TORSION
Simple torsion theory

When a uniform circular shaft is subjected to a torque it can be shown that every section of the shaft is subjected to a state of pure shear Figure (1), the moment of resistance developed by the shear stresses being everywhere equal to the magnitude, and opposite in sense, to the applied torque. For the purposes of deriving a simple theory, to make the following basic assumptions:

1. The material is homogeneous, i.e. of uniform elastic properties throughout.
2. The material is elastic, following Hooke's law with shear stress proportional to shear strain.
3. The stress does not exceed the elastic limit or limit of proportionality.
4. Circular Sections remain circular.
5. Cross-sections remain plane. (This is certainly not the case with the torsion of non-circular Sections.)
6. Cross-sections rotate as if rigid, i.e. every diameter rotates through the same angle.
Practical tests carried out on circular shafts have shown that the theory developed below on the basis of these assumptions shows excellent correlation with experimental results.

Figure (1) Shear system set up on an element in the surface of a shaft subjected to torsion.
(a) **Angle of twist**

Consider now the solid circular shaft of radius \( R \) subjected to a torque \( T \) at one end, the other end being fixed Figure (2). Under the action of this torque a radial line at the free end of the shaft twists through an angle \( \Theta \), point A moves to B, and AB subtends an angle \( \gamma \) at the fixed end. This is then the angle of distortion of the shaft, i.e. the shear strain.

\[
\text{angle in radians} = \frac{\text{arc}}{\text{radius}}
\]

\[
\text{arc } AB = R\Theta = L\gamma
\]

\[
\therefore \quad \gamma = \frac{R\Theta}{L} \quad \text{...(1)}
\]

From the definition of rigidity modulus

\[
G = \frac{\text{shear stress } \tau}{\text{shear strain } \gamma}
\]

**Figure (2)**
\[ \gamma = \frac{\tau}{G} \] ....(2)

where \( \tau \) is the shear stress set up at radius \( R \).

Therefore equating eqns. (1) and (2),

\[ \frac{R \theta}{L} = \frac{\tau}{G} \]

\[ \frac{\tau}{R} = \frac{G \theta}{L} \left( = \frac{\tau'}{r} \right) \] ....(3)

where \( \tau' \) is the shear stress at any other radius \( r \).
(b) Stresses

Let the cross-section of the shaft be considered as divided into elements of radius $r$ and thickness $(dr)$ as shown in Figure (3) each subjected to a shear stress ($\tau'$).
The force set up on each element,
\[ = \text{stress} \times \text{area} = \tau' \times 2\pi r \, dr \text{ (approximately)} \]
This force will produce a moment about the centre axis of the shaft, providing a contribution to the torque

\[ = (\tau' \times 2\pi r \, dr) \cdot r = \tau' \times 2\pi r^2 \, dr \]

The total torque on the section \( (T) \) will then be the sum of all such contributions across the section,

\[ T = \int_{0}^{R} 2\pi \tau' r^2 \, dr \]

Now the shear stress \( (\tau') \) will vary with the radius \( r \) and must therefore be replaced in terms of \( r \) before the integral is evaluated. From equation (3)
\[ \tau' = \frac{G \theta}{L} r \]

\[ T = \int_{0}^{R} 2\pi \frac{G \theta}{L} r^3 \, dr \]

\[ = \frac{G \theta}{L} \int_{0}^{R} 2\pi r^3 \, dr \]

The integral \[ \int_{0}^{R} 2\pi r^3 \, dr \] is called the polar second moment of area (J), and may be evaluated as a standard form for solid and hollow shafts.
\[ T = \frac{G\theta}{L} J \]

or

\[ \frac{T}{J} = \frac{G\theta}{L} \quad \text{...(4)} \]

Combining eqns. (3) and (4) produces the so-called simple theory of torsion:

\[ \frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{L} \quad \text{...(5)} \]
Polar second moment of area

As stated above the polar second moment of area $J$ is defined as

$$J = \int_0^R 2\pi r^3 \, dr$$

For solid shaft

For a solid shaft,

$$J = 2\pi \left[ \frac{r^4}{4} \right]_0^R$$

$$= \frac{2\pi R^4}{4} \quad \text{or} \quad \frac{\pi D^4}{32}$$

... (6)
For hollow shaft

For a hollow shaft of internal radius \( r \),

\[
J = 2\pi \int_{r}^{R} r^3 \, dr = 2\pi \left[ \frac{r^4}{4} \right]_{r}^{R}
\]

\[
= \frac{\pi}{2} (R^4 - r^4) \quad \text{or} \quad \frac{\pi}{32} (D^4 - d^4) \quad \text{.... (7)}
\]
For thin-walled hollow shafts the values of (D) and (d) may be nearly equal, and in such cases there can be considerable errors in using the above equation involving the difference of two large quantities of similar value. It is therefore convenient to obtain an alternative form of expression for the polar moment of area.

