CHAPTER 3: ONE-DIMENSIONAL, STEADY-STATE CONDUCTION

Objectives:

1. To determine expressions for the temperature distribution and heat transfer rate in common (planar, cylindrical, and spherical) geometries.
2. To introduce the concept of thermal resistance and the use of thermal circuits to model heat flow.
3. To consider the use of extended surfaces (fins) for heat transfer enhancement.

3.1 The Plane Wall

➢ Consider a plane wall that separates two fluids (one hot and one cold) as shown below.

➢ One-dimensional (1-D) heat conduction occurs in the plane wall. Therefore, the temperature in the wall is a function of \( x \) only and heat transfer is exclusively in the \( x \) direction.

3.1.1 Temperature Distribution

➢ The temperature distribution in the plane wall is to be determined by solving the heat equation.

➢ The appropriate form of the heat equation for steady 1-D conduction with no heat generation is

\[
\frac{d}{dx} \left( k \frac{dT}{dx} \right) = 0
\]  

(1)

➢ The boundary conditions are
Integrate equation (1) twice (assuming \( k \) is constant) and apply the boundary conditions to obtain the temperature distribution as

\[
T(x) = \left( T_s,2 - T_s,1 \right) \frac{x}{L} + T_s,1
\]  

(3)

**Note:** Equation (3) shows that for 1-D, steady-state conduction in a plane wall with no heat generation and constant \( k \), the temperature varies linearly with \( x \).

- The conduction heat transfer rate is obtained by using the Fourier’s law:

\[
q_x = -kA \frac{dT}{dx} = \frac{kA}{L} \left( T_s,1 - T_s,2 \right)
\]  

(4)

- The heat flux is

\[
q_x^* = \frac{q_x}{A} = \frac{k}{L} \left( T_s,1 - T_s,2 \right)
\]  

(5)

Note that both \( q_x \) and \( q_x^* \) are constants, independent of \( x \).

- If \( T_{s,1} \) and \( T_{s,2} \) are not known, convection boundary conditions should be applied in solving the heat equation.

- In the foregoing analysis we have used the **standard approach**. This involves solving the heat equation to obtain the temperature distribution and then using it with Fourier’s law to determine the heat transfer rate.

### 3.1.2 Thermal Resistance

- There exists an analogy between heat transfer and the flow of electric current.

- The driving potential for heat transfer is temperature drop, just like the driving potential for electric current is voltage drop.

- Resistance is the ratio of a driving potential to the corresponding transfer rate.
  - For electrical conduction, Ohm’s law provides an electrical resistance of the form

\[
R_e = \frac{\Delta E_e}{I}
\]  

(6)

- For the planar wall considered in section 3.1.1, the thermal resistance for conduction is

\[
R_{t,cond} = \frac{T_{s,1} - T_{s,2}}{q_x} = \frac{L}{kA}
\]  

(7)

- The thermal resistance for convection is obtained from Newton’s law of cooling \(( q = hA(T_s - T_e) )\) as
\[ R_{t,\text{conv}} = \frac{T_s - T_\infty}{q} = \frac{1}{h A} \]  

(8)

- The thermal circuit for heat transfer through the plane wall of section 3.1.1 is shown below.

\[ T_{x,1} \quad T_{s,1} \quad T_{s,2} \quad T_{x,2} \]

\[ q_x \quad \frac{1}{h_1 A} \quad \frac{L}{k A} \quad \frac{1}{h_2 A} \]

- Since \( q_x \) is constant throughout the network, it follows that

\[ q_x = \frac{T_{x,1} - T_{s,1}}{1/h_1 A} = \frac{T_{s,1} - T_{s,2}}{L/k A} = \frac{T_{s,2} - T_{x,2}}{1/h_2 A} \]  

(9)

- In terms of the overall temperature difference, the heat transfer rate may also be expressed as

\[ q_x = \frac{T_{x,1} - T_{x,2}}{R_{tot}} \]  

(10)

where \( R_{tot} = \frac{1}{h_1 A} + \frac{L}{k A} + \frac{1}{h_2 A} \) (resistances in series)  

(11)

- A thermal resistance for radiation may be defined as

\[ R_{t,\text{rad}} = \frac{T_s - T_{sur}}{q_{rad}} = \frac{1}{h_r A} \]  

(12)

where \( h_r \) is as defined in chapter 1.

