} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{1 - 3t^2}{1 - 2t}\right) \frac{dt}{dx} = \frac{d\left[\frac{1 - 3t^2}{1 - 2t}\right]}{dx} \frac{dt}{dx} = \frac{2 - 6t + 6t^2}{(1 - 2t)^3}
\]
Implicit differentiation: If we have implicit fun. for example 

\[ x^2 + 2xy + y^4 = 3, \]
then we must differentiate w.r.t. \( x \) then we have \( \frac{dy}{dx} \) as:

\[ 2x + 2y = 2x \frac{dy}{dx} + 4y^3 \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{2x + 2y}{2x + 4y^3} \]

or we differentiate w.r.t. \( y \) then we have \( \frac{dx}{dy} \) as:

\[ 2x + 2y = 2 \frac{dx}{dy} + 4y^3 \frac{dx}{dy} \Rightarrow 2x + 2y = \frac{2x + 4y^3}{\frac{dx}{dy}} \Rightarrow \frac{dx}{dy} = \frac{2x + 4y^3}{2x + 2y} = \frac{1}{\frac{dy}{dx}}. \]

Mean value theorem: If the fun. \( f \) is defined and continuous on closed interval \( [a, b] \) and \( f \) is differentiable at \( (a, b) \). Then there exist at least anumbe \( c \) \((a < c < b)\)

such that \( f'(c) = \frac{f(b) - f(a)}{b - a} \).

**EX.**

If \( f(x) = \frac{x^2}{6} \) on \( [2, 6] \) then \( f(x) \) is continuous, \( a \in [2, 6] \) and 

\[ f'(x) = \frac{x}{3}, \Rightarrow f'(x) \text{ differentiable} \]

\[ \frac{f(4)}{f(2)} = \frac{\frac{4}{3}}{\frac{2}{3}} = \frac{2}{1}, \quad f(6) = 6 \]

\[ f'(c) = \frac{c}{3} \]

\[ \frac{c}{3} = \frac{6 - \frac{c}{3}}{6 - 2} \Rightarrow \frac{c}{3} = \frac{\frac{16}{3}}{4} \]

\[ \Rightarrow c = 4 \]
Rolle's theorem: If the function $f$ is defined and continuous on closed interval $[a, b]$ and differentiable at $(a, b)$ and $f(a) = f(b) = 0$. Then there exists at least one number $c$ ($a < c < b$) such that $f'(c) = 0$ and

$$
\frac{f(b) - f(a)}{b - a} = \frac{0 - 0}{b - a} = 0.
$$

Ex.

If $f(x) = 6x^2 - 3x$ is a continuous function on $[0, 6]$ and differentiable on $(0, 6)$ and $f(x) = 12x - 3x^2$. Hence

$$
f'(c) = f(c) = 0 \Rightarrow f'(c) = 0
$$

$$
12c - 3c^2 = 0 \Rightarrow 3c(4c - 1) = 0
$$

$\Rightarrow c = 0$ or $c = \frac{1}{4}$

Derivatives of trigonometric functions:

- If $y = \cos \theta$ then $y = -\sin \theta \frac{d\theta}{dx}$
- If $y = \sin \theta$ then $y = \cos \theta \frac{d\theta}{dx}$
- If $y = \tan \theta$ then $y = \sec^2 \theta \frac{d\theta}{dx}$
- If $y = \cot \theta$ then $y = -\csc^2 \theta \frac{d\theta}{dx}$
- If $y = \sec \theta$ then $y = \sec \theta \tan \theta \frac{d\theta}{dx}$
- If $y = \csc \theta$ then $y = -\csc \theta \cot \theta \frac{d\theta}{dx}$

Ex. If $y = \cos x^2$ then $y = -\sin x^2 \cdot 2x$
\[ \text{Ex: If } y = \tan^2 \sqrt{x} \text{ then } y' = 2 \tan \sqrt{x} \cdot \sec^2 \sqrt{x} \cdot \frac{1}{2x} \]

\[ \text{Ex: If } 2 \tan y + 2 \sin x \cos x = \sec y + 1 \text{ then} \]

\[ \text{the derivative is } \sec y \frac{dy}{dx} + 2 \sin x \cdot \cos x \cdot \sec y \cdot \tan y = \sec y \tan y \]

\[ \Rightarrow \sec y \frac{dy}{dx} - \sec y \tan y \cdot \frac{dy}{dx} = 2 \sin x \cdot 2 \cos x \]