\[ J = \int_0^R 2\pi r^3 \, dr = \sum (2\pi r \, dr) r^2 \]

\[ = \sum A r^2 \]

Where; \( A = 2\pi r \, dr \) is the area of each small element of Figure (3). If a thin hollow cylinder is therefore considered as just one of these small elements with its wall thickness \( t = dr \), then

\[ J = Ar^2 = (2\pi r \, t) r^2 = 2\pi r^3 t \text{ (approximately)} \quad \ldots \quad (8) \]
Shear stress and shear strain in shafts

The shear stresses which are developed in a shaft subjected to pure torsion are indicated in Figure (1), their values being given by the simple torsion theory as

\[ \tau = \frac{G\theta}{L} R \]

Now from the definition of the shear or rigidity modulus \((G)\),

\[ \tau = G\gamma \]

It therefore follows that the two equations may be combined to relate the shear stress and strain in the shaft to the angle of twist per unit length, thus

\[ \tau = \frac{G\theta}{L} R = G\gamma \quad \text{...(9)} \]
or, in terms of some internal radius \( r \),

\[
\tau' = \frac{G\theta}{L} r = G\gamma \quad \text{.... (10)}
\]

These equations indicate that the shear stress and shear strain vary linearly with radius and have their maximum value at the outside radius Figure (4).
Section modulus
It is sometimes convenient to re-write part of the torsion theory formula to obtain the maximum shear stress in shafts as follows:

\[
\frac{T}{J} = \frac{\tau}{R}
\]
\[
\tau = \frac{TR}{J}
\]

With (R) the outside radius of the shaft the above equation yields the greatest value possible for T, Figure (4).

\[
\tau_{\text{max}} = \frac{TR}{J}
\]
\[
\tau_{\text{max}} = \frac{T}{Z}
\]

.... (11)
Where, $z = J/R$ is termed the polar section modulus. It will be seen from the preceding section that:

for solid shafts, 

$$Z = \frac{\pi D^3}{16} \quad \text{..... (12)}$$

and for hollow shafts, 

$$Z = \frac{\pi (D^4 - d^4)}{16D} \quad \text{..... (13)}$$
Torsional rigidity

The angle of twist per unit length of shafts is given by the torsion theory as

\[ \frac{\theta}{L} = \frac{T}{GJ} \]

The quantity (GJ) is termed the torsional rigidity of the shaft and is thus given by

\[ GJ = \frac{T}{\theta/L} \quad \text{.... (14)} \]

i.e. the torsional rigidity is the torque divided by the angle of twist (in radians) per unit length.
It has been shown above that the maximum shear stress in a solid shaft is developed in the outer surface, values at other radii decreasing linearly to zero at the centre. In applications where weight reduction is of prime importance as in the aerospace industry, for instance, it is often found advisable to use hollow shafts.
Composite shafts - series connection

If two or more shafts of different material, diameter or basic form are connected together in such a way that each carries the same torque, then the shafts are said to be connected in series and the composite shaft so produced is therefore termed *series-connected* Figure (5).

\[
T = \frac{G_1 J_1 \theta_1}{L_1} = \frac{G_2 J_2 \theta_2}{L_2} \quad \ldots \quad (15)
\]

\[
\frac{J_1}{L_1} = \frac{J_2}{L_2} \quad \ldots \quad (16)
\]
Composite shafts - parallel connection

If two or more materials are rigidly fixed together such that the applied torque is shared between them then the composite shaft so formed is said to be connected in parallel. Figure (6).

For parallel connection, 

\[ T = T_1 + T_2 \]  \hspace{1cm} \text{(17)}

In this case the angles of twist of each portion are equal and

\[ \frac{T_1 L_1}{G_1 J_1} = \frac{T_2 L_2}{G_2 J_2} \]  \hspace{1cm} \text{(18)}

\[ \text{i.e. for equal lengths (as is normally the case for parallel shafts)} \]

\[ \frac{T_1}{T_2} = \frac{G_1 J_1}{G_2 J_2} \]  \hspace{1cm} \text{(19)}

The maximum stresses in each part can then be found from

\[ \tau_1 = \frac{T_1 R_1}{J_1} \quad \text{and} \quad \tau_2 = \frac{T_2 R_2}{J_2} \]
Strain energy in torsion

The strain energy stored in a solid circular bar or shaft subjected to a torque (T) is given by the alternative expressions.

\[ U = \frac{1}{2} T \theta = \frac{T^2 L}{2GJ} = \frac{GJ \theta^2}{2L} = \frac{\tau^2}{4G} \times \text{volume} \quad \text{(20)} \]
Power transmitted by shafts