- Surface radiation and convection resistances act in parallel, and if \( T_\infty = T_{sur} \), they may be combined to obtain a single, effective surface resistance.
3.1.3 The Composite Wall

- More complex systems, such as composite walls, may also be modeled with thermal circuits.
- Consider a series composite wall separating two fluids as shown below.

The 1-D heat transfer rate for the system may be expressed as

\[ q_x = \frac{T_{\infty, 1} - T_{\infty, 4}}{\sum R_t} \]  \hspace{1cm} (13)

where

\[ \sum R_t = R_{tot} = \frac{1}{h_1 A} + \frac{L_A}{k_A A} + \frac{L_B}{k_B A} + \frac{L_C}{k_C A} + \frac{1}{h_4 A} \]  \hspace{1cm} (14)

- Alternatively, \( q_x \) can be related to the temperature difference and resistance associated with each element, i.e.

\[ q_x = \frac{T_{\infty, 1} - T_{s, 1}}{\left(\frac{1}{h_1 A}\right)} = \frac{T_{s, 1} - T_2}{(L_A/k_A A)} = \frac{T_2 - T_3}{(L_B/k_B A)} = \ldots \]  \hspace{1cm} (15)

- With composite systems it is often convenient to work with an overall heat transfer coefficient, \( U \), which is analogous to Newton’s law of cooling, i.e.

\[ q_x = U A \Delta T \]  \hspace{1cm} (16)

where \( \Delta T \) is the overall temperature difference.
From equations (13) and (16), we see that \( UA = 1/R_{tot} \).

In general, we may write

\[
R_{tot} = \sum R_I = \frac{\Delta T}{q} = \frac{1}{UA}
\]

(17)

A parallel composite of two materials is shown below.

- The heat transfer rate in the network is

\[
q_x = \frac{T_1 - T_2}{R_{tot}}
\]

(18)

where \( R_{tot} = \frac{1}{1/R_1 + 1/R_2} \)

- Alternatively, the heat transfer rate can be calculated as the sum of heat transfer rates in the individual materials, i.e.

\[
q_x = q_{1x} + q_{2x} = \frac{T_1 - T_2}{R_1} + \frac{T_1 - T_2}{R_2}
\]

(19)

Composite walls, such as the one shown below, may be characterized by series-parallel configurations.

Although the heat flow in the composite wall above is multidimensional, approximate solution can be obtained by assuming one-dimensional heat transfer. Two different thermal circuits (shown below) may be used:
a) Here, it is presumed that surfaces normal to the x direction are isothermal.

![Diagram of circuit with labeled components](image)

b) Here, it is assumed that surfaces parallel to the x direction are adiabatic.

![Diagram of circuit with labeled components](image)

- The actual value of $q$ lies between the values obtained with circuits (a) and (b).

### 3.1.4 Contact Resistance

- In composite systems, the interface between two layers is usually not perfect. This is due to surface roughness effect.
- Contact spots between the two layers are interspersed with gaps that are, in most instances, air filled.
- The additional resistance between the two layers, called thermal contact resistance, $R_{t,c}$, results in temperature drop across the interface, (see figure 3.4 of the textbook).
- The thermal contact resistance for the interface shown in figure 3.4 is

$$R_{t,c}^* = \frac{T_A - T_B}{q_x}$$

(20)

- Thermal contact resistance is dependent upon the solid materials, surface roughness, contact pressure, temperature, and interfacial fluid.
- The values of thermal contact resistance for some surface pairs are given in Tables 3.1 and 3.2.
3.2 An Alternative Conduction Analysis

- The standard approach for conduction analysis involves solving the heat equation to obtain the temperature distribution and then applying Fourier’s law to obtain the heat transfer rate.
- An alternative conduction analysis can be used to obtain the heat transfer rate if the following conditions are satisfied: steady-state, 1-D, and no heat generation.
- Consider conduction in the system shown below.

