Definition: A graph $G$ with $n$ vertices and $m$ edges consists of a vertex set $V(G) = \{v_1, ..., v_n\}$ and an edge set $E(G) = \{e_1, e_2, ..., e_m\}$, where each edge consists of two (possibly equal) vertices called its endpoints.

We write $uv$ for an edge $e = \{u, v\}$. If $uv \in E(G)$, then $u$ and $v$ are adjacent. We write "$u$ is adjacent to $v$".

**Example**

![Graph example]

**Definition: Loop**

A loop is an edge whose endpoints are equal. For example $e'$.

**Definition:** Parallel edges or multiple edges. Parallel edges are edges that have the same pair of endpoints. For example $e'$. 
Definition: order

The number of vertices in a graph is called the order of the graph.

Definition: P-graph

A graph in which no element of E appears more than P times is called a P-graph.

Example 2-graph

Such that $e_1 = e_2$.

Note: If the graph has no repeated edge, then it is I-graph.

$E = \{u, v, w, x, y\}$

$e_1 = (u, v)$

$e_2 = (u, w)$

$e_3 = (v, x)$

$e_4 = (w, y)$

$e_5 = (x, y)$
Definition A simple graph is a graph having no loops or multiple edges.

Definition A graph with a finite number of vertices as well as a finite number of edges is called a finite graph, otherwise, it is an infinite graph.

Definition When a vertex \( u \) is an end vertex of some edge \( e \), \( u \) and \( e \) are said to be incident with each other.

For example, in the graph:

- the edges \( e_2, e_6 \) and \( e_7 \) are incident with vertex \( v_4 \).

Note Two nonparallel edges are said to be adjacent if they are incident on a common vertex. \( v_5 \) and \( v_4 \) are adjacent.

but \( v_1 \) and \( v_4 \) are not.
Definition: The number of edges incident on a vertex \( v_i \) with self-loops counted twice is called the degree, \( d(v_i) \) of vertex \( v_i \).

For example, \( d(V_4) = 3 = d(V_3) \).

Theorem: The number of vertices of odd degree in a graph is always even.

Proof: If we consider the vertices with odd and even degrees separately,

\[
\sum_{i=1}^{n} d(v_i) = \sum_{\text{odd}} d(v_i) + \sum_{\text{even}} d(v_i) = 2m
\]

Note: Let us now consider a graph \( G \) with \( m \) edges and \( n \) vertices \( v_1, v_2, \ldots, v_n \). Since each edge contributes two degrees, the sum of the degrees of all vertices in \( G \) is twice the number of edges in \( G \). That is,

\[
\sum_{i=1}^{n} d(v_i) = 2m
\]

So the left-hand side in \( \sum \) is even, and the first expression on the right-hand is even.

S.t. \( \sum_{\text{odd}} d(v_k) = \text{an even number} \).
**Definition** Isolated Vertex

A vertex having no incident edge is called an isolated vertex. For example, $V_3$ is an isolated vertex.

**Note** Isolated vertices are vertices with zero degree.

**Definition** Pendant vertex

A vertex of degree one is called a pendant vertex or an end vertex. Vertex $V_1$ and $V_2$ are pendant vertices.

**Definition** Null graph

A graph $G=(V,E)$, if the set of edge $E$ is empty. Such a graph, without any edges.

**Notes** Every vertex in a null graph is an isolated vertex.

For example, $V_1$, $V_2$, $V_3$
Definition: Complete graph

A simple graph in which there exists an edge between every pair of vertices is called a complete graph, and it is denoted by $K_n$ if it has $n$ vertices and its edges are $m = n(n-1)/2$.

Example

\[ K_1, K_2, K_3 \]

Note: Sometimes referred to as a universal graph or a clique.

Definition: Bipartite graph

A graph $G = (V, E)$ is called a bipartite graph if the set of vertices $V$ can be partitioned into two nonempty subsets $V_1$ and $V_2$ such that there is no edge in $E$ joining two vertices in $V_1$ or two vertices in $V_2$.

Example

\[ W \]
Definition: Complete bipartite graph

A complete bipartite graph $K_{m,n}$ is a graph with $V = V_1 \cup V_2$, being the set of vertices such that there are no edges joining any two vertices in $V_1$ or any vertices in $V_2$, but there is an edge joining every vertex in $V_1$ with every vertex in $V_2$.

Example

\[ \begin{array}{c}
V_1 \\
V_2
\end{array} \]

\[ \begin{array}{c}
\vdots \\
\vdots
\end{array} \]

Definition: Regular Graph

A graph $G$ is said to be regular graph if all its vertices with the same degree.