If a shaft carries a torque \( T \) Newton metres and rotates at \( \omega \) rad/s it will do work at the rate of:

\[ T \cdot \omega \text{ Nm/s (or joule/s)}. \]

Now the rate at which a system works is defined as its power, the basic unit of power being the

\[ \text{Watt (1 Watt = 1 N.m/s)}. \]

Thus, the power transmitted by the shaft:

\[ = T \cdot \omega \text{ Watts}. \]

Since the Watt is a very small unit of power in engineering terms use is normally made of SI. multiples, i.e. kilowatts (kW) or megawatts (MW).
Combined bending and torsion - equivalent bending moment

For shafts subjected to the simultaneous application of a bending moment ($M$) and torque ($T$) the principal stresses set up in the shaft can be shown to be equal to those produced by an equivalent bending moment, of a certain value ($M_e$) acting alone.

From the simple bending theory the maximum direct stresses set up at the outside surface of the shaft owing to the bending moment ($M$) are given by

$$\sigma = \frac{M y_{\text{max}}}{I} = \frac{MD}{2I}$$

Similarly, from the torsion theory, the maximum shear stress in the surface of the shaft is given by

$$\tau = \frac{TR}{J} = \frac{TD}{2J}$$

But for a circular shaft $J = 2I$,

$$\tau = \frac{TD}{4I}$$
The principal stresses for this system can now be obtained by applying the formula derived in

\[ \sigma_1 \text{ or } \sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau^2]} \]

and, with \( \sigma_y = 0 \), the maximum principal stress (\( \sigma_1 \)) is given by

\[
\sigma_1 = \frac{1}{2}\left(\frac{MD}{2I}\right) + \frac{1}{2}\sqrt{\left(\frac{MD}{2I}\right)^2 + 4\left(\frac{TD}{4I}\right)^2}
\]

\[
= \frac{1}{2}\left(\frac{D}{2I}\right)[M + \sqrt{(M^2 + T^2)}]
\]

Now if \((M_c)\) is the bending moment which, acting alone, will produce the same maximum stress, then

\[
\sigma_1 = \frac{M_c y_{\text{max}}}{I} = \frac{M_c D}{2I}
\]

\[
\frac{M_c D}{2I} = \frac{1}{2}\left(\frac{D}{2I}\right)[M + \sqrt{(M^2 + T^2)}]
\]
i.e. the equivalent bending moment is given by

$$M_e = \frac{1}{2} [M + \sqrt{(M^2 + T^2)}] \quad \ldots (21)$$

and it will produce the same maximum direct stress as the combined bending and torsion effects.
Combined bending and torsion - equivalent torque

Again considering shafts subjected to the simultaneous application of a bending moment ($M$) and a torque ($T$) the maximum shear stress set up in the shaft may be determined by the application of an equivalent torque of value ($T_e$) acting alone. From the preceding section the principal stresses in the shaft are given by

\[
\sigma_1 = \frac{1}{2} \left( \frac{D}{2I} \right) [M + \sqrt{(M^2 + T^2)}] = \frac{1}{2} \left( \frac{D}{J} \right) [M + \sqrt{(M^2 + T^2)}] 
\]

\[
\sigma_2 = \frac{1}{2} \left( \frac{D}{2I} \right) [M - \sqrt{(M^2 + T^2)}] = \frac{1}{2} \left( \frac{D}{J} \right) [M - \sqrt{(M^2 + T^2)}] 
\]
Now the maximum shear stress is given by equation (12)
\[ \tau_{\text{max}} = \frac{1}{2} (\sigma_1 - \sigma_2) = \frac{1}{2} \left( \frac{D}{J} \right) \sqrt{M^2 + T^2} \]

But, from the torsion theory, the equivalent torque \( T_e \), will set up a maximum shear stress of

\[ \tau_{\text{max}} = \frac{T_e D}{2J} \]

Thus if these maximum shear stresses are to be equal,

\[ T_e = \sqrt{M^2 + T^2} \quad \ldots (22) \]
EXAMPLES

H.W.

PROBLEMS 1-2-3-4-5-6
BENDING
Simple bending theory

If a piece of rubber, most conveniently of rectangular cross-section, is bent between one’s fingers it is readily apparent that one surface of the rubber is stretched, i.e. put into tension, and the opposite surface is compressed. In order for this to be achieved it is necessary to make certain simplifying assumptions. The assumptions are as follows:

1. The beam is initially straight and unstressed.
2. The material of the beam is perfectly homogeneous and isotropic, i.e. of the same density and elastic properties throughout.
3. The elastic limit is nowhere exceeded.
4. Young's modulus for the material is the same in tension and compression.
5. Plane cross-sections remain plane before and after bending.
6. Every cross-section of the beam is symmetrical about the plane of bending, i.e. about an
7. There is no resultant force perpendicular to any cross-section.
If we now consider a beam initially unstressed and subjected to a constant (B.M.) along its length, i.e. pure bending, as would be obtained by applying equal couples at each end, it will bend to a radius (R) as shown in Figure (1b). As a result of this bending the top fibres of the beam will be subjected to tension and the bottom to compression. It is reasonable to suppose, therefore, that somewhere between the two there are points at which the stress is zero. The locus of all such points is termed the neutral axis (N.A.). The radius of curvature R is then measured to this axis. For symmetrical sections the N.A. is the axis of symmetry, but whatever the section the N.A. will always pass through the centre of area or centroid.