- Under steady-state conditions with no heat generation and no heat loss from the sides, the conservation of energy requirements on a differential element gives \( q_x = q_{x+dx} \). This shows that \( q_x \) is independent of \( x \).
- For the general case in which \( A = A(x) \) and \( k = k(T) \), the integral form of Fourier’s law may be expressed as

\[
q_x \int_{x_o}^{x} \frac{dA}{A(x)} = - \int_{T_o}^{T} k(T) dT \tag{21}
\]

- If, at point \( x = x_o, T_o = T(x_o) \) is known, integration of the above equation provides the functional form of \( T(x) \).
- If, at another point \( x = x_1, T_1 = T(x) \) is known, integration of the equation between \( x_o \) and \( x_1 \) provides an expression from which \( q_x \) may be computed.
- For the specific case in which area \( A \) is uniform and \( k \) is independent of \( T \), the above equation reduces to

\[
q_x \frac{\Delta x}{A} = - k \Delta T \tag{22}
\]

where \( \Delta x = x_1 - x_o \) and \( \Delta T = T_1 - T_o \).
### 3.3 Radial Systems

- Cylindrical and spherical systems often experience temperature gradients in the radial direction only and may therefore be treated as one dimensional.

#### 3.3.1 The Cylinder

- Consider a hollow cylinder of length L (shown below), whose inner and outer surfaces are exposed to fluids at different temperatures. The system is analyzed by the standard method as follows:

#### For steady-state conditions with no heat generation, the heat equation for the system is

\[
\frac{1}{r} \frac{d}{dr} \left( k \frac{dT}{dr} \right) = 0
\]

(23)

- The boundary conditions are

\[
\begin{align*}
T(r_1) &= T_{s,1} \\
T(r_2) &= T_{s,2}
\end{align*}
\]

(24)

- Integrating the heat equation (assuming constant \(k\)) and using the boundary conditions yield

\[
T(r) = \frac{T_{s,1} - T_{s,2}}{\ln(r_1/r_2)} \ln \left( \frac{r}{r_2} \right) + T_{s,2}
\]

(25)

Therefore, the temperature distribution associated with radial conduction through a cylindrical wall is logarithmic.
The heat transfer rate is obtained by using the temperature distribution with Fourier’s law:

\[ q_r = -k A \frac{dT}{dr} = -k (2\pi r L) \frac{dT}{dr} \]

\[ = \frac{2\pi L k (T_s,1 - T_s,2)}{\ln \left( \frac{r_2}{r_1} \right)} \]  

Equation (26) shows that the heat transfer rate \( q_r \) is a constant in the radial direction.

From equation (26), the thermal resistance for radial conduction in a cylindrical wall is

\[ R_{t,\text{cond}} = \frac{\Delta T}{q_r} = \frac{\ln \left( \frac{r_2}{r_1} \right)}{2\pi L k} \]  

(See the thermal circuit in the figure above.)

Since \( q_r \) is independent of \( r \), the above analysis could also be performed by using the alternative method.

Consider a composite cylindrical wall of length \( L \) shown below.

Neglecting interfacial contact resistances, the heat transfer rate may be expressed as
\[ q_r = \frac{T_{\infty,1} - T_{\infty,4}}{2\pi r_1 L h_1 + \frac{\ln\left(r_2/r_1\right)}{2\pi k_A L} + \frac{\ln\left(r_3/r_2\right)}{2\pi k_B L} + \frac{\ln\left(r_4/r_3\right)}{2\pi k_C L} + \frac{1}{2\pi r_4 L h_4}} \]  

Equation (28) may be expressed in terms of an overall heat transfer coefficient as

\[ q_r = \frac{T_{\infty,1} - T_{\infty,4}}{R_{tot}} = U A \left(T_{\infty,1} - T_{\infty,4}\right) \]  

If \( U \) is defined in terms of the inside area \( A_1 \), equations (28) and (29) may be equated to yield

\[ U_1 = \frac{1}{1 + \frac{1}{h_1 A_1} \ln \frac{r_1}{k_A} + \frac{1}{r_1} \ln \frac{r_2}{k_B} + \frac{r_1}{r_2} \ln \frac{r_3}{k_C} + \frac{r_1}{r_3} \ln \frac{r_4}{h_4} + \frac{1}{r_4 h_4}} \]  

Similar equations could be written for \( U_2, U_3, \) etc. Note that

\[ U_1 A_1 = U_2 A_2 = U_3 A_3 = U_4 A_4 = \sum R_t \]  

**Critical radius**

- For a plane wall exposed to a fluid, an increase in the thickness of the wall results in an increase in the conduction resistance \( R_k = L/(kA) \) but does not change the convection resistance \( R_c \). Hence, the heat transfer rate will reduce as the wall thickness increases.