For example

\[ \begin{array}{c}
\vdots \\
\vdots
\end{array} \]

Definition: Complementary Isomorphism

Two graphs $G$ and $\overline{G}$ are said to be isomorphic if they have the same number of vertices, the same number of edges, and an equal number of vertices with a given degree, and we write $G \cong \overline{G}$. 

Note: A graph $G$ is $r$-regular if $\deg(v) = r \forall v \in V$. 

Examples

Cubic graph $G$

Petersen graph $K_5$

Regular graph $r = 3$

Definitions: Subgraphs

A graph $H$ is said to be a subgraph of a graph $G$ if all the vertices and all the edges of $H$ are in $G$, and each edge of $H$ has the same end vertices in $H$ as in $G$. We write $H \subseteq G$.

Ex

Graph $b$ is a subgraph of $a$.

Notes:

1. Every graph is its own subgraph.
2. A subgraph of a subgraph of $G$ is a subgraph of $G$.
3. A single vertex in a graph $G$ is a subgraph of $G$.
4. A single edge in $G$, together with its end vertices, is also a subgraph of $G$. 
Special type of subgraph of graph $G$:

If the subgraph obtained by deleting a vertex or edge:

If $v \in V(G)$ and $|V(G)| \geq 3$ then $G-v$ denotes a subgraph with vertex set $V(G)-\{v\}$ and whose edge are all those of $G$ not incident with $v$.

If $e \in E(G)$, the $G-e$ is the subgraph having vertex set $V(G)$ and edge set $E(G)-\{e\}$.

**Example:**

Note: The subgraph of $G$ has the same order of $G$ is said to be a spanning subgraph of $G$.