\[ \Rightarrow \sec y \frac{dy}{dx} = 2 \sin^2 x - 2 \cos^2 x \]

\[ \therefore \frac{dy}{dx} = \frac{2 \sin^2 x - 2 \cos^2 x}{\sec^2 y - \sec y \tan y} = \frac{-2 \cos(2x)}{\sec^2 y - \sec y \tan y} \]

\[ \text{5. Integration} \]

\[ \text{Def.} \ 0 \ (\text{Indefinite integral}) \text{ Let } F(x) = y \text{ be a fun. whose } \]

\[ \text{derivative is given by the equation } \frac{dy}{dx} = F'(x) \text{ also } \]

\[ F(x) + C \text{ is an antiderivative of } f(x). \text{ The set of all } \]

\[ \text{antiderivative of } f(x) \text{ is called the indefinite integral of } f \text{ wrt } x \text{ and is denoted by the symbol } \int f(x) \, dx. \]

\[ \text{Hence the law of indefinite integrals is} \]

\[ \int f(x) \, dx = F(x) + C \text{ where } C \text{ is constant} \]

\[ \text{Ex: Solve the differential equation} \]

\[ \frac{dy}{dx} = 3x^2 \]

\[ \text{Solution: we have} \]

\[ \frac{d(x^3)}{dx} = 3x^2 \]

\[ \therefore y = \int 3x^2 \, dx = x^3 + C. \]
properties of indefinite integrals

1. \[ \int [f(x) + g(x)] \, dx = \int f(x) \, dx + \int g(x) \, dx \]

2. \[ \int k \cdot f(x) \, dx = k \int f(x) \, dx \] if \( k \) is any constant.

3. \[ \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad \text{if} \quad n \neq -1 \]

4. \[ \int \left( \frac{df(x)}{dx} \right) \, dx = f(x) + C \quad n \neq -1 \]

5. \[ \int dx = x + C \]

6. \[ \int \sin u \, du = -\cos u + C \]

7. \[ \int \cos u \, du = \sin u + C \]

8. \[ \int \sec^2 u \, du = \tan u + C \]

9. \[ \int \csc^2 u \, du = -\cot u + C \]

10. \[ \int (\csc u \cot u) \, du = -\csc u + C \]

11. \[ \int \sec u \tan u \, du = \sec u + C \]

\[ \text{Ex.} \quad 1. \int (x^3 + 2)^2 \, dx = \int (x^6 + 4x^3 + 4) \, dx = \frac{x^7}{7} + \frac{4}{3}x^4 + 4x + C \]

2. \[ \int (x+1)^3 \, dx = \frac{(x+1)^4}{4} + C \]

3. \[ \int \frac{\sin 5x}{2\sqrt{x}} \, dx = -2 \cos \sqrt{x} + C \]

4. \[ \int \sec^2 (x+1) \, dx = \tan(x+1) + C \]
Definite integrals

If \( f \) is a continuous function on \([a, b]\) and \( F \) is any antiderivative of \( f \) on \([a, b]\) then

\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]

\[\text{Ex.}\]

\[
\int_{-1}^1 x^2 \, dx = \frac{x^3}{3} \bigg|_{-1}^1 = \frac{1}{3} - \frac{-1}{3} = \frac{2}{3}
\]

Properties of definite integrals

If \( a, b, c \) and \( k \) are real numbers and \( f \) is a continuous function then

1. \( \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx \)

2. \( \int_a^b f(x) \, dx = 0 \quad \text{if} \quad f \quad \text{is an even function} \)

3. \( \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx \)

4. \( \int_a^b k \cdot f(x) \, dx = k \cdot \int_a^b f(x) \, dx \)

5. \( \int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \)

6. \( \int_a^b f(x) \, dx = 2 \cdot \int_0^b f(x) \, dx \quad \text{if} \quad f \quad \text{is an odd function} \)

\[\text{Ex.}\]

\[
\int_{-1}^1 (x^2 - 2x^3) \, dx = 0 \quad \text{the function is odd.}
\]

\[\text{Ex. (2)}\]

\[
\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \frac{\cos x}{\sqrt{1 - \sin x}} \, dx = -\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} (1 - \sin x)^{\frac{1}{2}} \cos x \, dx = \left[ \frac{(1 - \sin x)^{\frac{3}{2}}}{\frac{3}{2}} \right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}}
\]