- However, for geometries with non-constant cross-sectional area (e.g. a cylinder), increase in the wall thickness does not always bring about a decrease in the heat transfer rate.

- The critical radius of insulation for a cylinder exposed to convection is

\[ r_{cr} = \frac{k}{h} \]

where \( k \) is the of thermal conductivity of the insulation material and \( h \) is the convection heat transfer coefficient on the insulation. This is illustrated in example 3.5.
3.3.2 The Sphere

- Conduction in a spherical shell can be analyzed with the alternative conduction analysis method.
- Consider a differential control volume in a spherical shell (below):

  ![Diagram of a spherical shell with control volume](image)

  - The appropriate form of Fourier’s law is
    
    \[ q_r = -k A \frac{dT}{dr} = -k \left( 4\pi r^2 \right) \frac{dT}{dr} \]  
    
    (32)

  - Conservation of energy requires that \( q_t = q_{t+dr} \) for steady-state, 1-D conduction with no heat generation. Therefore, \( q_t \) is a constant independent of \( r \). Equation (32) may therefore be expressed in integral form as
    
    \[ \frac{q_r}{4\pi} \int_{r_1}^{r_2} \frac{d\tau}{r^2} = -\int_{T_{s,1}}^{T_{s,2}} k(T) dT \]  
    
    (33)

  - The heat transfer rate is then (assuming constant \( k \))
    
    \[ q_r = \frac{4\pi k (T_{s,1} - T_{s,2})}{(1/r_1) - (1/r_2)} \]  
    
    (34)

  - Therefore, the thermal resistance is
    
    \[ R_{t,\text{cond}} = \frac{\Delta T}{q_r} = \frac{1}{4\pi k} \left( \frac{1}{r_1} - \frac{1}{r_2} \right) \]  
    
    (35)

- Spherical composites may be treated the same way as composite walls and cylinders.
- The critical radius of insulation for a sphere is
  
  \[ r_{cr} = \frac{2k}{h} \]

3.4 Summary of 1-D Conduction Results

- See Table 3.3 for a summary of the steady-state solutions to the heat equation with no generation for plane, cylindrical and spherical walls.
3.5 Conduction with Thermal Energy Generation

- Thermal energy generation within a medium is a result of conversion from some other energy forms.
- A common source of energy generation is electrical energy due to a current $I$ passing through a medium of electrical resistance $R_e$.
  - The rate of energy generation is
    \[
    \dot{E}_g = I^2 R_e
    \]  
    (36)
  - If energy generation occurs uniformly throughout the medium of volume $V$, the volumetric generation rate is
    \[
    \dot{q} = \frac{\dot{E}_g}{V} = \frac{I^2 R_e}{V}
    \]  
    (37)
- Other sources of energy conversion are nuclear, chemical, and electromagnetic energy.

3.5.1 The Plane Wall

- Consider the plane wall shown below, in which there is uniform energy generation per unit volume and the surfaces are maintained at $T_{s,1}$ and $T_{s,2}$.

- For constant thermal conductivity $k$, the heat equation for the system is
  \[
  \frac{d}{dx} \left( k \frac{dT}{dx} \right) + \dot{q} = 0
  \]  
  (38)
- The boundary conditions are
\[
\begin{align*}
T(-L) &= T_{s,1} \\
T(L) &= T_{s,2}
\end{align*}
\]  

(39)

- Integrating the heat equation twice and using the boundary conditions yield the temperature distribution

\[
T(x) = \frac{\dot{q} L^2}{2k} \left( 1 - \frac{x^2}{L^2} \right) + \frac{T_{s,2} - T_{s,1}}{2} \frac{x}{L} + \frac{T_{s,1} + T_{s,2}}{2}
\]

(40)

- When both surfaces of the plane wall are maintained at the same temperature \( T_{s,1} = T_{s,2} = T_s \), the temperature distribution is symmetrical about the midplane.