Consider now two cross-sections of a beam, HE and GF, originally parallel Figure (1a) When the beam is bent Figure (1b). it is assumed that these sections remain
plane; i.e. $H'E'$ and $G'F'$, the final positions of the sections, are still straight lines. They will then subtend some angle ($\theta$). Consider now some fibre $AB$ in the material, distance $y$ from the N.A. When the beam is bent this will stretch to $A'B'$.

Strain in fibre $AB = \frac{\text{extension}}{\text{original length}} = \frac{A'B' - AB}{AB}$

But $AB = CD$, and, since the N.A. is unstressed, $CD = C'D'$.

\[ \text{strain} = \frac{A'B' - C'D'}{C'D'} = \frac{(R + y)\theta - R\theta}{R\theta} = \frac{y}{R} \]

\[ \frac{\text{stress}}{\text{strain}} = \text{Young's modulus} \ E \]

\[ \text{strain} = \frac{\sigma}{E} \]

Equating the two equations for strain,

\[ \frac{\sigma}{E} = \frac{y}{R} \quad \text{or} \quad \frac{\sigma}{y} = \frac{E}{R} \]

\[ \ldots...(1) \]
Consider now a cross-section of the beam Figure (2) From equation (1) the stress on a fibre at distance \( y \) from the N.A. is

\[
\sigma = \frac{E}{R} y
\]

If the strip is of area \( \delta A \) the force on the strip is

\[
F = \sigma \delta A = \frac{E}{R} y \delta A
\]

This has a moment about the N.A. of

\[
Fy = \frac{E}{R} y^2 \delta A
\]
The total moment for the whole cross-section is therefore

\[ M = \sum \frac{E}{R} y^2 \delta A = \frac{E}{R} \sum y^2 \delta A \]

since \((E)\) and \((R)\) are assumed constant.

The term \(\sum y^2 \delta A\) is called the second moment of area of the cross-section and given the symbol \((I)\).

\[ \therefore M = \frac{E}{R} I \quad \text{and} \quad \frac{M}{I} = \frac{E}{R} \quad \ldots (2) \]

Combining eqns. (1) and (2) we have the bending theory equation

\[ \frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R} \quad \ldots (3) \]
Neutral axis

In bending, one surface of the beam is subjected to tension and the opposite surface to compression there must be a region within the beam cross-section at which the stress changes sign, i.e. where the stress is zero, and this is termed the neutral axis (N.A.). Further, equation (3) may be re-written in the form

\[ \sigma = \frac{M}{I} y \quad \text{----(4)} \]

which shows that at any section the stress is directly proportional to \( y \), the distance from the N.A., i.e. \((s)\) varies linearly with \((y)\), the maximum stress values occurring in the outside surface of the beam where \((y)\) is a maximum. The force on the small element of area is \((s.dA)\) acting perpendicular to the cross-section, i.e. parallel to the beam axis. The total force parallel to the beam axis is therefore \(\int \sigma dA\). The tensile force on one side of the N.A. must exactly balance the compressive force on the other side.
Substituting from equation (1)

\[ \int \frac{E}{R} y dA = 0 \quad \text{and hence} \quad \frac{E}{R} \int y dA = 0 \]

Typical stress distributions in bending are shown in Figure (4). In order to obtain the maximum resistance to bending it is advisable therefore to use sections which have large areas as far away from the N.A. as possible. For this reason beams with I- or T-sections find considerable favour in present engineering applications, such as girders, where bending plays a large part. Such beams have large moments of area about one axis and, provided that it is ensured that bending takes place about this axis, they will have a high resistance to bending stresses.
Section modulus

From equation (4) the maximum stress obtained in any cross-section is given by

\[ \sigma_{\text{max}} = \frac{M}{I y_{\text{max}}} \quad \text{...(5)} \]

For any given allowable stress the maximum moment which can be accepted by a particular shape of cross-section is therefore

\[ M = \frac{I}{y_{\text{max}}} \sigma_{\text{max}} \]

