2.1 Partial differential equations

**Definition:** Any equation of the form

\[ F \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \ldots, \frac{\partial^2 u}{\partial x \partial y}, \ldots \right) = 0 \]

is called partial differential equation.

**Definition:** The order of a partial differential equations is the highest order occurring in the equation.

- \[ \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial t} \] second degree in two variable
- \[ (\frac{\partial u}{\partial x})^3 + \frac{\partial u}{\partial t} = 0 \] first order in two variable
- \[ x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + \frac{\partial u}{\partial t} = 0 \] first order in three variable

**Definition:** The equation of the form \( F(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \ldots) = 0 \) is called partial differential equation of first order.

**Note:** If \( p = \frac{\partial u}{\partial x}, q = \frac{\partial u}{\partial y} \) then the P.D.E can be written as
\[ f(x, y, t, p, q) = 0 \]

2.2 Linear equation of the first order

**Definition:** The equation of the form - \( Pu + Qq = R \)

where \( P, Q \) and \( R \) are given function of \( x, y, t \) (which do not involve \( P \) or \( q \)) is called linear equation of the first order (of Lagrange equation).

In general

Linear P.D.E of first order in \( n \) independent variable
\[ x_1 P_1 + x_2 P_2 + \ldots + x_n P_n = 0 \]

when \( x, x_1, \ldots, x_n \) - 1, \( V \), \( P_1, P_2, \ldots, P_n \) - 1
variable \( x_1, x_2, \ldots, x_n \) and dependent variable \( f \) is
\[
\frac{\partial f}{\partial x_1} = P_i \quad (i = 1, 2, \ldots, n)
\]

**Example**

\[
\frac{\partial^2 f}{\partial x^2} + y \frac{\partial^2 f}{\partial y^2} = 2^2 + x^2
\]

L. P. D. E. of first order.

\[
\frac{\partial f}{\partial x} = 2^2 + x^2
\]

is not L. P. D. E.

**Theorem**

The general solution of the L. P. D. E \( P \frac{\partial u}{\partial x} + Q \frac{\partial u}{\partial y} = R \)

is \( F(u,v) = 0 \) where \( F \) is an arbitrary function and \( u(x,y) = c_1 \) and \( v(x,y) = c_2 \) form a solution of the equation.

**Example**

\[
\frac{dx}{x} = \frac{dy}{y} = \frac{dt}{(x+y)^2}
\]

Find the general solution of D.E.

\[
x^2 \frac{\partial^2 z}{\partial x^2} + y^2 \frac{\partial^2 z}{\partial y^2} = (x+y)^2
\]

**Solution**

\[
\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dt}{(x+y)^2}
\]

\[
\frac{dx}{x^2} - \frac{dy}{y^2} = \frac{dt}{(x+y)^2} \Rightarrow \frac{1}{x} - \frac{1}{y} = c_1 \Rightarrow \frac{x - y}{xy} = c_1
\]

\[
\frac{dx}{x^2} - \frac{dy}{y^2} = \frac{dt}{(x-y)^2} \Rightarrow \frac{dx}{x} - \frac{dy}{y} = \frac{dt}{(x-y)^2}
\]

\[
\Rightarrow \ln(x - y) = \ln(y) + c \Rightarrow \ln\left(\frac{x - y}{y}\right) = c
\]

\[
\Rightarrow \frac{x - y}{y} = c_2
\]

From \( 1 \) & \( 2 \) \( \frac{x}{y} = c_3 \)

The solution

1) \( F\left(\frac{xy}{c}, \frac{x - y}{c}\right) = 0 \)

**Example**

Solve \( (x^2 - y^2 - z^2) \frac{\partial z}{\partial x} + 2xy \frac{\partial z}{\partial y} = 2x \frac{\partial z}{\partial x} \)

**Solution**

\[
\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2x^2}
\]

\[
\frac{dz}{2x^2} = \frac{dz}{2x^2} = 0 \frac{dx}{y} = \frac{dz}{x^2}
\]
Fourier Transform

Definition:
The Fourier transform of \( f(t) \) is defined as:
\[
F(p) = \mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-ipt} \, dt
\]

Definition:
The inverse Fourier transform of \( F(p) \) is defined as:
\[
f(t) = \mathcal{F}^{-1}[F(p)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(p) e^{ipt} \, dp
\]

Example:
Find Fourier transform of \( f(t) \) if:
\[
f(t) = \begin{cases} 
1 - t^2 & \text{if } |t| < 1 \\
0 & \text{if } |t| \geq 1
\end{cases}
\]

Then prove that \( \int_{0}^{\infty} \left( \frac{\sin p - p \cos p}{p^3} \right) \cos \frac{p}{p^2} \, dp = \frac{3\pi}{75}\).