- The temperature distribution is given by

\[
T(x) = \frac{\dot{q} L^2}{2k} \left( 1 - \frac{x^2}{L^2} \right) + T_s
\]

(41)

- The maximum temperature exists at the midplane

\[
T(0) = T_0 = \frac{\dot{q} L^2}{2k} + T_s
\]

(42)

- Equations (41) and (42) can be combined to give

\[
\frac{T(x) - T_0}{T_s - T_0} = \left( \frac{x}{L} \right)^2
\]

(43)

- At the plane of symmetry, the temperature gradient is zero, \( (dT/dx)_{x=0} = 0 \). Therefore, there is no heat transfer across this plane, i.e. the plane is equivalent to an adiabatic surface. Equation (41), therefore, also applies to a plane wall of thickness \( L \) with one side insulated and the other maintained at a fixed temperature \( T_s \).
If the surface temperature $T_s$ is unknown, it can be obtained by applying energy balance at the surface, i.e.

$$-k \frac{dT}{dx} \bigg|_{x=L} = h(T_s - T_\infty)$$  \hspace{1cm} (44)

### 3.5.2 Radial Systems

- Heat generation may occur in a variety of radial geometries.
- Consider the long solid cylinder shown below. This could represent a current-carrying wire or a fuel element in a nuclear reactor.

![Radial System Diagram](image.png)

- At steady-state, the rate of heat generation within the cylinder must equal the rate at which heat is convected from the surface of the cylinder to a moving fluid. This condition allows the surface temperature to be maintained at a fixed value of $T_s$.
- For constant thermal conductivity $k$, the heat equation for the system is

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) + \frac{\dot{q}}{k} = 0$$  \hspace{1cm} (45)

- The boundary conditions are

$$\frac{dT}{dr} \bigg|_{r=0} = 0 \quad \text{and} \quad T(r_o) = T_s$$  \hspace{1cm} (46)
Integrating the heat equation twice and using the boundary conditions yield the temperature distribution

\[ T(r) = \frac{\dot{q} r_o^2}{4k} \left( 1 - \frac{r^2}{r_o^2} \right) + T_s \]  

(47)

The centerline temperature is

\[ T(0) = T_o = \frac{\dot{q} r_o^2}{4k} + T_s \]  

(48)

Equations (47) and (48) can be combined to give

\[ \frac{T(r) - T_s}{T_o - T_s} = 1 - \left( \frac{r}{r_o} \right)^2 \]  

(49)

To relate \( T_s \) to \( T_\infty \), apply an overall energy balance on the cylinder to obtain

\[ \dot{q} \left( \pi r_o^2 L \right) = h \left( 2\pi r_o L \right) (T_s - T_\infty) \]  

(50)

### 3.5.3 Application of Resistance Concepts

- When heat generation is present within a medium, the heat transfer rate is not constant within the medium. Therefore, the conduction resistance concept does not apply. It should never be used.
3.6 Heat Transfer from Extended Surfaces

- The term *extended surface* is commonly used to depict an important special case involving heat transfer by conduction within a solid and heat transfer by convection (and/or radiation) from the boundaries of the solid.
- A strut that connects two walls at different temperatures as shown in Figure 3.11 is an example of extended surface.
- The most frequent application of conduction-convection effects is one in which an extended surface is used specifically to enhance heat transfer between a solid and an adjoining fluid. Such an extended surface is termed a *fin*. [See Figure 3.12.]
- Examples of fin applications can be found in the engine heads on motorcycles and lawn mowers, electric power transformers and the condenser of an air conditioner.
- There are different fin configurations (see figure 3.14).

3.6.1 A General Conduction Analysis

- To determine the heat transfer rate associated with a fin, we must first obtain the temperature distribution along the fin.
- Consider the extended surface shown below. Heat is transferred from the base by conduction and removed from the extended surface by convection.

```
\[ q_x = q_x + dx + dq_{conv} \]  \hspace{1cm} (51)
```

- The analysis of the extended surface is simplified by the following assumptions:
  - 1-D conduction (because the temperature changes in the y, z directions are small)
  - Steady-state conditions
  - Thermal conductivity is constant
  - Radiation from the surface is negligible
  - Heat transfer coefficient \( h \) is uniform over the surface.
By applying Fourier’s law on \( q_x \) and \( q_x+dx \) and Newton’s law of cooling on \( dq_{conv} \) (see details in the textbook), the energy balance equation becomes

\[
\frac{d^2 T}{dx^2} + \left( \frac{1}{\frac{A_c}{dx}} \right) \frac{dT}{dx} - \left( \frac{h}{A_c k} \frac{dA_s}{dx} \right) (T - T_\infty) = 0
\]  

(52)

where \( A_c \) is the cross-sectional area and \( dA_s \) is the surface area of the differential element.

