Optimal Control Theory

Chapter one:

1) Laplace Transform
2) Inverse Laplace Transform
3) Special Functions
4) Solving Initial Value Problem

1) Introduction Control theory including optimal control theory
Example: Solve the following I.V.P laplace transform via Laplace transform

\[
\dot{x} + 4x = 2u_x(t) \quad x(0) = 0 \quad \dot{x}(0) = 0
\]

\[
\text{Sol: } \mathcal{L}\{\dot{x} + 4x\} = \mathcal{L}\{2u_x(t)\} = 2
\]

\[
\Rightarrow \mathcal{L}\{\dot{x}\} + 4 \mathcal{L}\{x\} = \frac{2}{s} \Rightarrow s^2 X(s) - s X(0) - x(0) + 4X(s) = \frac{2}{s}
\]

\[
s^2 X(s) + 4X(s) = \frac{2}{s}
\]

\[
x(s) = \frac{\frac{2}{s}}{s^2 + 4} = \frac{2}{s(s^2+4)} \Rightarrow x(s) = \frac{\frac{1}{5}}{s^2+4}
\]

**Theorem:** Shift on s-axis s

\[
\mathcal{L}\{e^{\alpha t} f(t)\} = F(s+\alpha)
\]

**Theorem 2:**

\[
\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)
\]

And in general for \( n = 1,2, \ldots \)

\[
\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)
\]

**Theorem 3:** (Convolution theorem)

\[
\hat{f}(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{G(s)H(s)\} = (g \ast h)(t) = (h \ast g)(t) \text{ read } "\text{Convolution of } g \text{ and } h"
\]

**Example:** Determine the inverse Laplace transform

\[
F(s) = \frac{1}{5} \cdot \frac{1}{s+1}
\]

\[
\text{Sol: } G(s) = \frac{1}{5} \Rightarrow g(t) = 1
\]
\[ H(s) = \frac{1}{s+1} \Rightarrow h(t) = e^{-t} \]

\[ p(t) = F^{-1}(F(s)) \]

\[ p(t) = (g * h)(t) = \int_{0}^{t} g(t - \tau) h(\tau) d\tau = \int_{0}^{t} e^{-\tau} d\tau \]

\[ p(t) = (h * g)(t) = \int_{0}^{t} e^{-\tau} d\tau \]
Chapter One

The Laplace transformation in system analysis, one often encounters time-invariant linear differential equations of second order or higher order, and it is generally difficult to obtain solutions to these equations in closed form via the solution methods in ordinary differential equations. One way to circumvent this difficulty however is to use Laplace Transformation.

The Laplace Transformation is defined by:

\[ F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) \, dt \quad (1.1) \]

Theorem 1: Existence of Laplace Transform Given that conditions:

\( C1) \) \( f(t) \) is piecewise continuous on every finite (time) interval of \([0, \infty)\)

\( C2) \) the magnitude of \( f(t) \) is bounded by an exponential function \( |f(t)| \leq me^at \)

For some real constants \( a \) and \( m \) and \( \forall t \in [0, \infty) \)

Hold for \( f(t) \) then its Laplace transform \( F(s) \) exists for \( \forall \text{Re}\{s\} > a \)

Inverse Laplace Transform:

The inverse Laplace Transform is represented by \( \mathcal{L}^{-1} \):

\[ f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F(s)\} \quad (1.2) \]
Special Functions:

1) Unit-step Function $U_s(t)$, $U(t)$
   
   $U_s(t) = 1$

2) Unit-ramp Function $U_r(t)$
   
   $U_r(t) = t$

3) Unit-pulse Function $U_p(t)$
   
   $U_p(t) = \begin{cases} 1 & 0 < t < t_1 \\ 0 & t < 0 \quad \text{or} \quad t > t_1 \end{cases}$

4) Unit-sinusoidal Function
   
   $S(t) = \begin{cases} \sin \omega t & t > 0 \\ 0 & t < 0 \end{cases}$

   $G(t) = \begin{cases} \cos \omega t & t > 0 \\ 0 & t < 0 \end{cases}$

Exponential Function

$P(t) = \begin{cases} e^{-at} & t > 0 \\ 0 & t < 0 \end{cases}$

Solving Initial Value Problems (I.V.P)

I.V.P

$x(t) = \text{dependent variable}$

Laplace Transform

using initial conditions

algebraic equation of $X(s) = \text{transform of } x(t)$

Rearrange and solve for $X(s)$

Inverse Laplace Transform

$x(t)$