**Definition:** The complement $\bar{G}$ of a graph $G$ is the graph with vertex set $V(G)$ such that two vertices are adjacent in $\bar{G}$ if and only if those vertices are not adjacent in $G$. 

```
Hence, if $G$ is a $(p, q)$ graph, then $G$ is a $(\overline{p}, \overline{q})$ graph, where $q + \overline{q} = \binom{p}{2}$.

$G_0$  

$G_1$  

$G_2$  

$G_3$

The regular graph of order 4.

$G_0$ and $G_3$ are complementary.

$G_1$ and $G_2$  

Note: A graph $G$ is self-complementary if $G \cong \overline{G}$.

Definition: $n$-partite, $n \geq 1$

If it is possible to partition $V(G)$ into mutually $V_1, V_2, \ldots, V_n$ (called partite sets) such that every element of $E(G)$ joins a vertex of $V_i$ to a vertex of $V_j$, $i \neq j$, then $G \cong \overline{K_p}$.
A complete \( n \)-partite graph \( G \) is a graph with partite sets \( V_1, V_2, \ldots, V_n \) having the added property that if \( u \in V_i \) and \( v \in V_j \), then \( uv \in E(G) \).

If \( |V_i| = p_i \), then this graph is denoted by \( K(P_{p_1}, P_{p_2}, \ldots, P_{p_n}) \).

**Example:**

which is called a star

---

The binary operations are defined on graphs:

Let \( G_1 \) and \( G_2 \) are two graphs with disjoint vertex sets.

- The union \( G = G_1 \cup G_2 \) has
  
  \[
  V(G) = V(G_1) \cup V(G_2)
  \]
  
  and
  
  \[
  E(G) = E(G_1) \cup E(G_2)
  \]

**Example:**

\[ K(1,3) \cup K_2 \cup K_1(1,3) \]
3. The join \( G = G_1 + G_2 \)

has \( V(G) = V(G_1) \cup V(G_2) \)

\[ E(G) = E(G_1) \cup E(G_2) \cup \{(u,v) \in V(G_1) \times V(G_2) \mid u \in V(G_1), v \in V(G_2) \} \]

and \( G \subseteq V(G) \).

\[ G_1 \quad G_2 \quad G_1 + G_2 \]

4. The cartesian product \( G = G_1 \times G_2 \)

has \( V(G) = V(G_1) \times V(G_2) \)

and two vertices \((u_1, v_1)\) and \((u_2, v_2)\) of \( G \) are adjacent if and only if either

- \( u_1 = u_2 \) and \( v_1, v_2 \in E(G_2) \)

\[ G_1 \quad G_2 \quad G_1 \times G_2 \]

or \( v_1 = v_2 \) and \( u_1, u_2 \in E(G_1) \)

\[(u_1, v_1), (u_2, v_2), (u_3, v_3)\]

Exercises:

Determine all non-isomorphic graphs of order 5.

If \( H \subseteq G \), does it follow that \( H \subseteq \overline{G} \) ?

3. How many subgraphs of the graph containing four vertices and for edges?
The union of two graphs is:

If the two graphs are taken to be $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$, where $V(G_1)$ and $V(G_2)$ are assumed to be disjoint, then their union $G_1 \cup G_2$ is defined as the graph with vertex-set $V(G_1) \cup V(G_2)$ and edge-family $E(G_1) \cup E(G_2)$.

**Deletions and contractions.**

If $e$ is an edge of a graph $G$, we denote by $G - e$ the graph obtained from $G$ by deleting the edge $e$.

If $F$ is any set of edges in $G$, we denote by $G - F$ the graph obtained by deleting the edges in $F$.

Similarly, if $v$ is a vertex $v$ together with the edges incident to $v$.

If $S$ is any set of vertices in $G$, we denote by $G - S$ the graph obtained by deleting the vertices in $S$ and all edges incident to any of them.

$G/e$ the graph obtained by taking an edge $e$ and removing $e$ and identifying its ends $v$ and $w$ in such a way that the resulting vertex is incident to these edges, which were originally incident to $v$ and $w$. 
A contraction of $G$ is then defined to be any graph which results from $G$ after a succession of such edge-contractions.

**Definition.** A connected graph is said to be connected if it cannot be expressed as the union of two graphs; otherwise it is disconnected.

**Definition.** Components of $G$

Any connected graph is called a component.

**Ex.**

![Graphs Example](image)

**Definition.** Circuit graphs and wheels

A connected graph which is regular of degree two is called a circuit graph. The circuit graph on $n$ vertices, denoted by $C_n$, is formed by joining each vertex to a new vertex $v$ is called the wheel of $n$ vertices and is written $W_n$. 
Definitions - Digraphs

A directed graph or digraph \( D \) is a finite nonempty set of objects called vertices together with a (possibly empty) set of ordered pairs of distinct vertices of \( D \) called arcs or directed edges. 

The cardinality of \( w \) is the number of its arcs, \( q \).

\( D = (P, q) \).

\( V(D) = \{u, v, w\} \)

\( E(D) = \{(u, v), (w, u), (u, v)\} \)

If \( a = (u, v) \) is an arc of a digraph \( D \), then \( a \) is said to join \( u \) and \( v \). We say that \( a \) is incident from \( u \) and incident to \( v \).

The outdegree \( \text{od} v \) of a vertex \( v \) of a digraph \( D \) is the number of vertices of \( D \) that are adjacent from \( v \).

The indegree \( \text{id} v \) of \( v \) is the number of vertices of \( D \) adjacent to \( v \).

The degree \( \text{deg} v \) of a vertex \( v \) of \( D \) is defined by \( \text{deg} v = \text{od} v + \text{id} v \).
Theorem. If $G$ is a digraph of order $p$ and