### 3.6.2 Fins of Uniform Cross-Sectional Area

Consider the straight rectangular and pin fins of uniform cross-section shown below.

- For those fins, \( A_c \) is a constant and \( A_s = P \times x \), where \( A_s \) is the surface area measured from the base and \( P \) is the fin perimeter. Therefore,

\[
dA_c/dx = 0 \text{ and } dA_s/dx = P
\]

(53)

- Substitute (53) into (52) to obtain

\[
\frac{d^2 T}{dx^2} - \frac{h P}{k A_c} (T - T_\infty) = 0
\]

(54)

- To simplify the form of the equation, let us define a new variable called *excess temperature* \( \theta \) as

\[
\theta (x) = T_b - T_x
\]

(55)

- Substitute (55) into (54) to obtain

\[
\frac{d^2 \theta}{dx^2} - m^2 \theta = 0
\]

(56)

where

\[
m^2 = \frac{h P}{k A_c}
\]

(57)
Equation (56) is the differential equation for the variation of excess temperature from the base to the tip of the fin. It is a linear, homogenous, 2nd-order differential equation with constant coefficients. Its general solution is of the form
\[ \theta(x) = C_1 e^{mx} + C_2 e^{-mx} \]  
(58)

Two boundary conditions are needed to evaluate the constants \( C_1 \) and \( C_2 \) in equation (58).

The first boundary condition is that of the temperature at the base of the fin \( (x = 0) \), i.e.
\(0) = T_b - T_\infty = \theta_b \]

The second boundary condition is specified at the fin tip \( (x = L) \). Four different possible cases at the tip are considered.

- Case A: convection heat transfer from the fin tip
- Case B: adiabatic fin tip
- Case C: prescribed temperature at the fin tip
- Case D: very long fin.

The fin tip boundary condition corresponding to each case is presented in Table 3.4. The table also contains the resulting temperature distribution and heat transfer rate.

**Note:** The heat transfer rate is obtained by applying either the Fourier's law at the fin base or Newton's law of cooling over the fin surface.

### 3.6.3 Fin Performance

**Effectiveness**

- An assessment of the usefulness of fin for a particular application is made by evaluating the fin effectiveness \( \varepsilon_f \).
- The fin effectiveness \( \varepsilon_f \) is the ratio of the fin heat transfer rate to the heat transfer rate that would exist without the fin.

\[ \varepsilon_f = \frac{q_f}{h A_{c,b} \theta_b} \]  
(59)

where \( A_{c,b} \) is the fin cross-sectional area at the base.

- In equation (59), \( q_f \) is the heat transfer rate through the fin. For example, for infinitely long fin of uniform cross-section (case D of Table 3.4), the fin effectiveness is

\[ \varepsilon_f = \frac{\sqrt{h P k A_c \theta_b}}{h A_{c,b} \theta_b} = \left( \frac{k P}{h A_c} \right)^{1/2} \]  
(60)

- Equation (60) shows that fin effectiveness is enhanced by the following factors:
  - High thermal conductivity of fin material
  - High ratio of perimeter to cross-sectional area
Also, equation (60) shows that the use of fins is better justified in cases in which \( h \) is small, e.g. in free convection.

In general, the use of fin is justified only when \( \varepsilon_f \geq 2 \).

Fin performance may also be quantified in terms of a thermal resistance.

- The fin thermal resistance is

\[
R_{t,f} = \frac{\theta_b}{q_f}
\]

(61)

- The thermal resistance due to convection at the exposed base (if there were no fin) is

\[
R_{t,b} = \frac{1}{hA_{c,b}}
\]

(62)

- Substituting (61) and (62) into (59), it follows that

\[
\varepsilon_f = \frac{R_{t,b}}{R_{t,f}}
\]

(63)

Note: The usefulness of fin can also be assessed with the use of a parameter called the Biot number, \( Bi \),

\[
Bi = \frac{hl}{k}
\]

where \( l \) is the ratio of the fin surface area to the fin perimeter, \( l = A/P \).