Solution:
\[
F(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-ipt} \, dt = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - t^2) e^{-ipt} \, dt
\]

\[
= \frac{1}{\sqrt{2\pi}} \left[ \int_{-1}^{1} (1 - t^2) e^{-ipt} \, dt \right]
\]

\[
= \frac{1}{\sqrt{2\pi}} \left[ \int_{-1}^{1} 1 e^{-ipt} \, dt - \int_{-1}^{1} t^2 e^{-ipt} \, dt \right]
\]

\[
= \frac{1}{\sqrt{2\pi}} \left[ \frac{i}{p} e^{-ip} - \frac{2}{p^2} e^{-ip} + \frac{2}{p^3} e^{-ip} \right]
\]

\[
= \frac{1}{\sqrt{2\pi}} \left[ \frac{4}{p^2} e^{ip} - \frac{4}{p^3} e^{-ip} \right]
\]

\[
= 2\sqrt{\frac{2}{\pi}} \left[ \frac{\sin p - p \cos p}{p^3} \right]
\]

From inverse Fourier transform of \( F(p) \) we have:
\[
f(t) = \mathcal{F}^{-1}[F(p)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}} \left[ \frac{\sin p - p \cos p}{p^3} \right] e^{ipt} \, dp
\]

\[
= \begin{cases} 
1 - t^2 & \text{if } |t| < 1 \\
0 & \text{if } |t| \geq 1
\end{cases}
\]
\[ e^{-\frac{t}{2}} \sum_{n=0}^{\infty} \left( \sin^2 \frac{\pi n}{N} \right) e^{-\frac{t^2}{4}} \]

Since the function in second integral is odd, then it value is zero.

\[ \frac{4}{\pi} \int_{0}^{\infty} \frac{\sin \frac{\pi p}{N} \cos \frac{\pi p t}{N}}{\rho^3} \cos t p \cos \rho p \, dp = \begin{cases} 0 & \text{if } t < 1 \\ t - t^2 & \text{if } t > 1 \end{cases} \]

If \( t = \frac{1}{2} \),

\[ \frac{4}{\pi} \int_{0}^{\infty} \frac{\sin \frac{\pi p}{N} \cos \frac{\pi p t}{N}}{\rho^3} \cos \frac{t}{2} \, dp = \frac{3}{16} \]

\[ \int_{0}^{\infty} \frac{\sin \frac{\pi p}{N} \cos \frac{\pi p t}{N}}{\rho^3} \cos \frac{t}{2} \, dp = \frac{3}{16} \]

**Fourier Cosine Transform**

Suppose \( f(t) \) is an even function \((i.e. f(-t) = f(t))\). Then Fourier transform of \( f(t) \) is:

\[ \mathcal{F}[f(t)] = F(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-ipt} \, dt \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(t) e^{-ipt} \, dt + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} f(t) e^{-ipt} \, dt \]

Put \( t = -T \) in first integral:

\[ \mathcal{F}[f(t)] = -\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(-T) e^{ipt} \, dT + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} f(T) e^{ipt} \, dT \]

\[ = -\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(T) e^{ipt} \, dT + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(T) e^{-ipt} \, dT \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(T) [e^{ipt} + e^{-ipt}] \, dT \]

So Fourier transform to an even function:

\[ \mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(t) \cos pt \, dt \]

\[ \mathcal{F}[f(t)] = \sqrt{\frac{\pi}{2}} \int_{0}^{\infty} f(t) \cos pt \, dt \]
Recall that a PDE is called homogeneous if all the partial deriva-tions have the same order.

\[ \begin{align*}
\frac{\partial^2 z}{\partial x \partial y} - x \frac{\partial^2 z}{\partial x^2} &= 0 \\
x^2 \frac{\partial^2 z}{\partial x^2} + 5xy \frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2} &= x^2 + y^2 \\
x^2 \frac{\partial^2 z}{\partial y^2} + 5xy \frac{\partial^2 z}{\partial y \partial x} + \frac{\partial z}{\partial x} &= x^2
\end{align*} \]

are homogeneous.

\[ \begin{align*}
\frac{\partial^2 z}{\partial x^2} + xz &= y \\
\frac{\partial^2 z}{\partial y^2} + xz &= \frac{3z}{x^2}
\end{align*} \]

are not homogeneous.

