Numerical analysis

Ch. 1 Numerical solution of higher order I.V.P.

Previously we have treated only the case of the first order differential equation. In this chapter we will consider the higher order I.V.P. The following two steps can be used to find the numerical solution for the higher-order I.V.P.

1. We replace the higher-order I.V.P by a system of first order equation.
2. We use the methods previously studied to treat each equation in the system.

Taylor Series method:-

The general form of taylor series is:-

\[ y(x) = y(x_0) + h y'(x_0) + \frac{h^2}{2} y''(x_0) + \frac{h^3}{6} y'''(x_0) + \cdots + \frac{h^n}{n!} y^{(n)}(x_0) + \cdots \]

\[ x(t) = x(t_0) + h x'(t_0) + \frac{h^2}{2} x''(t_0) + \cdots \]

\[ y(t) = y(t_0) + h y'(t_0) + \frac{h^2}{2} y''(t_0) + \cdots \]
Radiological Investigations:

Histopathological Investigations:

Medications & Procedures Before Referral:

[Signatures and dates]
Example 1.

Use Taylor series to solve the equation for \( t = 0.1 \)

\[ y'' = y' y' \quad y(0) = 1 \quad y'(0) = -1 \]

Solve:

\[ \frac{d^2 y}{dt^2} = y \frac{dy}{dt} \]

Let \( \frac{dy}{dt} = z \quad y(0) = 1 \)

\[ z' = \frac{d^2 y}{dt^2} = y z \quad z(0) = -1 \]

Using Taylor series we get

\[ y'(0) = -1 \quad \quad \quad z'(0) = -1 \]
\[ y''(0) = -1 \quad \quad \quad z''(0) = z y' + z' y \]
\[ y'''(t) = z y' + z' y \quad \quad \quad z'''(0) = 0 \]
\[ y'''(0) = -1(-1) + (-1)(0) = 0 \quad \quad \quad z'''(0) = 3 \]

So

\[ y(t) = y(0) + y'(0) t + \frac{y''(0)}{2} t^2 + \frac{y'''(0)}{3!} t^3 \]
\[ y(0.1) = 0.8950 \]

\[ z(t) = z(0) + z'(0) t + \frac{z''(0)}{2} t^2 + \frac{z'''(0)}{3!} t^3 \]
\[ z(0.1) = -1.0995 \]
1. Use Taylor Series method to find $y(0.2)$ for the equation

$$y'' + ty' - 2y = t$$ $\quad y(0) = y''(0) = 0$
$$y'(0) = 1$$

2. Use Taylor Series to compute $y(0.1)$ for the equation

$$y'' = -8 \sin y$$ $\quad y(0) = \frac{\pi}{4}$ $\quad y'(0) = 1$

2 - Euler Predictor-Corrector:

In this method we want to find two values for $x_p$ and $x_c$ the first is called the predictor value and $(x_c)$ is called the corrector value for $x$. Where

we have by Euler Predictor-Corrector method

$$x_{n+1} = x_n + h \frac{x'_n + x'_{n+1}}{2}$$
$$y_{n+1} = y_n + h \frac{y'_n + y'_{n+1}}{2}$$

Then we use these values to find the corrector values using the following equations:

$$x_{n+1} = x_n + h \frac{x'_n + x'_{n+1}}{2}$$
$$y_{n+1} = y_n + h \frac{y'_n + y'_{n+1}}{2}$$
Example: Use Euler method to solve the system at \( t = 0.5 \) where \( h = 0.5 \)

\[
\frac{dx}{dt} = y \\
\frac{dy}{dt} = -x
\]

\( x(0) = 0 \quad y(0) = +1 \)

Solution:

\[
x_p(0.5) = x(0) + 0.5 \left( x'(0) \right) \\
= 0.5
\]

\[
y_p(0.5) = y(0) + 0.5 \left( y'(0) \right) \\
= 1
\]

\[
x_c(0.5) = x(0) + 0.5 \left( \frac{x'(0) + x'(0.5)}{2} \right) \\
= 0.8
\]

\[
y_c(0.5) = y(0) + 0.5 \left( \frac{y'(0) + y'(0.5)}{2} \right) \\
= 0.875
\]