As an approximation, the Biot number can be taken as the ratio between the conduction resistance and the convection resistance

\[
Bi = \frac{R_{t,cond}}{R_{t,conv}}
\]

For a fin to be considered effective, the following condition is to be satisfied

\[
Bi < 0.1
\]

Efficiency

- Another measure of fin thermal performance is the fin efficiency \( \eta_f \), defined as

\[
\eta_f = \frac{q_f}{q_{max}} = \frac{q_f}{hA_f \theta_b}
\]

(64)

where \( A_f \) is the surface area of the fin and \( q_{max} \) is the heat transfer rate if the entire fin surface were at the base temperature.

For example, for a straight fin of uniform cross-section and an adiabatic tip, the fin efficiency is

\[
\eta_f = \frac{M \tanh mL}{hPL \theta_b} = \frac{\tanh mL}{mL}
\]

(65)
Corrected length

➤ In lieu of the cumbersome expression for heat transfer from a straight fin with an active tip (case A of Table 3.4), approximate prediction may be obtained by using the adiabatic tip result (case B of Table 3.4) with a corrected fin length of the form

\[ L_c = L + \left( \frac{t}{2} \right) \quad \text{for a rectangular fin} \]

\[ L_c = L + \left( \frac{D}{4} \right) \quad \text{for a pin fin} \]

➤ Hence, for fin with convection at the tip, the heat transfer rate and efficiency may be approximated as

\[ q_f = M \tanh mL_c \quad (66) \]

\[ \eta_f = \frac{\tanh mL_c}{mL_c} \quad (67) \]

where \( M \) and \( m \) are as defined in Table 3.4.

➤ The efficiencies of various fin configurations are presented as functions of \( L_c^{3/2}(h/k A_p)^{1/2} \) in figures 3.18 and 3.19 of the textbook. These figures may be used to estimate fin efficiency for a particular application (see example 3.10).

3.6.4 Fins of Nonuniform Cross-Sectional Area

➤ Analysis of fin thermal behavior becomes more complex if the fin is of nonuniform cross-section.

➤ Expressions for the efficiency and surface area of several common fin geometries are summarized in Table 3.5. Once the efficiency is determined, the heat transfer rate can be calculated from equation (64), i.e.

\[ \eta_f = \frac{q_f}{h A_f \theta_b} \quad (68) \]

3.6.5 Overall Surface Efficiency

➤ The fin efficiency, \( \eta_f \), (equation 64) characterizes the performance of a single fin.

➤ For an array of fins, the performance is characterized with the overall surface efficiency \( \eta_o \).

➤ Consider the fin arrays shown below.

   - The distance between the fins (i.e., \( S \)) is called the fin pitch.
   - The overall efficiency is defined as

\[ \eta_o = \frac{q_t}{q_{\text{max}}} = \frac{q_t}{h A_f \theta_p} \quad (69) \]
where $A_t$ is the sum of the surface areas of the fins and the exposed portion of the base (often termed the prime surface). $q_t$ is the total heat transfer rate from the surface area $A_t$.

- If there are $N$ fins in the array, each of surface area $A_f$ and the area of the prime surface is designated as $A_b$, the total surface area is

$$A_t = NA_f + A_b$$  \hspace{1cm} (70)

- The total rate of heat transfer by convection is

$$q_t = N \eta_f h A_f \theta_b + h A_b \theta_b$$  \hspace{1cm} (71)

where $\eta_f$ is the efficiency of a single fin.

- Use (70) to eliminate $A_b$ from (71). Hence

$$q_t = h A_t \left[ 1 - \frac{N A_f}{A_t} (1-\eta_f) \right] \theta_b$$  \hspace{1cm} (72)

- Substituting (72) into (69), it follows that

$$\eta_o = 1 - \frac{N A_f}{A_t} (1-\eta_f)$$  \hspace{1cm} (73)

- Once $\eta_o$ is known, the total heat transfer rate for the fin array may be calculated using equation (69).

- From (69), an expression may be inferred for the thermal resistance of a fin array
\[ R_{t,o} = \frac{\theta_B}{q_t} = \frac{1}{\eta \cdot h A_t} \]  

(74)

where \( R_{t,o} \) is an effective resistance that accounts for parallel heat flow paths by conduction/convection in the fins and by convection from the prime surface.

- The figures below illustrate the thermal circuits corresponding to the parallel paths and their representation in terms of an effective resistance. Two possible cases are considered:
  
a) Fins that are integral with the base


b) Fins that are attached to the base (in which case the thermal contact resistance between fin and base has to be accounted for)