For ready comparison of the strength of various beam cross-sections this is sometimes written in the form

\[ M = Z \sigma_{\text{max}} \quad \text{...(6)} \]
where \( Z = \frac{I}{y_{\text{max}}} \) is termed the section modulus. The higher the value of \( Z \) for a particular cross-section the higher the B.M. which it can withstand for a given maximum stress. This is particularly important in the case of unsymmetrical sections such as T-sections where the values of \( y_{\text{max}} \) will also be different on each side of the N.A. Figure (4) and here two values of section modulus are often quoted,

\[
Z_1 = \frac{I}{y_1} \quad \text{and} \quad Z_2 = \frac{I}{y_2} \quad ...(7)
\]

each being then used with the appropriate value of allowable stress.
Second moment of area

Consider the rectangular beam cross-section shown in Figure (5) and an element of area (dA), thickness (dy), breadth (B) and distance (y) from the N.A. which by symmetry passes through the centre of the section. The second moment of area (I) has been defined earlier as

\[ I = \int y^2 \, dA \]
Thus for the rectangular section the second moment of area about the N.A., i.e. an axis through the centre, is given by

\[ I_{\text{N.A.}} = \int_{-D/2}^{D/2} y^2 B \, dy = B \int_{-D/2}^{D/2} y^2 \, dy \]

\[ = B \left[ \frac{y^3}{3} \right]_{-D/2}^{D/2} = \frac{BD^3}{12} \]  \hspace{1cm} \text{(8)}

Similarly, the second moment of area of the rectangular section about an axis through the lower edge of the section would be found using the same procedure but with integral limits of 0 to D.

\[ I = B \left[ \frac{y^3}{3} \right]_0^D = \frac{BD^3}{3} \]  \hspace{1cm} \text{(9)}
These standard forms prove very convenient in the determination of \( I_{NA} \) values for built-up sections which can be conveniently divided into rectangles. For symmetrical sections as, for instance, the I-section shown in Figure (6)

\[
I_{NA} = I \text{ of dotted rectangle} - I \text{ of shaded portions}
\]

\[
= \frac{BD^3}{12} - 2\left(\frac{bd^3}{12}\right) \quad \text{...(10)}
\]
It will be found that any symmetrical section can be divided into convenient rectangles with the N.A. running through each of their centroids and the above procedure can then be employed to effect a rapid solution. For unsymmetrical sections such as the T-section of Figure (7) it is more convenient to divide the section into rectangles with their edges in the N.A., when the second type of standard form may be applied.

\[ I_{\text{N.A.}} = I_{\text{ABCD}} - I_{\text{shaded areas}} + I_{\text{EFGH}} \]

(each of these quantities may be written in the form \( BD^3/3 \)).
As an alternative procedure it is possible to determine the second moment of area of each rectangle about an axis through its own centroid \((I_G = 8D^3/12)\) and to “shift” this value to the equivalent value about the N.A. by means of the parallel axis theorem.

\[
I_{NA} = I_G + Ah^2  \quad \ldots (11)
\]

Where; (A) is the area of the rectangle and (h) the distance of its centroid (G) from the N.A. Whilst this is perhaps not so quick or convenient for sections built-up from rectangles.
Bending of composite or flitched beams

(a) A composite beam is one which is constructed from a combination of materials. If such a beam is formed by rigidly bolting together two timber joists and reinforcing steel plate, then it is termed a flitched beam.

Since the bending theory only holds good when a constant value of Young’s modulus applies across a section it cannot be used directly to solve composite-beam problems where two different materials, and therefore different values of \( E \), are present. The method of solution in such a case is to replace one of the materials by an equivalent section of the other.
Consider, therefore, the beam shown in Figure (8) in which a steel plate is held centrally in an appropriate recess between two blocks of wood. Here it is convenient to replace the steel by an equivalent area of wood, retaining the same bending strength, i.e. the moment at any section must be the same in the equivalent section as in the original so that the force at any given (dy) in the equivalent beam must be equal to that at the strip it replaces.

\[ 
\sigma t \, dy = \sigma' t' \, dy \\
\sigma t = \sigma' t' \\
\varepsilon E t = \varepsilon' E' t' \\
\text{since} \quad \frac{\sigma}{\varepsilon} = E \\
\text{Again, for true similarity the strains must be equal,} \\
\therefore \quad \varepsilon = \varepsilon' \\
\therefore \quad E t = E' t' \quad \text{or} \quad \frac{t'}{t} = \frac{E}{E'} \\
i.e. \quad t' = \frac{E}{E'} \, t 
\]
Thus, to replace the steel strip by an equivalent wooden strip, the thickness must be multiplied by the modular ratio $E/E'$. The equivalent section is then one of the same material throughout and the simple bending theory applies. The stress in the wooden part of the original beam is found directly and that in the steel found from the value at the same point in the equivalent material as follows:

From eqn. (12) \[ \frac{\sigma}{\sigma'} = \frac{l'}{l} \]

And from eqn. (13) \[ \frac{\sigma}{\sigma'} = \frac{E}{E'} \quad \text{or} \quad \sigma = \frac{E}{E'} \sigma' \quad \text{......(15)} \]

Stress in steel = modular ratio x stress in equivalent wood
Strain energy in bending

For beams subjected to bending the total strain energy of the system is given by

\[ U = \int_0^L \frac{M^2 ds}{2EI} \]

For uniform beams, or parts of beams, subjected to a constant (B.M).

\[ U = \frac{M^2 L}{2EI} \]
EXAMPLES  1-2-3

H.W.

