Partial Derivatives

15.1 Function of two or more variables:

\[ y = 0 \quad \text{x-axis} \]
\[ y = x \quad \text{Line} \]
\[ y = x^2 \quad \text{Parable} \]

Here: For one value of \( x \) get a unique and only value of \( y \).

Such relations are called single-valued if one variable function and we write \( y = f(x) \) one value \( x \) give two value \( y \) then we have multi-valued.

**Example 1:** \[ z = x^2 + y^2 \]
\[ z = f(x,y) = 3^2 + 4^2 = 25 \quad \text{single-valued} \]

**Example 2:** \[ z^2 = x^2 + y^2 \]
\[ z^2 = f^2 (3,4) = 9 + 16 = 25 \rightarrow z = \pm 5 \quad \text{multi-valued} \]

**Example 3:**
(1) Area (A) = \( x \cdot y \)
(2) Volume (V) = \( \pi r^2 h \)

In the function \( V \), the dependent variable is \( V \). The independent variables are \( r \) and \( h \).

**Domain of** \( f(x,y) \):

**Example 4:** \[ f(x,y) = \frac{1}{x^2 - y^2} \]
\[ x^2 - y^2 \neq 0 \rightarrow x^2 \neq y^2 \rightarrow y \neq \pm x \]
\[ D = \{ (x,y) \mid y \neq \mp x^3 \} \quad y^2 \neq x^2. \]

**Example 5:** \[ Z = \sqrt{y - x^2} \]
\[ y - x^2 \geq 0 \Rightarrow y \geq x^2 \]
\[ D = \{ (x,y) \mid y \geq x^2 \} \]
\[ R: \text{is the set of all non-negative numbers} \]
\[ Z \geq 0 \]

15.2 **Limits and Continuity**

Def: Let \( Z = f(x,y) \) be a function defined in some neighborhood of the point \((x_0, y_0)\) and let \( L \) be a number, then \( L \) is the limit of \( f \) at \((x_0, y_0)\) as \( x \) approaches \( x_0 \) and \( y \) approaches \( y_0 \), if for every \( \varepsilon > 0 \), there is a number \( \delta > 0 \) such that if

\[ 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta \]

then

\[ |f(x,y) - L| < \varepsilon \]

**Example 6:** \[ \lim_{(x,y) \to (1,2)} \frac{xy}{x^2 + y^2} = \frac{(1)(2)}{(1^2 + 2^2)} = \frac{2}{5} \]

**Example 7:** \[ \lim_{(x,y) \to (0,0)} \frac{x^3 + y^3}{x^2 + y^2} \]

1. Let \((x,y)\) approach \((0,0)\) along \( y = 0 \)
\[ \lim_{(x,y) \to (0,0)} \frac{x^3}{x^2} = 0 \]

The limit is exist and equal 0.

2. \( x = 0 \), \( \lim_{(x,y) \to (0,0)} \frac{y^2}{x^2} = 0 \)
Continuity: **Def.** A function \( f \) of two variables is continuous at \((x_0, y_0)\) if

1. \( f \) is defined at \((x_0, y_0)\)
2. \( \lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0) \)

**Example 8:** Show that \( f(x, y) = x^2 + y^2 \) is continuous at \((1, 2)\).

**Solution:**

1. \( f(1, 2) = 1^2 + 2^2 = 5 \)
2. \( \lim_{(x_1, y_1) \to (1, 2)} x_1^2 + y_1^2 = 1^2 + 2^2 = 5 \)

\( f \) is cont. at \((1, 2)\)

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**Partial Derivatives**

15.3: **Def.** If \( f \) is a function of variables then the first partial derivatives of \( f \) with respect to \( x \) and \( y \) are functions \( f_x \) and \( f_y \) defined as:

\[
\begin{align*}
f_x &= \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \\
f_y &= \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}
\end{align*}
\]

Provide the limits exist:

\[
\begin{align*}
f_x &= \frac{\partial f}{\partial x} \\
f_y &= \frac{\partial f}{\partial y}
\end{align*}
\]
Example 9: If \( f(x, y) = x^3 y^2 - 2x^2 y + 3x \)

Find \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \)

Solution: 
\[
\begin{align*}
\frac{\partial f}{\partial x} &= 3x^2 y^2 - 4xy + 3 \\
\frac{\partial f}{\partial y} &= 2x^3 y - 2x^2 + 0
\end{align*}
\]

If \( f \) is a function of two variables \( x \) and \( y \) the second partial derivatives of \( f \) are denoted as follows:

1. \( \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} \) or \( f_{xx} \)
2. \( \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x \partial y} \) or \( f_{xy} \)
3. \( \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} \) or \( f_{yx} \)
4. \( \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} \) or \( f_{yy} \)

Note: \( \frac{\partial^2 f}{\partial x \partial y} \) differentialale first w.r.t. \( y \),

\( f_{yx} \) differentialale first w.r.t. \( y \).