\[\text{Ex. (3)}\]

\[
\int_1^2 (2 + u) \sqrt{u} \, du = \int_1^2 2 \sqrt{u} \, du + \int_1^2 u \, du = \left[ \frac{2u^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^2 + \left[ \frac{u^2}{2} \right]_1^2 = \frac{10}{3} + \frac{1}{3} = \frac{11}{3}
\]
Definite integral: An integral of $f(x)$ on a closed interval $[a, b]$ is called a definite integral of $f(x)$ from $x = a$ to $x = b$ and denoted by 

$$
\int_{a}^{b} f(x)\,dx
$$

the number $a$ and $b$ are called the lower and upper limits of integration respectively.

Theorem: If $f(x)$ is non-negative and continuous on the interval $[a, b]$ then a definite integral of $f(x)$ from $x = a$ to $x = b$ exists and equals the area bounded by $f(x)$, x-axis, the lines $x = a$ and $x = b$. In other words, the definite integral $\int_{a}^{b} f(x)\,dx$ represents the area under the curve $f(x)$ on the interval.

$$
\text{Area} = A = \int_{a}^{b} f(x)\,dx
$$

Mean value theorem for integrals: If $f(x)$ is continuous on the interval $[a, b]$ then there exists a number $c$ in $[a, b]$ at which

$$
\frac{F(b) - F(a)}{b - a} = f(c)
$$

where

$$
F(x) = \int_{a}^{x} f(t)\,dt
$$

Example: Evaluate the mean value theorem of $f(x) = \sin x \cos x$ on the interval $[0, \pi]$. Solution

$$
\frac{1}{\pi} \int_{0}^{\pi} \sin x \cos x\,dx = \frac{1}{\pi} \left[ \frac{\sin x \cos x}{2} \right]_{0}^{\pi} = \frac{1}{\pi} \left( \frac{0}{2} - \frac{0}{2} \right) = 0
$$
The logarithmic and exponential functions

**Definition**
A natural logarithm is a function from the domain $x \in \mathbb{R}^+$ to the range $x \in \mathbb{R}$, denoted by $y = \ln x$

Graph of $y = \ln x$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = \ln x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty$</td>
<td>$\ln x \to \infty$ as $x \to \infty$</td>
</tr>
<tr>
<td>$1$</td>
<td>$\ln 1 = 0 \Rightarrow (1,0)$</td>
</tr>
<tr>
<td>$-\infty$</td>
<td>$\lim_{x \to -\infty} \ln x = -\infty$</td>
</tr>
</tbody>
</table>

Properties

1. $\ln u + v = \ln uv$
2. $\ln \left(\frac{u}{v}\right) = \ln u - \ln v$
3. $\ln u^v = v \ln u$

Example: $\ln \sqrt[3]{25} = \frac{1}{3} \ln 25 = \frac{1}{3} \ln 5^2 = \frac{2}{3} \ln 5$
Def. (2) [Exponential Fun.]: The function \( y = \ln x \) for \( x > 0 \) have inverse function called exponential fun. denoted by \( y = e^x \) with domain \( \{ x : x > 0 \} \) and range \( \mathbb{R} \).

For \( \text{Ex.} \), \( y = e^{x^2}, \ y = e^x \).

Properties:

1. \( e^{x+y} = e^x e^y \)
2. \( e^{x-y} = \frac{e^x}{e^y} \)
3. \( [e^x]'' = e^x \)
4. \( \frac{d}{dx} e^x = 1 \)
5. \( \ln e = 1 \) \( \ln e^x = x \)

Graph of \( y = e^x \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = e^x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\infty)</td>
<td>( e^{-\infty} )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( e^0 = 1 )</td>
</tr>
<tr>
<td>( \infty )</td>
<td>( e^\infty )</td>
</tr>
<tr>
<td>( 1 )</td>
<td>( e^1 = e )</td>
</tr>
</tbody>
</table>

Laws of derivative for \( y = e^x, \ y = \ln x \):

1. If \( y = e^{ux} \) then \( y' = u e^{ux} \)
2. If \( y = \ln u \) then \( y' = \frac{1}{u} \) \( \frac{du}{dx} \)
Ex. Find $\frac{dy}{dx}$ for the following.