PROBLEMS 1-2-3-4-5-6-7-8
In practically all engineering applications limitations are placed upon the performance and behavior of components and normally they are expected to operate within certain set limits of for example, stress or deflection. The stress limits are normally set so that the component does not yield or fail under the most severe load conditions which it is likely to meet in service.
Consider a beam (AB) which is initially horizontal when unloaded. If this deflects to a new position (A'B') under load, the slope at any point C is

\[ i = \frac{dy}{dx} \]

Figure (1) Unloaded beam AB deflected to A'B' under load
This is usually very small in practice, and for small curvatures

\[ ds = dx = R di \]

\[ \therefore \frac{di}{dx} = \frac{1}{R} \]

But \[ i = \frac{dy}{dx} \]

\[ \therefore \frac{d^2 y}{dx^2} = \frac{1}{R} \] \hspace{1cm} ...... (1)

Now from the simple bending theory

\[ \frac{M}{I} = \frac{E}{R} \]

\[ \therefore \frac{1}{R} = \frac{M}{EI} \]

Therefore substituting in eqn. (1)

\[ M = EI \frac{d^2 y}{dx^2} \] \hspace{1cm} ...... (2)
This is the basic differential equation for the deflection of beams.

If the beam is now assumed to carry a distributed loading which varies in intensity over the length of the beam, then a small element of the beam of length ($dx$) will be subjected to the loading condition shown in Figure (2). The parts of the beam on either side of the element (EFGH) carry the externally applied forces, while reactions to these forces are shown on the element itself. Thus for vertical equilibrium of (EFGH),

$$Q - wdx = Q - dQ$$

$$dQ = wdx$$

Figure (2) Small element of beam subjected to non-uniform loading (effectively uniform over small length $dx$).
and integrating,

\[ Q = \int w \, dx \] 

..... (3)

Also, for equilibrium, moments about any point must be zero. Therefore taking moments about F,

\[(M + dM) + wdx \frac{dx}{2} = M + Q \, dx\]

Therefore neglecting the square of small quantities,

\[ dM = Q \, dx \]

\[ M = \int Q \, dx \]

deflection = y

slope = \( \frac{dy}{dx} \)

bending moment = \( EI \frac{d^2y}{dx^2} \)

shear force = \( EI \frac{d^3y}{dx^3} \)

load distribution = \( EI \frac{d^4y}{dx^4} \)
Figure (3) Sign conventions for load, S.F., B.M., slope and deflection.
If the value of the B.M. at any point on a beam is known in terms of $x$, the distance along the beam, and provided that the equation applies along the complete beam, then integration of eqn. (5.4a) will yield slopes and deflections at any point, i.e.

$$M = EI \frac{d^2y}{dx^2} \quad \text{and} \quad \frac{dy}{dx} = \int \frac{M}{EI} \, dx + A$$

$$y = \int \int \left( \frac{M}{EI} \, dx \right) \, dx + Ax + B$$

where $A$ and $B$ are constants of integration evaluated from known conditions of slope and deflection for particular values of $x$. 

(a) Cantilever with concentrated load at the end

\[ M_{xx} = EI \frac{d^2 y}{dx^2} = -Wx \]

\[ EI \frac{dy}{dx} = -\frac{WX^2}{2} + A \]

assuming \( EI \) is constant.

\[ EIy = -\frac{WX^3}{6} + Ax + B \]

Now when \( x = L, \frac{dy}{dx} = 0 \) \( \therefore A = \frac{WL^2}{2} \)

and when \( x = L, y = 0 \) \( \therefore B = \frac{WL^3}{6} - \frac{WL^2}{2} L = -\frac{WL^3}{3} \)

\[ y = \frac{1}{EI} \left[ -\frac{WX^3}{6} + \frac{WL^2 x}{2} - \frac{WL^3}{3} \right] \]

\[ \ldots (5) \]
This gives the deflection at all values of x and produces a maximum value at the tip of the cantilever when x = 0,

\[ \text{Maximum deflection} = y_{\text{max}} = -\frac{W.L^3}{3E.I} \]  

\[ \text{...... (6)} \]