In this chapter we find the solution of non-homogeneous.

\[ f(Dx, Dy) \neq f(x, y) \]

this solution contains two parts:

1. General solution (Z1) of the equation \( f(Dx, Dy) Z = 0 \)
2. Special solution (Z2) which uses the same formulas as the method in the previous chapter.

**First case:** PDE with constant coefficients.

1. When \( f(Dx, Dy) = aD_x + bD_y + c \)

The solution of \((aD_x + bD_y + c) Z = 0\) is \( Z = e^{b} + (ax - bx)\)

When \( a, b, c \) are constants.

\[ \begin{align*}
\text{Ex} & \quad (2D_x - D_y + 3) Z = 0 \\
a &= 2 \\ b &= 1 \\ c &= 3 \\
Z_1 &= e^{2y} + (2x + x)
\end{align*} \]

\[ \begin{align*}
\text{Ex} & \quad (D_x - 3D_y + 2) Z = 0 \\
a &= 1 \\ b &= -3 \\ c &= 2 \\
Z_1 &= e^{\frac{3y}{2}} + (y + 3x)
\end{align*} \]

\[ \begin{align*}
\text{Ex} & \quad (D_y + 2) Z = 0 \\
a &= 0 \\ b &= 1 \\ c &= 2 \\
Z_1 &= -e^{2y} + (x)
\end{align*} \]
The solution of \((aD_x + bD_y + c)^k z = 0\) is:

\[ z = e^{-\frac{y}{k}} \left[ \phi_1(ay-bx) + x\phi_2(ay-bx) + x^2\phi_3(ay-bx) + \ldots + x^{k-1}\phi_k(ay-bx) \right] \]

where \(\phi_1, \phi_2, \ldots, \phi_k\) are arbitrary functions.

**Example:**

\[ (D_x - 2D_y + 0) z = 0 \]

\(a = 1, b = -2, c = 1, k = 4\)

\[ z_1 = e^{\frac{y}{4}} \left[ \phi_1(y + 2x) + x\phi_2(y + 2x) + x^2\phi_3(y + 2x) + x^3\phi_4(y + 2x) \right] \]

**Example:**

\[ (D_x + 4D_y + 5) z = 0 \]

\(a = 3, b = 4, c = 5, k = 3\)

\[ z_1 = e^{\frac{y}{3}} \left[ \phi_1(3y-4x) + x\phi_2(3y-4x) + x^2\phi_3(3y-4x) \right] \]

**Example:**

\[ (2D_x - D_y + 1)^2 (D_x + 3D_y + 4) z = 0 \]

\(k = 2, a = 2, b = -1, c = 4\)

\[ z_1 = e^{\frac{y}{2}} \left[ \phi_1(2y + x) + x\phi_2(2y + x) \right] + e^{\frac{y}{2}} \left[ \phi_3(y - 3x) \right] \]

**Example:**

\[ (2D_x - D_y + 6)(D_x - 5D_y + 8)^2 z = 0 \]

\(k = 2, a = 2, b = -5, c = 8\)

\[ z = e^{6y} \left[ \phi_1(2y + x) \right] + e^{6y} \left[ \phi_2(y + 5x) + x\phi_3(y + 5x) \right] \]

\[ (2D_x + 3D_y - 5)(D_y + 2D_y)(D_x + 5D_y) (D_y - 2) z = 0 \]

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The text appears to be a continuation of a mathematics derivation or explanation, possibly related to differential equations or linear algebra, given the notation and operations used.
Formation of PDE

The PDE can be formed by one of:

1. elimination of arbitrary constant.
2. elimination of arbitrary function from a relation involving three or more variables.