Example 10: Find the second partial derivatives 
\( \frac{\partial^2 f}{\partial y \partial x} \) of \( f(x, y) = x^3 y^2 - 2x^2 y + 3x \)

So:
1. \( f_{xx} = \frac{\partial}{\partial x} \left( 3x^2 y^2 - 4xy + 3 \right) = 6xy^2 - 4y \)
2. \( f_{yy} = \frac{\partial}{\partial y} \left( 2x^3 - 2x^2 + 0 \right) = 2x^3 = 2x^3 \)
\( \frac{\partial^2 f}{\partial x \partial y} = \frac{1}{y} \left( \frac{\partial f}{\partial y} \right) = \frac{1}{y} \left( 2x^2 y - 2x^2 \right) = 6x^2 y - 4x \)

\( \frac{\partial^2 f}{\partial y \partial x} = \frac{1}{x} \left( \frac{\partial f}{\partial x} \right) = \frac{1}{x} \left( 3x^2 y^2 - 4xy + 3 \right) = 6x^2 y - 4x \)

The mixed derivative theorem

Example:

\( w = e^{x^3} \sin z^2 - \ln x \frac{y}{z} \) Show that \( \frac{\partial^2 w}{\partial z \partial x} = \frac{\partial^2 w}{\partial x \partial z} \)

So:

\[ \frac{\partial^2 w}{\partial x \partial z} = \frac{\partial^2 w}{\partial x \partial z} = \frac{\partial w}{\partial x} \left( \frac{\partial w}{\partial z} \right) = \left[ 2z e^{x^3} \cos z^2/y \right] \]

\[ \frac{\partial^2 w}{\partial z \partial x} = \left[ 2z e^{x^3} \cos z^2/y \right] \]

Chain Rule

Case (1)

\( f(x, y, z) \) let \( x = x(t), y = y(t), z = z(t) \)

Then:

\[ \frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt} \]

Example:

Find \( \frac{df}{dt} \) using

6) The chain rule

6) other methods (old method substitution)

For \( f(x, y) = x^2 e^y - xy^3 \)

\( x(t) = \cos t \) \( y(t) = \sin t \).

6) \( \frac{df}{dt} = (2x e^y - y^3)(-\sin t) + (x^2 e^y - 3xy^2)(\cos t) \)

6) \( f = \cos^2 t \cdot e^{\sin t} - \cos t \cdot \sin^3 t \)
\[ e^{\cos^2 t} \cdot \sin^2 t \cdot (\cos t) + (e^{\sin t}) (2 \cos t \cdot \sin t) - [ \cos \theta \cdot (\cos \theta \cdot \sin t) + \sin^2 \theta (-\sin t)] \]

Case (2): Let \( w = f(x, y, z) \) and \( x = x(r, s) \), \( y = y(r, s) \), \( z = z(r, s) \).

Then \( w = f(x(r, s), y(r, s), z(r, s)) \).

\[ \frac{\partial w}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial r} \]
\[ \frac{\partial w}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial s} \]

**Implicit Differentiation**

Suppose \( F(x, y) \) and its partial derivatives \( F_x \) and \( F_y \) are continuous and the equation \( F(x, y) = 0 \) defines \( y \) as a differentiable function of \( x \). Then at any point where \( F_y \neq 0 \),

\[ \left[ \frac{dy}{dx} = -\frac{F_x}{F_y} \right] \]

**Example:** Find \( \frac{dy}{dx} \) if \( x^2 + \sin y = -2y \)

So:

\[ \frac{dy}{dx} = -\frac{F_x}{F_y} \]

\( F_x = 2x + 0 = 0 \)
\( F_y = \cos y - 2y \) \( \therefore \left[ \frac{dy}{dx} = \frac{-2x}{\cos y - 2} \right] \)
Directional Derivatives and Gradient Vectors:

Def: If the partial derivatives of \( f(x,y,z) \) are defined at \( P_0(x_0, y_0, z_0) \), then the gradient of \( f \) at \( P_0 \) is the vector:

\[
\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k},
\]

in the domain of \( f \) obtained by evaluating the partial derivatives.