The negative sign indicates that deflection is in the negative y direction, i.e. downwards.

\[ \frac{dy}{dx} = \frac{1}{EI} \left[ -\frac{Wx^2}{2} + \frac{WL^2}{2} \right] \]  

\[ \text{...... (7)} \]

and produces a maximum value again when x = 0.

\[ \text{Maximum slope} = \left( \frac{dy}{dx} \right)_{\text{max}} = \frac{WL^2}{2EI} \text{ (positive)} \]  

\[ \text{...... (8)} \]
This gives the deflection at all values of \( x \) and produces a maximum value at the tip of the cantilever when \( x = 0 \),

\[
\text{Maximum deflection} = y_{\text{max}} = -\frac{W L^3}{3EI}
\]  

\[\text{...... (6)}\]

The negative sign indicates that deflection is in the negative \( y \) direction, i.e. downwards.

\[
\frac{dy}{dx} = \frac{1}{EI} \left[ -\frac{W x^2}{2} + \frac{W L^2}{2} \right]
\]  

\[\text{...... (7)}\]

and produces a maximum value again when \( x = 0 \).

\[
\text{Maximum slope} = \left( \frac{dy}{dx} \right)_{\text{max}} = \frac{W L^2}{2EI} \quad \text{(positive)}
\]  

\[\text{...... (8)}\]
(b) Cantilever with uniformly distributed load

\[ M_{xx} = EI \frac{d^2y}{dx^2} = -\frac{wx^2}{2} \]

\[ EI \frac{dy}{dx} = -\frac{wx^3}{6} + A \]

\[ EIy = -\frac{wx^4}{24} + Ax + B \]

Again, when \( x = L \), \( \frac{dy}{dx} = 0 \) and \( A = \frac{wL^3}{6} \)
\[ x = L, \quad y = 0 \quad \text{and} \quad B = \frac{wL^4}{24} - \frac{wL^4}{6} = -\frac{wL^4}{8} \]

\[ y = \frac{1}{EI} \left[ -\frac{wx^4}{24} + \frac{wL^3x}{6} - \frac{wL^4}{8} \right] \quad \ldots \quad (9) \]

At \( x = 0 \), \n\[ y_{\text{max}} = -\frac{wL^4}{8EI} \quad \text{and} \quad \left( \frac{dy}{dx} \right)_{\text{max}} = \frac{wL^3}{6EI} \quad \ldots \quad (10) \]
(c) Simply-supported beam with uniformly distributed load

\[ M_{xx} = EI \frac{d^2 y}{dx^2} = \frac{wLx}{2} - \frac{wx^2}{2} \]

\[ EI \frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + A \]

\[ EI y = \frac{wLx^3}{12} - \frac{wx^4}{24} + Ax + B \]
At \( x = 0 \), \( y = 0 \) \quad \therefore \quad B = 0

At \( x = L \), \( y = 0 \) \quad \therefore \quad 0 = \frac{wL^4}{12} - \frac{wL^4}{24} + AL

\therefore \quad A = -\frac{wL^3}{24}

\therefore \quad y = \frac{1}{EI} \left[ \frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24} \right]

\ldots \quad (11)

In this case the maximum deflection will occur at the centre of the beam where \( x = L/2 \).

\[ y_{\text{max}} = \frac{1}{EI} \left[ \frac{wL}{12} \left( \frac{L^3}{8} \right) - \frac{w}{24} \left( \frac{L^4}{16} \right) - \frac{wL^3}{24} \left( \frac{L}{2} \right) \right] \]

\[ = -\frac{5wL^4}{384EI} \quad \ldots \quad (12) \]

\[ \left( \frac{dy}{dx} \right)_{\text{max}} = \frac{wL^3}{24EI} \quad \text{at the ends of the beam.} \quad \ldots \quad (13) \]
In order to obtain a single expression for B.M. which will apply across the complete beam in this case it is convenient to take the origin for x at the centre, then:

\[ M_{xx} = EI \frac{d^2y}{dx^2} = \frac{W}{2} \left( \frac{L}{2} - x \right) = \frac{WL}{4} - \frac{WX}{2} \]

\[ EI \frac{dy}{dx} = \frac{WL}{4} x - \frac{WX^2}{4} + A \]

\[ EIy = \frac{WLx^2}{8} - \frac{WX^3}{12} + Ax + B \]
At \( x = 0, \frac{dy}{dx} = 0 \quad \therefore \quad A = 0 \)

\[ x = \frac{L}{2}, \quad y = 0 \quad \therefore \quad 0 = \frac{WL^3}{32} - \frac{WL^3}{96} + B \]

\[ \therefore \quad B = -\frac{WL^3}{48} \]

\[ y = \frac{1}{EI} \left[ \frac{WLx^2}{8} - \frac{Wx^3}{12} - \frac{WL^3}{48} \right] \]

\[ y_{\text{max}} = -\frac{WL^3}{48EI} \quad \text{at the centre} \quad \ldots \quad (14) \]

\[ \left( \frac{dy}{dx} \right)_{\text{max}} = \pm \frac{WL^2}{16EI} \quad \text{at the ends} \quad \ldots \quad (16) \]
(e) Cantilever subjected to non-uniform distributed load

