Three-Dimensional Force Systems:

0. Rectangular Components:
- When the force $F$' acting at point $P$ shown has the rectangular Components $F_x$, $F_y$, and $F_z$ with the direction cosines $\cos \alpha$, $\cos \beta$, and $\cos \gamma$, these are known:

$$F_x = F \cos \alpha,$$

$$F_y = F \cos \beta,$$

$$F_z = F \cos \gamma,$$

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

- $F_{\text{net}} = F_{x_{\text{net}}} + F_{y_{\text{net}}} + F_{z_{\text{net}}}$

0. In solving three-dimensional problems, one must usually find the $x$, $y$, and $z$ scalar components of a force, the direction of force described.

a) By two points on the line of action of the force thus:
   - If the coordinates of points $A$ and $B$ shown are known, the force $F$ may be written as
     $$F = F \cdot \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = F \cdot \frac{(x_B - x_A) i + (y_B - y_A) j + (z_B - z_A) k}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}$$

b) By two angles which orient the line of action.
   - Consider the geometry shown. We assume that the angles $\alpha$ and $\phi$ are known. First resolve $F$ into horizontal and vertical Components as $F_{xy} = F \cos \phi$

$$F_z = F \sin \phi$$

Then resolve the horizontal $F_{xy}$ into $x$, $y$ Components.

$$F_x = F_{xy} \cos \theta = F \cos \phi \cos \theta$$

$$F_y = F_{xy} \sin \theta = F \cos \phi \sin \theta$$
Note: Dot product: We can express the rectangular Components of a force \( S \) (or any other vector) with the aid of the vector operation known as the dot or scalar product.

\[
\mathbf{i} \cdot \mathbf{i} = j \cdot j = k \cdot k = 1
\]

\[
\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0 \quad \text{or} \quad k \cdot \mathbf{i} = \mathbf{j} \cdot k = 0
\]

\( \theta \) projection of force on the other line: The projection of \( \mathbf{F} \) in direction \( \mathbf{n} \) or any line shown in Figure has the magnitude

\[
F_n = \mathbf{F} \cdot \mathbf{n} - \text{The } \mathbf{F}_n \text{- scalar projection}
\]

\[
\mathbf{F}_n = \mathbf{F} \cdot \mathbf{n} - \text{The } \mathbf{F}_n \text{- vector projection}
\]

\( \theta \) angle between the force \( \mathbf{F} \) and the direction which size by the unit vector \( \mathbf{n} \) is \( \theta \), we may written as

\[
\theta = \cos^{-1} \left( \frac{\mathbf{F} \cdot \mathbf{n}}{F} \right) \Rightarrow \theta = \cos^{-1} \left( \frac{F_n}{F} \right)
\]

In general, the angle between two vectors \( \mathbf{A} \) and \( \mathbf{B} \) is

\[
\theta = \cos^{-1} \left( \frac{\mathbf{A} \cdot \mathbf{B}}{||\mathbf{A}|| ||\mathbf{B}||} \right)
\]

Sample problem:

A force \( \mathbf{F} \) with a magnitude of 100N is applied at the origin \( O \) of the axes (x-y-z) as shown. The line action of \( \mathbf{F} \) passes through a point \( A \) whose coordinates are \( 3 \text{m}, 4 \text{m}, \) and \( 5 \text{m} \). Determine

a) the x, y, and z scalar components of \( \mathbf{F} \)

b) the projection \( \mathbf{F}_{xy} \) of \( \mathbf{F} \) on the (x-y) plane

c) The projection \( \mathbf{F}_{xz} \) of \( \mathbf{F} \) along line AB

Solution:

- The coordinates points \( O, A, B \)

\( O(0,0,0) \), \( A(3,4,5) \), \( B(6,6,2) \)

\( \mathbf{F}_{xy} = 100 \mathbf{F}_{xy} = 100 \left( \frac{3(4)+4j+5k}{\sqrt{3^2+4^2+5^2}} \right) = 100 \left( \frac{12+4j+5k}{10} \right) \)

\( \mathbf{F}_{xz} = 100 \mathbf{F}_{xz} = 100 \left( \frac{3(2)+4j+5k}{\sqrt{3^2+4^2+5^2}} \right) \)

Figure 10
b) The projection $F_{xy}$ of $F$ on the $x$-$y$ plane.

$$F_{xy} = F \cdot \cos \theta_{xy} \quad \text{from eqn.} \ 3$$

$$\cos \theta_{xy} = \frac{xy}{OA} = \frac{\sqrt{3^2 + 4^2}}{\sqrt{3^2 + 4^2 + 5^2}} \approx 0.707$$

$$\therefore F_{xy} = (0.707 \cdot 10) = 7.07 \, \text{N} \quad \text{Ans}$$

C) projection $F_{OB}$ from Figure shown.