1. $y = e^x \Rightarrow \frac{dy}{dx} = e^x \cos x$

2. $y = \ln \sqrt{x} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$

3. $y = e^{\ln x^2} \Rightarrow y = x^2 \Rightarrow \frac{dy}{dx} = 2x$

4. $y = \frac{x(x+1)}{(x-1)^2(x^2+1)} \Rightarrow \ln y = \ln \left( \frac{x(x+1)}{(x-1)^2(x^2+1)} \right)$

   $\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{x(x+1) - 2(x^2+1)}{(x-1)^2(x^2+1)}$

   $\ln y = \ln x(x+1) - \ln (x-1)^2 - \ln (x^2+1)$

   $\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} + \frac{1}{x+1} - \frac{1}{x-1} - \frac{2x}{x^2+1}$

   $\frac{dy}{dx} = y \left[ \frac{1}{x} + \frac{1}{x+1} - \frac{2x}{x^2+1} \right]

   = \frac{x(x+1)}{(x-1)^2(x^2+1)} \left[ \frac{1}{x} + \frac{1}{x+1} - \frac{2x}{x^2+1} \right]

5. $y = x^x \Rightarrow \ln y = x \ln x$

   $\Rightarrow \frac{1}{y} \frac{dy}{dx} = x \frac{1}{x} + \ln x + 1 = 1 + \ln x$

   $\frac{dy}{dx} = y [1 + \ln x] = x^x [1 + \ln x]$

Laws of integrations

1. $\int \frac{du}{u} = \ln |u| + C$

2. $\int e^u \, du = e^u + C$

Ex. Evaluate

1. $\int \frac{dx}{x+1} = \ln |x+1| + C$

2. $\int x^2 \, dx = \frac{1}{3} x^3 + C$

$\int e^x \, dx = e^x + C$
3. \( \int \tan \alpha \, dx = -\int \frac{\sin \alpha}{\cos \alpha} \, dx = -\ln |\cos \alpha| + C \)

4. \( \int \frac{\ln x}{x} \, dx = \frac{\ln x^2}{2} + C \)

5. \( \int \sec x \, dx = \int \frac{\sec x + \tan x}{\sec x + \tan x} \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \)
   
   \( = \ln |\sec x + \tan x| + C \)

---

The logarithm function and \( e \) function

We have the functions

1. \( y = \log_a x = \frac{\ln x}{\ln a} \)

2. \( y = x = e^1 \)

Properties:

1. \( \log_a (xy) = \log_a x + \log_a y \)

2. \( \log_a x^y = y \log_a x \)

3. \( \log_a (\frac{x}{y}) = \log_a x - \log_a y \)

4. \( x \cdot a^y = ax^y \)

5. \( \frac{a^x}{a^y} = a^{x-y} \)

6. \( [a^x]^y = a^{xy} \)

7. \( d = a \Rightarrow a = e = x = a, a \cdot d = a \cdot a = a^2, \ldots \)

---

Laws of derivative:

1. \( \frac{dy}{dx} = \log_a y = \frac{1}{u} \cdot \frac{du}{dx} \)

2. \( \frac{dy}{dx} = \frac{u}{a} = \frac{u}{a} \cdot \frac{du}{dx} \cdot \ln a \)
Ex. Find \( \frac{dy}{dx} \) for the following:

1. \( y = \log x^2 \Rightarrow y = \frac{1}{x^2} \cdot 2x = \frac{2}{x} \)

2. \( y = \log \sin x \Rightarrow y = \frac{1}{\sin x} \cdot \cos x = \frac{1}{\ln 3} \)

3. \( y = 5 \Rightarrow y = 5 \cos x \ln 5 \)

4. \( y = 2^\ln x \Rightarrow y = 2^x \cdot \frac{1}{x} \ln 2 \)

5. \( y = (\ln x)^{\tan x} \Rightarrow y = (\ln x)^{\tan x} \cdot \tan x \cdot \sec^2 x \cdot \frac{1}{x} \ln 10 \)

Laws of Integrations

1. \( \int \frac{du}{u \ln a} = \log |u| + C \)

2. \( \int u \, du = \frac{u^2}{2} + C \)

Ex. Evaluate

1. \( \int \frac{dx}{x \ln x} = \log |x| + C \)

2. \( \int \sin x \cos x \, dx = \frac{\sin x}{2} + C \)

Limit of the Rationals

Ex. Evaluate

1. \( \lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = \frac{1}{\infty} = 0 \)

2. \( \lim_{x \to -\infty} x = \lim_{x \to -\infty} \frac{x}{-1} = \frac{\infty}{-1} = -\infty \)