The loading at section $XX$ is

$$w' = EI \frac{d^4y}{dx^4} = - \left[ w + (3w-w) \frac{x}{L} \right] = -w \left( 1 + \frac{2x}{L} \right)$$

$$EI \frac{d^3y}{dx^3} = -w \left( x + \frac{x^2}{L} \right) + A \tag{1}$$

$$EI \frac{d^2y}{dx^2} = -w \left( \frac{x^2}{2} + \frac{x^3}{3L} \right) + Ax + B \tag{2}$$
\[
EI \frac{dy}{dx} = -w \left( \frac{x^3}{6} + \frac{x^4}{12L} \right) + \frac{Ax^2}{2} + Bx + C 
\]

(3)

\[
EIy = -w \left( \frac{x^4}{24} + \frac{x^5}{60L} \right) + \frac{Ax^3}{6} + \frac{Bx^2}{2} + Cx + D
\]

(4)

Thus, before the slope or deflection can be evaluated, four constants have to be determined; therefore four conditions are required. They are:

At \( x = 0 \), S.F. is zero

from (1); \( A = 0 \)

At \( x = 0 \), B.M. is zero

from (2); \( B = 0 \)

At \( x = L \), slope \( dy/dx = 0 \) (slope normally assumed zero at a built-in support)

from (3)

\[
0 = -w \left( \frac{L^3}{6} + \frac{L^3}{12} \right) + C
\]

\[
C = \frac{wL^3}{4}
\]

At \( x = L \), \( y = 0 \)
from (4) \[ 0 = -w \left( \frac{L^4}{24} + \frac{L^4}{60} \right) + \frac{wL^4}{4} + D \]

\[ D = -\frac{23wL^4}{120} \]

\[ EIy = -\frac{wx^4}{24} - \frac{wx^5}{60L} + \frac{wL^3x}{4} - \frac{23wL^4}{120} \]

Then, for example, the deflection at the tip of the cantilever, where \( x = 0 \), is

\[ y = -\frac{23wL^4}{120EI} \]
Fatigue

Fracture of components due to fatigue is the most common cause of service failure, particularly in shafts, axles, aircraft wings, etc., where cyclic stressing is taking place. With static loading of a ductile material, plastic flow precedes final fracture, the specimen necks and the fractured surface reveals a fibrous structure, but with fatigue, the crack is initiated from points of high stress concentration on the surface of the component such as sharp changes in cross-section, slag inclusions, tool marks, etc., and then spreads or propagates under the influence of the load cycles until it reaches a critical size when fast fracture of the remaining cross-section takes place.

The Stress to number of cycles (S/N) curve

Fatigue tests are usually carried out under conditions of rotating - bending and with a zero mean stress as obtained by means of a Wohler machine. From Figure (1), it can be seen that the top surface of the specimen, held “cantilever fashion” in the machine, is in tension, whilst the bottom surface is in compression. As the specimen rotates, the top surface moves to the bottom and hence each segment of the surface moves continuously from tension to compression producing a stress-cycle curve as shown in Figure (2). In order to understand certain terms in common usage, let us consider a stress-cycle curve where there is a positive tensile mean stress as may be obtained using other types of fatigue machines such as a Haigh “push-pull” machine.
Figure (1) Single point load arrangement in a Wohler machine for zero mean stress fatigue testing

Figure (2) Simple sinusoidal (zero mean) stress fatigue curve, “reversed-symmetrical”
The stress-cycle curve is shown in Figure (3), and from this diagram it can be seen that:

Stress range, $\sigma_r = 2\sigma_a$. \hspace{1cm} (1)

Mean stress, $\sigma_m = \frac{\sigma_{\text{max}} + \sigma_{\text{min}}}{2}$. \hspace{1cm} (2)

Alternating stress amplitude, $\sigma_a = \frac{\sigma_{\text{max}} - \sigma_{\text{min}}}{2}$. \hspace{1cm} (3)
If the mean stress is not zero, we sometimes make use of the "stress ratio" $R$, where

$$R = \frac{\sigma_{\text{min}}}{\sigma_{\text{max}}}$$

(4)

The most general method of presenting the results of a fatigue test is to plot a graph of the stress amplitude as ordinate against the corresponding number of cycles to failure.

Figure (4) Typical S/N curve fatigue life curve