$$F_{OB} = \frac{\vec{F}}{\vec{OA}} \cdot \vec{OB}$$

$$\vec{F} = 42.4 \, \text{i} + 56.6 \, \text{j} + 70.7 \, \text{k}$$

$$\vec{OA} = \frac{\vec{OB}}{\vec{OB}} = \frac{6 \, \text{i} + 6 \, \text{j} + 2 \, \text{k}}{\sqrt{6^2 + 6^2 + 2^2}}$$

$$= 0.688 \, \text{i} + 0.688 \, \text{j} + 0.229 \, \text{k}$$

$$\therefore F_{OB} = (42.4 \, \text{i} + 56.6 \, \text{j} + 70.7 \, \text{k})(0.688 \, \text{i} + 0.688 \, \text{j} + 0.229 \, \text{k})$$

$$F_{OB} = 42.4 \cdot (0.688) + 56.6 \cdot (0.688) + 70.7 \cdot (0.229) \quad \text{Ans}$$

$$F_{OB} = 84.4 \, \text{N}$$

The projection as a vector is

$$\vec{F}_{OB} = F_{OB} \cdot \frac{\vec{OA}}{OA}$$

$$\vec{F}_{OB} = 84.4 \cdot (0.688 \, \text{i} + 0.688 \, \text{j} + 0.229 \, \text{k})$$

$$\vec{F}_{OB} = 58.1 \, \text{i} + 58.1 \, \text{j} + 19.35 \, \text{k} \quad \text{Ans}$$
Moments in Three Dimensions:

Consider a force \( \vec{F} \) with a given line of action acting on the body shown in figure. The vector \( \vec{M} \) has magnitude and direction by the vector cross-product relation. Therefore, we may write the moment of \( \vec{F} \) about the axis through point \( O \) as

\[
\vec{M}_o = \vec{r} \times \vec{F}
\]

- Evaluating the cross product:

From the definition of the cross product, using the right-handed coordinate system, we get

\[
\begin{align*}
    \hat{i} \times \hat{j} &= \hat{k} \\
    \hat{j} \times \hat{i} &= -\hat{k} \\
    \hat{i} \times \hat{k} &= \hat{j} \\
    \hat{k} \times \hat{i} &= -\hat{j} \\
    \hat{i} \times \hat{i} &= 0 \\
\end{align*}
\]

- Moment about an arbitrary axis:

We can obtain an expression for the moment \( \vec{M}_A \) of \( \vec{F} \) about any axis \( \lambda \) through \( O \) as shown in Fig. 15. \( \hat{n} \) is a unit vector in the \( \lambda \)-direction, then we can use the dot-product expression. Thus:

\[
\vec{M}_A = \vec{r} \times \vec{F} \\
\vec{M}_A = \vec{M}_o \cdot \hat{n} \\
\vec{n}_A = \frac{\vec{r}}{\lambda} \quad \text{when} \quad \lambda = 2
\]
Varignon's Theorem in three Dimensions:
Figure shows a system of concurrent forces \( \mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3 \ldots \) the sum of moment about \( O \) of these forces is

\[
\sum_{i=1}^{\infty} \mathbf{r} \times \mathbf{F}_i = \mathbf{r} \times \sum_{i=1}^{\infty} \mathbf{F}_i
\]

This equation states that the sum of the moment of a system of forces about a given point equals the moment of their sum about the same point.

Couples in Three Dimensions:
In figure shows two equal and opposite forces \( \mathbf{F} \) and \( -\mathbf{F} \) acting on the body, the vector \( \mathbf{r} \) runs from any point \( B \) on the line of action of \( -\mathbf{F} \) to any point \( A \) on line of action of \( \mathbf{F} \). Point \( A \) and \( B \) are located by position vectors \( \mathbf{r}_A \) and \( \mathbf{r}_B \) from any point \( O \). The Combined moment of two forces about \( O \) is

\[
\mathbf{M} = \mathbf{r}_A \times \mathbf{F} + \mathbf{r}_B \times (-\mathbf{F}) = (\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F}
\]

However, \( \mathbf{r}_A - \mathbf{r}_B = \mathbf{r} \) so that all moment Center \( O \) and the moment as the couple becomes

\[
\mathbf{M} = \mathbf{r} \times \mathbf{F}
\]

The difference \( \mathbf{r}_A - \mathbf{r}_B \) between the two vectors is easily obtained by adding \( -\mathbf{r}_B \) to \( \mathbf{r}_A \) as shown in figure - where either the triangle procedure may be used. the difference \( \mathbf{r} \) between the two vectors is expression by the vector equation

\[
\mathbf{r} = \mathbf{r}_A - \mathbf{r}_B
\]
Sample problem 1:

A tension \(T\) of magnitude 10 kW is applied to the top, \(A\), and secant to the ground at \(B\). Determine the moment \(M_2\) of \(T\) about the \(z\)-axis passing through the base, \(O\).