If \( f(x,y,z) \) contains partial derivatives at \( P_0(x_0, y_0, z_0) \) and \( \mathbf{u} \) is a unit vector then the derivative of \( f \) at \( P_0 \) in the direction of \( \mathbf{u} \) is the (it)

number:

\[
\frac{\partial}{\partial \mathbf{u}} f = (\nabla f)_{P_0} \cdot \mathbf{u}
\]

Example: Find \( \nabla f \) for \( f(x,y,z) = e^{xy} - x \cos(yz^2) \)

\[
\begin{align*}
\frac{\partial f}{\partial x} &= ye^{xy} - \cos(yz^2) \\
\frac{\partial f}{\partial y} &= xe^{xy} + xz^2 \sin(yz^2) \\
\frac{\partial f}{\partial z} &= -2xyz \sin(yz^2) \\
\n\nabla f &= (ye^{xy} - \cos(yz^2)) \hat{i} + (xe^{xy} + xz^2 \sin(yz^2)) \hat{j} - 2xyz \sin(yz^2) \hat{k}.
\end{align*}
\]
Tangent plane and normal line

Let \( f(x, y, z) = 0 \) be the equation of the surface \( S \).

The equation of the tangent plane at the point \( P_0(x_0, y_0, z_0) \) is

\[
\frac{\partial f}{\partial x} |_{P_0}(x-x_0) + \frac{\partial f}{\partial y} |_{P_0}(y-y_0) + \frac{\partial f}{\partial z} |_{P_0}(z-z_0) = 0
\]

The equation of the normal line to \( S \) at \( P_0 \) is the line perpendicular to the tangent plane and parallel to \( \nabla f \) at \( P_0 \), given by:

\[
\begin{align*}
\frac{dx}{\frac{\partial f}{\partial x} |_{P_0}} &= \frac{dy}{\frac{\partial f}{\partial y} |_{P_0}} = \frac{dz}{\frac{\partial f}{\partial z} |_{P_0}}
\end{align*}
\]

If none of the partial derivatives of \( f \) is zero at \( P_0 \), then the normal line is also given by:

\[
\begin{align*}
\frac{x-x_0}{\frac{\partial f}{\partial x} |_{P_0}} &= \frac{y-y_0}{\frac{\partial f}{\partial y} |_{P_0}} = \frac{z-z_0}{\frac{\partial f}{\partial z} |_{P_0}}
\end{align*}
\]

Example: find the tangent plane and normal line to the surface \( x^2 + xy - z^2 + 1 \) at the point \( P_0(1, 1, 1) \).

So:

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 2x + yz, \quad \frac{\partial f}{\partial x}(1, 1, 1) = 3 \\
\frac{\partial f}{\partial y} &= xz, \quad \frac{\partial f}{\partial y}(1, 1, 1) = 1 \\
\frac{\partial f}{\partial z} &= xz - 3z^2, \quad \frac{\partial f}{\partial z}(1, 1, 1) = -2
\end{align*}
\]

tangent plane

\[
3(x-1) + (y-1) - 2(z-1) = 0
\]

\[
3x + y - 2z = 2
\]
Maximum and Minimum and

Saddle points:

Second derivative test

If \( f \) has a continuous first and second partial derivative on some open interval containing \((a,b)\) and \( f_x'(a,b) = f_y'(a,b) = 0 \) then:

1. \( f_{xx} < 0 \) and \( f_{xx} f_{yy} - f_{xy}^2 > 0 \)
   \( \Rightarrow \) Local max

2. \( f_{xx} > 0 \) and \( f_{xx} f_{yy} - f_{xy}^2 > 0 \) at \((a,b)\) \( \Rightarrow \) Local min.

3. \( f_{xx} f_{yy} - f_{xy}^2 < 0 \) at \((a,b)\) \( \Rightarrow \) Saddle point.

4. No information if \( f_{xx} f_{yy} - f_{xy}^2 = 0 \) at \((a,b)\)

Testing for Extreme Values:

If \( z = f(x,y) \) is continuous the extrem values of \( f \) may occur only at:

1. Boundary point of the Domain of \( f \).
2. Interior point of the Domain of \( f \) where
   \( f_x = f_y = 0 \).

PROBLEMS

1. Find the absolute max. and min. value of \( f(x,y) = 2 + 2x + 2y - x^2 - y^2 \) on the triangular plate in the first quadrant bounded by \( x = 0, y = 0 \)
   \( y = 9 - x \).