Center of mass and Centroids:

*Center of mass:
Consider a three-dimensional body of any size and shape, having mass $m$. If we suspend the body, as shown in Figure, from any point such as $A$, the body will be in equilibrium under the action of tension in the cord and the resultant of the gravitational forces acting on all particles of the body, and suspending the body from other points such as $B$ and $C$, the line of action of the resultant force purposes at a single point $G$ which is called the Center of gravity of the body.

![Diagram of three bodies suspended from different points]

The weight of the body given by the sum $W = \int \text{d}W$. If we apply the moment principle about the $y$-axis, for example, the moment of the sum thus $\bar{x}W = \int x \text{d}W$. We may express the coordinates of the Center of gravity $G$ as

$$
\bar{x} = \frac{\int x \text{d}W}{W} \quad \bar{y} = \frac{\int y \text{d}W}{W} \quad \bar{z} = \frac{\int z \text{d}W}{W}
$$

with the substitution of $W = mg$, and $\text{d}W = g \text{d}m$, the coordinates of the Center of gravity become

$$
\bar{x} = \frac{\int x g \text{d}m}{m} \quad \bar{y} = \frac{\int y g \text{d}m}{m} \quad \bar{z} = \frac{\int z g \text{d}m}{m}
$$
Centroids of Lines, Areas, and Volumes:

The calculation of centroids depends on whether we can model the shape of the body as a line, an area, or a volume.

1. For a slender rod or wire of length \( L \), cross-sectional area, and density \( \sigma \), \( dm = \sigma A \) dl. If the \( \sigma \) and \( A \) are constant, the coordinates of the centroid \( C \) of the line may be written:

\[
\bar{x} = \frac{\int x \, dm}{m} = \frac{\int x \sigma A \, dl}{\sigma A L} = \frac{\int x \sigma A \, dl}{\sigma A L} = \frac{\int x \, dl}{L} \]

\[
\bar{y} = \frac{\int y \, dl}{L} \quad \bar{z} = \frac{\int z \, dl}{L}
\]

2. Areas: when a body of density \( \sigma \) has a small but constant thickness \( h \). The mass of an element becomes \( dm = \sigma h \, dA \). The coordinates of the centroid \( C \) of the surface area may be written:

\[
\bar{x} = \frac{\int x \sigma h \, dA}{A} \quad \bar{y} = \frac{\int y \sigma h \, dA}{A} \quad \bar{z} = \frac{\int z \sigma h \, dA}{A}
\]

3. Volume: for a general body of volume \( V \) and density \( \sigma \), the element has a mass \( dm = \sigma \, dV \). The density \( \sigma \) constant, and the coordinates of the centroid of mass also become:

\[
\bar{x} = \frac{\int x \sigma \, dV}{\sigma V} \quad \bar{y} = \frac{\int y \sigma \, dV}{\sigma V} \quad \bar{z} = \frac{\int z \sigma \, dV}{\sigma V}
\]
Centroid of Areas:

1. Centroid of nonuniform area:

   with following two examples determine the coordinates of the Centroid shaded area given.

\[
\bar{x} = \frac{\int x \, dA}{\int dA}
\]

\[
\bar{y} = \frac{\int y \, dA}{\int dA}
\]

Sample problem (x):

locate the Centroid of the shaded area under the Curve \( x = ky^3 \) from \( x=0 \) to \( x=a \).

Solution:

\[
\bar{x} = \frac{\int x \, dA}{\int dA} = \frac{\int y \, dA}{\int dA} = x
\]

\[
\bar{y} = \frac{\int y \, dA}{\int dA} = \frac{\int y \, dA}{\int dA} = \frac{3}{\sqrt[3]{1}}
\]

\[
\bar{x} = \frac{\int x^2 \, dA}{\int dA}
\]

\[
\int x^2 \, dA = \int x \, dA \cdot \frac{x^2}{2} = \int x^1 \cdot \frac{x^2}{2} = \int \frac{x^3}{2} dx = \frac{x^{4/3}}{4/3}
\]

\[
\bar{y} = \frac{\int y \, dA}{\int dA} = \frac{\int y \, dA}{\int dA} = \frac{3a^2 b}{7}
\]
\[
-x = \frac{3a^2b}{4} \Rightarrow x = \frac{3a^2b}{4} \times \frac{4}{3a^2} = \frac{1}{3} \Rightarrow x = \frac{4}{14} a
\]

The \( E \) is:
\[
E = \frac{\int_{S}dA}{\int_{A}dA} = \frac{\int_{0}^{a} \frac{b}{2} y \, dx}{\int_{0}^{a} \frac{b}{2} \, dx} \Rightarrow \int_{0}^{a} \left( \frac{3a^2b}{4} \right) y \, dx
\]

\[
= \frac{1}{2} \int_{0}^{a} y^2 \, dx = \frac{1}{2} \int_{0}^{a} \left( \frac{3a^2b}{4} \right) ^2 \, dx = \frac{1}{2} \int_{0}^{a} \frac{x^{2/3}}{x^{2/3}} \, dx
\]

\[
= \frac{1}{2} \left[ \frac{1}{a^{2/3}} \int_{0}^{a} x^{2/3} \, dx \right] = \frac{1}{2} \left[ \frac{1}{a^{2/3}} \left( \frac{x^{5/3}}{5/3} \right) \right]_{0}^{a}
\]

\[
= \frac{1}{2} \cdot \frac{b^{2}}{a^{2/3}} \cdot \frac{a^{5/3}}{5/3} = \frac{1}{2} \cdot \frac{3}{5} \cdot b^{2} \cdot \frac{3a^2}{20}
\]

\[
E = \frac{3ab^2}{20} \Rightarrow \frac{3a^2b}{4}
\]

\[
\bar{y} = \frac{3ab^2}{20} \times \frac{4}{2ab} \Rightarrow \bar{y} = \frac{2b}{5}
\]
2. **Centroid of uniform area:**

