Area Moments of Inertia:

When forces are distributed continuously over an area which they act, it is often necessary to calculate the moment of these forces about some axis either in or perpendicular to the plane of the area. Therefore, that the total an integral of form \( S \text{(distance)}^2 \text{(area)} \), this integral called the moment of inertia.

1. Nonuniform area Moments of Inertia.

- When the slide area is horizontal strip. Calculated the moment of inertia \( I_x \) by law as:

\[
I_x = \int y^2 \, dA = \int x \, dy
\]

2. When the slide area is vertical strip. Calculated the moment of inertia \( I_x \) by law as:

\[
I_x = \frac{1}{3} \int y^3 \, dx
\]

Sample problems:

Determine the moment of inertia of the area under parabola about the x-axis. Solve by using a) a horizontal strip of area. b) a vertical strip of area.

Solution:

\[
x = ky^2 \Rightarrow k = \frac{x}{y^2} \Rightarrow k = \frac{4}{3} \Rightarrow \left| k \right| = \frac{4}{9}
\]

All measured lengths...
\[ I_x = \int y^2 \, dx = \int y^2 \left( 1 - \frac{y^2}{3} \right) dy \]
\[ = \int 4y^2 \left( 1 - \frac{y^2}{3} \right) dy = 4 \left( \frac{y^3}{3} - \frac{y^5}{8} \right) \bigg|_{0}^{3} \]
\[ = \frac{4}{3} (3)^3 \left( 3 - \frac{3^3}{2 \cdot 8} \right) = \frac{172}{3} = 14.4 \text{ (units)}^4 \]

b) Vertical Strip
\[ I_x = \frac{1}{3} \int y^3 \, dx \]
when \[ x = \frac{4}{3} y^2 \rightarrow y = \frac{3}{4} \frac{x}{2} = \frac{3}{8} x \]
\[ \Rightarrow y = \sqrt{\frac{3}{4}} x \rightarrow y = \frac{3}{2} \sqrt{x} \]
\[ I_x = \frac{1}{3} \int \left( \frac{3}{2} \sqrt{x} \right)^3 \, dx = \frac{1}{3} \int \frac{27}{8} x^{3/2} \, dx \]
\[ = \frac{1}{3} \left( \frac{3}{2} \right)^3 \left[ \frac{2}{15} x^{5/2} \right]_0^4 = \frac{2}{15} \left( \frac{3}{2} \right)^3 \left( \frac{5}{2} \right)^{5/2} \]
\[ = \frac{1}{3} \left( \frac{3}{2} \right)^3 \left( 4^{3/2} \right) = \frac{2}{15} \left( \frac{3}{2} \right)^3 \left( 4 \right)^{3/2} = 0.445 \times 32 \]
\[ = 14.4 \text{ (units)}^4 \]

- **Uniform Area Moment of Inertia:**
  - Rectangle Area Moment of Inertia
    - If the rectangle is based on the \( x \)-axis moment of inertia written as
    \[ I_{xb} = \frac{bh^3}{12} \]
    - If the rectangle area in space it uses low
    \[ I_{xc} = \frac{bh^3}{12} \text{ but with addition of the amount} \]
    \[ \text{of parallel axis} (Ad^2) \]
2. Triangle area moment of inertia.
   - when the area based on x-axis
     \( I_{x_b} = \frac{bh^3}{12} \)
   - when the area in space
     \( I_{x_c} = \frac{bh^3}{36} \) and addition \((Ad^2)\) theory of parallel-axis.

3. Circle area moment of inertia.
   - only one law \( I_{x_c} = \frac{\pi r^4}{4} \)
   but when the circle area in space must addition \((Ad^2)\) theory parallel-axis

4. Semicircle area moment of inertia.
   - when the semicircle area is horizontal there are two laws thus
     \( I_{x_b} = \frac{r^4 \pi}{8} \)
     \( I_{x_c} = 0.11 r^4 \)
   - when the semicircle area is vertical there is only one law
     \( I_{x_c} = \frac{r^4 \pi}{8} \)
   this law when the semicircle area in space must addition \((Ad^2)\) theory parallel-axis

5. Quarter-circle area moment of inertia.
   - there are two laws
     \( I_{x_b} = \frac{r^4 \pi}{16} \)
     \( I_{x_c} = \frac{r^4 \pi}{8} \) when base is on the axis

6. Composite Area Moment of Inertia:

\[ I_x = I_{x1} + I_{x2} + I_{x3} + I_{x4} + \ldots + I_{xn} \]

Sample problems:

Calculate the moment of inertia about the x-axis for the shaded area shown.

1. Rectangle area:

\[ I_{x1} = \frac{bh^3}{3} = \frac{80 \times 60^3}{3} = 5760000 \text{ mm}^4 \]

or \( 5.76 \times 10^6 \text{ mm}^4 \)

2. Quarter-circle area:

\[ d = \frac{d}{2} = \left( 1 - \frac{4r}{\pi} \right) + 30 \Rightarrow d = \left( 30 - \frac{4 \times 30}{2 \times 3.14} \right) + 30 \Rightarrow d = 41.72 \text{ mm} \]

\[ A_2 = \frac{\pi d^2}{4} = \left( \frac{30 \times 3.14}{4} \right) = 706.5 \text{ mm}^2 \]

\[ I_{xc} = 0.056 (30)^4 + 706.5 \times (41.72)^2 \]

\[ I_{x2} = 16560 + 160653.3 = 160620.8 \text{ mm}^4 \text{ or } 1.6 \times 10^6 \text{ mm}^4 \]

3. Triangle area:

\[ I_{x3} = \frac{bh^3}{12} = \frac{40 \times 30^3}{12} = 90000 \text{ mm}^4 \text{ or } 9.0 \times 10^6 \text{ mm}^4 \]

\[ I_x = I_{x1} - I_{x2} - I_{x3} \]

\[ I_x = 5760000 - 1606208 - 90000 = 4064792 \text{ mm}^4 \]

or \( 4.06 \times 10^6 \text{ mm}^4 \)