Ex.
Derive a PDE (by eliminating the constant) from the equation \( z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \)

Set: 

Differentiate with respect to \( x \) and \( y \) we get:

\[
\begin{align*}
2 \frac{\partial z}{\partial x} &= \frac{2x}{a^2} \Rightarrow \frac{1}{a^2} &= \frac{1}{x} \frac{\partial z}{\partial x} = \frac{p}{x} \\
2 \frac{\partial z}{\partial y} &= \frac{2y}{b^2} \Rightarrow \frac{1}{b^2} &= \frac{1}{y} \frac{\partial z}{\partial y} = \frac{q}{y}
\end{align*}
\]

\[ z = x p + y q \]

Note:

\[ p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad \frac{\partial^2 z}{\partial x \partial y} = s \quad \frac{\partial^2 z}{\partial y \partial x} = t \]

Ex.
Find a PDE from the relation \( z = a x^2 + b y^2 \)
where \( a \) and \( b \) are arbitrary constant.

Sol.

\[
\begin{align*}
\frac{\partial z}{\partial x} &= 2ax^1, \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{3} \frac{\partial z}{\partial x} \\
\frac{\partial z}{\partial y} &= 2by^1, \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{1}{3} \frac{\partial z}{\partial y}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} &= 3z
\end{align*}
\]

Ex.
Find a PDE from the relation \( z = a x^2 + b x y + c y^2 \)
where \( a, b, \) and \( c \) be \( n \) constant.

Sol.

\[
\begin{align*}
\frac{\partial z}{\partial x} &= 2ax + by \quad \frac{\partial^2 z}{\partial x^2} = 2a x + b y \\
\frac{\partial z}{\partial y} &= bx + 2cy \quad \frac{\partial^2 z}{\partial y^2} = 2c y \\
\frac{\partial^2 z}{\partial x \partial y} &= b x + 2c y \\
\end{align*}
\]

\[ z = x y z, \quad x y z = z \]
Derive a PDE by eliminating the arbitrary fund

1) \( z = f(x^2 + y^2) \)
\[
\frac{\partial^2 z}{\partial x^2} = f''(x^2 + y^2) (2x) \quad \Rightarrow \quad \frac{\partial^2 z}{\partial x^2} = \lambda x \Rightarrow \nabla^2 z = \lambda z = 0
\]
\[
\frac{\partial^2 z}{\partial y^2} = f''(x^2 + y^2) (2y) \quad \Rightarrow \quad \frac{\partial^2 z}{\partial y^2} = \lambda y \Rightarrow \nabla^2 z = \lambda z = 0
\]

2) \( z = f(x + iy) + g(x - iy) \)
\[
\frac{\partial z}{\partial x} = f'(x + iy) + g'(x - iy) \]
\[
\frac{\partial^2 z}{\partial x^2} = f''(x + iy) + g''(x - iy) \]
\[
\frac{\partial z}{\partial y} = if'(x + iy) - ig'(x - iy) \]
\[
\frac{\partial^2 z}{\partial y^2} = -f''(x + iy) - g''(x - iy) \]
\[
\frac{\partial^2 z}{\partial x \partial y} = -i f'(x + iy) + i g'(x - iy) \]
\[
\frac{\partial^2 z}{\partial y \partial x} = -i g'(x - iy) + i f'(x + iy) \]
\[
\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0
\]

3) \( z = f(x) + g(y) \)
\[
\frac{\partial z}{\partial x} = f'(x) \quad \Rightarrow \quad \frac{\partial^2 z}{\partial x^2} = 0
\]
\[
\frac{\partial z}{\partial y} = g'(y) \quad \Rightarrow \quad \frac{\partial^2 z}{\partial y^2} = 0
\]

4) \( z = f(x + 2y) + g(x - 3y) \)
\[
\frac{\partial z}{\partial x} = f'(x + 2y) + g'(x - 3y) \]
\[
\frac{\partial^2 z}{\partial x^2} = f''(x + 2y) + g''(x - 3y) \]
\[
\frac{\partial z}{\partial y} = 3 f'(x + 2y) - 3 g'(x - 3y) \quad \Rightarrow \quad \frac{\partial^2 z}{\partial y^2} = 9 \frac{\partial^2 z}{\partial x^2} \]
\[
\frac{\partial^2 z}{\partial x \partial y} = 9 f'(x + 2y) + 9 g'(x - 3y) \]
\[
\frac{\partial z}{\partial x} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}
\]