Solution:

- The coordinates of points \(O, B, A\):
  
  \(O(0, 0, 0)\) \(A(0, 15, 0)\) \(B(12, 0, 9)\)

\[
\begin{align*}
\vec{M}_o &= \vec{M}_o \times \vec{r}_O \\
\vec{M}_o &= \vec{r}_{OA} \times \vec{T}_{AB} \\
\vec{T}_{AB} &= \vec{T} \times \vec{r}_{AB} = 10 \left( \frac{12^2 - (-15)^2 + 9^2}{\sqrt{12^2 + (-15)^2 + 9^2}} \right) \\
\vec{T}_{AB} &= 10 \left( 0.566 \hat{i} - 0.737 \hat{j} + 0.424 \hat{k} \right) \text{N} \\
\end{align*}
\]

Then \(\vec{r}_{OA} = 15 \hat{j}\).

The \(\vec{M}_o = \vec{r}_{OA} \times \vec{T}_{AB} \)

\[
\begin{align*}
\vec{M}_o &= 15 \hat{j} \times 10 \left( 0.566 \hat{i} - 0.737 \hat{j} + 0.424 \hat{k} \right) \\
\vec{M}_o &= 150 \left( -0.566 \hat{k} + 0.424 \hat{i} \right) \text{N}\cdot\text{m}
\end{align*}
\]

The value \(M_2\) is the scalar component of \(\vec{M}_o\) in the \(z\)-direction

\[
\begin{align*}
M_2 &= \vec{M}_o \cdot \hat{k} = 150 \left( -0.566 \right) + 0.424 \hat{i} \text{N}\cdot\text{m} \\
M_2 &= 84.9 \text{N}\cdot\text{m}
\end{align*}
\]

Sample problem 2:

Determine the magnitude and direction of the couple \(M\) which will replace the two given couples, \(F_1\) and \((F_1')\), applied in the two forces at the block parallel to the \((y, z)\) plane. The 30 N forces act parallel to the \(y - z\) plane.
Solution:
1. The coordinates of the points A, B, C, and D are
   A(0+0.1i+0.05j); B(0+0.6+0.05j); C(0+0.4+0.160j)
   D(0+0.160j)
2. \( \vec{M}_y = \vec{F} \times \vec{F}_y \); \( \vec{F} = \vec{F}_{AB} = 0.1i - 0.1j \)
   \( \vec{F}_y = -26j \)
   Then \( \vec{M}_{y1} = (0.1i - 0.1j) \times (-26j) = -2.65 \vec{k} \text{ N.m} \)
3. \( \vec{M}_y = \vec{F} \times \vec{F}_y \); \( \vec{F} = \vec{F}_{CD} = 0.06i \)
   \( \vec{F}_y = 30 \cos 60^\circ \vec{j} \text{ N} \)
   Then \( \vec{M}_{y2} = 0.06i \times 30 \cos 60^\circ \vec{j} \)
   \( \vec{M}_{y2} = 0.9 \vec{k} \text{ N.m} \)
4. \( \vec{M}_z = \vec{F} \times \vec{F}_z \); \( \vec{F} = \vec{F}_{CD} = 0.06i \)
   \( \vec{F}_z = -30 \sin 60^\circ \vec{k} \text{ N} \)
   Then \( \vec{M}_z = 0.06i \times (-30 \sin 60^\circ \vec{k}) = 1.359 \vec{j} \text{ N.m} \)
   \( \vec{M} = \vec{M}_{y1} + \vec{M}_{y2} + \vec{M}_z = -2.65 \vec{k} + 0.9 \vec{k} + 1.359 \vec{j} \)
   \( \vec{M} = 1.359 \vec{j} - 1.65 \vec{k} \text{ N.m} \)
   \( M = \sqrt{(-1.65)^2 + (1.359)^2} = 2.23 \text{ N.m} \)
   \( \theta = \tan^{-1} \frac{1.359}{1.6} = \tan^{-1} 0.857 = 44.3^\circ \)
   The two forces \( F \) and \( -F \) are found by \( M = F \times d \)
   \( F = \frac{M}{d} = \frac{2.23}{0.10} = 22.3 \text{ N} \)
   The force \( 22.3 \text{ N} \)
   and the direction \( \theta = 44.3^\circ \)
To replace a force by its equivalent force-couple system, where the force $F$ acting on a rigid body at point $A$ is replaced by an equal force at point $B$ and the couple $M = r \times F$. By the adding equal and opposite forces $F$ and $-F$ at point $B$, we obtain the couple the couple Composed of $-F$ and the original $F$.

Sample problem (3):
A force of 40 lb is applied at $A$ as shown. Replace the force by an equivalent force at $O$ and a couple. Describe the couple as a vector $\mathbf{\bar{M}}$.

Solution:
The coordinates of points $O$ and $A$ thus
$O(0,0,0)$, $A(0,8,5)$
The vector $\mathbf{\bar{r}} = \mathbf{\bar{PO}} = 8\mathbf{j} + 5\mathbf{k}$
The couple in (B-D) equal thus
$\mathbf{\bar{M}} = \mathbf{\bar{r}} \times \mathbf{\bar{F}}$ when $\mathbf{\bar{F}} = -40\mathbf{i}$ as show in figure
So $\mathbf{\bar{M}} = (8\mathbf{j} + 5\mathbf{k}) \times (-40\mathbf{i})$
$\mathbf{\bar{M}} = 320\mathbf{k} - 200\mathbf{j}$ lb. in