(2) Find \( x(1) \) and \( y(1) \)

\[
x_p(1) = x_c(0.5) + 0.5 \left( x'(0.5) \right) \\
= 1
\]

\[
y_p(1) = y(0.5) + 0.5 \left( y'(0.5) \right) \\
= 0.75
\]

\[
x_c(1) = x(0.5) + 0.5 \left( \frac{x'(0.5) + x'(1)}{2} \right) \\
= 0.9063
\]

\[
y_c(1) = y(0.5) + 0.5 \left( \frac{y'(0.5) + y'(1)}{2} \right) \\
= 0.5234
\]
Exercises:

0. \[ \frac{dx}{dt} = xy + t \quad x(0) = 0 \]
\[ \frac{dy}{dt} = x - t \quad y(0) = 1 \]

Find \( x(0.2) \) and \( y(0.2) \) by using Euler predictor-corrector.

3. Runge-Kutta Method:

The general form of Runge-Kutta second order for \( x \) and \( y \) are:

\[ x_{n+1} = x_n + \frac{1}{2} (k_1 x + k_2 x) \]
\[ y_{n+1} = y_n + \frac{1}{2} (k_1 y + k_2 y) \]

where

\[ k_1 x = h f(t_n, x_n, y_n) \]
\[ k_2 x = h f(t_n + h, x_n + k_1 x, y_n + k_1 y) \]
\[ k_1 y = h g(t_n, x_n, y_n) \]
\[ k_2 y = h g(t_n + h, x_n + k_1 x, y_n + k_1 y) \]
Example: Use Runge-Kutta second order to compute \( y(0.2) \).

For the equation:

\[
y'' + y' - ty - 2y = t \quad y(0) = y''(0) = 0, \quad y'(0) = 1
\]

Solution:

Let \(\frac{dy}{dt} = z \quad y(0) = 0\)

\[
\frac{dz}{dt} = w \quad z(0) = 1
\]

\[
\frac{dw}{dt} = t - tw + tz + 2y \quad w(0) = 0
\]

\[
p(t, y, z, w) = \frac{dy}{dt} = z \quad y(0) = 0
\]

\[
g(t, y, z, w) = \frac{dz}{dt} = w \quad z(0) = 1
\]

\[
s(t, y, z, w) = \frac{dw}{dt} = t - tw + tz + 2y \quad w(0) = 0
\]

Using Runge-Kutta 2nd order:

\[
k_1 y = 0.2 \quad p(0.0, 0.0, 0.0) = 0.2
\]

\[
k_1 z = 0.2 \quad g(0.0, 0.0, 0.0) = 0
\]

\[
k_1 w = 0.2 s(0.0, 0.0, 0.0) = 0
\]

\[
k_2 y = 0.2 \quad p(0.2, 0.2, 0.2, 0.2) = 0.2
\]

\[
k_2 z = 0.2 \quad g(0.2, 0.2, 0.2, 0.2) = 0
\]

\[
k_2 w = 0.2 s(0.2, 0.2, 0.2, 0.2) = 0.16
\]

So

\[
y(0.2) = y(0) + \frac{1}{2} (k_1 y + k_2 y) = 0.2
\]

\[
z(0.2) = z(0) + \frac{1}{2} (k_1 z + k_2 z) = 1
\]

\[
w(0.2) = w(0) + \frac{1}{2} (k_1 w + k_2 w) = 0.68
\]
1. \[ \frac{dx}{dt} = y \quad x(0) = 0 \]

\[ \frac{dy}{dt} = -x \quad y(0) = 1 \]

Find \( x \) and \( y \) at \( t = 0.5 \) using Runge-Kutta 2nd order.