$$
\sum_{v \in V} d_v = \sum_{v \in V} \text{id} v = q
$$

Definition

A digraph $D_1$ is isomorphic to a digraph $D_2$ if there exists a one-to-one mapping $\phi$, called an isomorphism, from $V(D_1)$ onto $V(D_2)$ such that $(u, v) \in E(D_1)$ if and only if $(\phi u, \phi v) \in E(D_2)$.

The relation "is isomorphic to" is an equivalence relation on digraphs. We write $D_1 \cong D_2$.

There is only one digraph with order 1; this is the trivial digraph.

If $p = 2$, the directed graphs are

- $D_1$
- $D_2$
- $D_3$

If $p = 3$, $q = 0$, the $(3, 0)$ digraph

Two digraphs $D_1$ and $D_2$ are identical, written $D_1 \cong D_2$, if

- $V(D_1) = V(D_2)$ and
- $E(D_1) = E(D_2)$. 

Two digraphs are isomorphic if there is an isomorphism:

\[ D_1 \rightarrow D_2 \]

Two identical digraphs are necessarily isomorphic but not conversely.

Ex.

\[ \begin{array}{c}
\text{D}_1 \\
\end{array} \quad \begin{array}{c}
\text{D}_2 \\
\end{array} \]

Isomorphic nonidentical digraphs

**Definition**

A digraph \( D_1 \) is a subdigraph of a digraph \( D \) if \( V(D_1) \subseteq V(D) \) and \( E(D_1) \subseteq E(D) \).

**Note** A subdigraph \( D_1 \) of \( D \) is a spanning subdigraph if \( D_1 \) has the same order as \( D \).

**Definition** A digraph \( D \) is called complete if for every two distinct vertices \( u \) and \( v \) of \( D \), at least one of the arcs \((u,v)\) and \((v,u)\) is present in \( D \).

**Definition** The complete symmetric digraph of order \( p \) has both arcs \((u,v)\) and \((v,u)\) for every two distinct vertices \( u \) and \( v \), and is denoted by \( K_p \).
If $G$ is a graph, then $G^*$ denotes the symmetric digraph obtained by replacing each edge of $G$ by a symmetric pair of arcs.

The digraph $K^*_n$ has size $n(n-1)$ and $v\in :d^v=v-1$ for every vertex $v$ of $D$.

Complete symmetric digraphs

$K^*_1$ $K^*_2$ $K^*_3$ $K^*_4$

Regular digraphs

$D_4$
Adjacency Matrix.

A graph $G$ with vertex $V(G) = \{V_1, V_2, \ldots, V_p\}$ can also be described by means of matrices:

$$A(G) = [a_{ij}]$$

where $a_{ij} = \begin{cases} 1 & \text{if } V_iV_j \in E(G) \\ 0 & \text{if } V_iV_j \notin E(G). \end{cases}$

Thus, the adjacency matrix of a graph $G$ is a symmetric $(0,1)$ matrix having zero entries along the main diagonal.

Ex.

\[
\begin{array}{c}
V_1 \\
V_2 \\
V_3 \\
V_4
\end{array}
\quad
\begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0
\end{bmatrix}
\]

Incidence Matrix.

a. Let $G$ be a graph with $n$ vertices, $e$ edges, and no self-loops. Define an $n \times e$ matrix

$$A = [a_{ij}]$$

whose $n$ rows correspond to the $n$ vertices and the $e$ columns correspond to the $e$ edges, as follows.

The matrix element

$$a_{ij} = \begin{cases} 1 & \text{if } j\text{th edge } e_j \text{ is incident on } i\text{th vertex } V_i \\ -1 & \text{if } j\text{th edge } e_j \text{ is incident on } i\text{th vertex } V_j \\ 0 & \text{otherwise.} \end{cases}$$
b. The node-arc incidence matrix of a digraph $D$ with $m$ nodes and $n$ edges is the $mn \times n$ matrix $A$ with

$$A_{ij} = \begin{cases} 
1 & \text{if } e_j = (k,i) \text{ for some } k \in V \setminus \{i\} \\
-1 & \text{if } e_j = (i,k) \text{ for some } k \in V \setminus \{i\} \\
0 & \text{otherwise}
\end{cases}$$
Exercises (1) 

1. Draw
   a) a simple graph, b) a non-simple graph with no loops
   c) a non-simple graph with no multiple edges, each having
      5 vertices, each having 5 vertices and 8 edges.

2. Draw a graph on six vertices whose degrees are
   a) 5, 5, 5, 5, 3, 3 does there exist a simple graph with
      these degrees?
   b) whose degrees are 5, 5, 4, 3, 3, 2?

3. Find, up to isomorphism, all the simple graphs on
   three or four vertices.

4. Show that the two graphs in figure are isomorphic.

5. Explain why the two graph in figure are not isomorphic.

6. Which of the graphs in figure are subgraphs of the
   graph?
   △ □ □ □
7. If $G$ is a graph without loops, what can you say about:
   1. the sum of the entries in any row or column of the adjacency matrix of $G$,
   2. the sum of the entries in any row of the incidence matrix of $G$?
   3. the sum of the entries in any column of the incidence matrix of $G$?

8. Let $G$ be a simple graph with at least two vertices. Prove that $G$ must contain two or more vertices of the same degree.

9. Draw the following:
   1. $N_5$, $K_6$, $K_{2,3}$, $K_{1,8}$, $W_4$, $C_5$

10. How many edges has each of the following $K_{10}$, $K_{3,7}$, $W_7$

11. Give an example (if it exists) of each of the following:
   1. a bipartite graph which is regular of degree 5,
   2. a cubic graph with eleven vertices
   3. all regular graphs of degree 4.

12. Let $G$ be a graph with $n$ vertices and $m$ edges, and let $u$ be a vertex of $G$ of degree $k$. Let $e$ be an edge of $G$. How many vertices and edges have $G-u$, $G-e$ and $G\setminus e$?