(a) To the area of a rectangle:

\[ \bar{x} = \frac{b}{2} \quad ; \quad \bar{y} = \frac{h}{2} \]

**Examples:** Find the centroid of the following areas.

\[ \bar{x} = \frac{b}{2} + 4 = \frac{6}{2} + 4 = 7 \text{m} \]
\[ \bar{y} = \frac{h}{2} + 3 = \frac{6}{2} + 3 = 6 \text{m} \]

(b) **Centroid of triangle area:**

\[ \bar{x} = \frac{b}{3} \quad \bar{y} = \frac{h}{3} \]

**Examples:** Find the centroid of the following triangle areas.

\[ \bar{x} = \frac{2}{3} b + 6 = \frac{2}{3} \times 6 + 6 = 10 \text{m} \]
\[ \bar{y} = \frac{h}{3} + 4 = \frac{9}{3} + 4 = 7 \text{m} \]
\[ \bar{x} = \frac{b}{3} + 4 = \frac{6}{3} + 4 = 6 \text{m} \]
\[ \bar{y} = \frac{h}{3} = \frac{9}{3} = 3 \text{m} \]
Equilateral triangle ribs:

\[ \bar{x} = \frac{1}{2} b + z = \frac{1}{2} \cdot 6 + 2 = 5 \text{ m} \]
\[ \bar{y} = \frac{h}{3} = \frac{9}{3} = 3 \text{ m} \]

Triangle various ribs:

\[ \bar{x}_1 = \frac{2}{3} b + 3 = 7 \text{ m} \]
\[ \bar{y}_1 = \frac{h}{3} + 5 = \frac{6}{3} + 5 = 7 \text{ m} \]

\[ \bar{x}_2 = \frac{2}{3} \cdot 3 + 2 = 4 \text{ m} \]

6) To the area of a Circle:

\[ \bar{x} = r \quad \bar{y} = r \]

7) To the area of Semicircle:

Here the difference between the two cases must:

1° Semicircle applicable on the x-axis.

\[ \bar{x} = r \quad \bar{y} = \frac{4r}{3\pi} \]

2° Semicircle applicable on the y-axis.

\[ \bar{x} = \frac{4r}{3\pi} \]
Examples: Find the centroid of the following semicircle areas.

\[
\bar{x} = \frac{4r}{3\pi}, \quad \bar{y} = 0
\]

6. To the area of quarter-circle:

\[
\bar{x} = \frac{4r}{8\pi}, \quad \bar{y} = \frac{4r}{3\pi}
\]

Examples of quarter-circle areas:

\[
\bar{x} = \frac{4r}{3\pi} + 3, \quad \bar{y} = (r - \frac{4r}{3\pi}) + 2
\]

Centroid of Composite Areas:

To find this centroid we divide the total area to uninform area, thus:

\[
\bar{x} = \frac{\sum A \bar{x}}{\sum A} = \frac{A_1 \bar{x}_1 + A_2 \bar{x}_2 + A_3 \bar{x}_3 + \ldots}{A_1 + A_2 + A_3 + \ldots}
\]

\[
\bar{y} = \frac{\sum A \bar{y}}{A} = \frac{A_1 \bar{y}_1 + A_2 \bar{y}_2 + A_3 \bar{y}_3}{A_1 + A_2 + A_3}
\]
Sample problem (2):

Determine the \( x \) - and \( y \) -coordinates of the Centroid of the shaded area.

\[
\bar{x} = \frac{\sum A \bar{x}}{\sum A} = \frac{A_1 \bar{x}_1 - A_2 \bar{x}_2}{A_1 - A_2}
\]

\( A_1 = \frac{\pi r^2}{4} = \frac{\pi a^2}{4} = \frac{1}{4} a^2 \pi \) quarter-circle area

\( \bar{x}_1 = \frac{4r}{3\pi} \) \\
\( A_2 = \frac{1}{2} \cdot \frac{a^2}{2} = \frac{1}{4} a^2 \) triangle area

\( \bar{x}_2 = \frac{b}{3} = \frac{a}{3} = \frac{1}{6} a \)

\[
\bar{y} = \frac{\sum A \bar{y}}{\sum A} = \frac{A_1 \bar{y}_1 - A_2 \bar{y}_2}{A_1 - A_2}
\]

\( \bar{y}_1 = \frac{4r}{3\pi} = \frac{4a}{3\pi} \) \\
\( \bar{y}_2 = \frac{h}{3} = \frac{a}{3} = \frac{1}{3} \)

\[
\bar{x} = \frac{1}{3} a \pi - \frac{1}{4} a^2 \frac{1}{6} a = \frac{1}{3} a^3 - \frac{1}{2} a^3 = \frac{8a^3 - a^3}{24} = \frac{4a^3 - a^3}{4a^2(\pi - 1)}
\]

\[
\bar{y} = \frac{1}{4} a \pi - \frac{1}{4} a^2 \frac{a}{3} = \frac{4a^3 - a^3}{4} = \frac{3a^3}{12} - \frac{12}{12} = \frac{3a^3}{12} = \frac{1}{4} a^2 (\pi - 1)
\]

\[
\bar{y} = \frac{3a^3}{4} \pi = \frac{1}{2} a \pi (\pi - 1) (\pi - 1)
\]