Eigenvalue problems

• Eigenvalues have their greatest importance in *dynamic problems*.

• *The solution of* $\frac{dy}{dt} = Ay$ *is changing with time* growing or decaying or oscillating. We can’t find it by elimination.

• Eigenvalue problems occur in many areas of science and engineering, such as structural analysis.

• Eigenvalues are also important in analyzing numerical methods.
Eigenvalue and Eigenvectors

- Let $A = [a_{ij}]$ be a given $n \times n$ matrix and consider the vector Eq.

$$Ax = \lambda x \quad \text{...............}(1)$$

Here, $x$ is an unknown vector and $\lambda$ an unknown scalar, and we want to determine both.

Clearly, the zero vector $x = 0$ is a solution of Eq. (1) for any value of $\lambda$. This is of no practical interest.

A value of $\lambda$ for which Eq. (1) has a solution $x \neq 0$ is called an eigenvalue or characteristic value of the matrix $A$.

The corresponding solutions $x \neq 0$ of Eq.(1) are called eigenvectors or characteristic vectors.
Eq. (1) can be written in the following form

$$Ax - \lambda x = 0 \quad \text{or} \quad (A - \lambda I)x = 0$$

The system of linear equations has a nontrivial solution if and only if

$$\det(A - \lambda I) = 0$$

The expansion of $\det(A - \lambda I) = 0$ gives a polynomial of degree $n$ in $\lambda$. 
**Eigenvectors**

To each distinct eigenvalue of a matrix $A$ there will correspond at least one eigenvector which can be found by solving the appropriate set of homogenous equations. If $\lambda_i$ is an eigenvalue then the corresponding eigenvector $x_i$ is the solution of $(A - \lambda_i I)x_i = 0$

**Summary**

To solve the eigenvalue problem for an $n$ by $n$ matrix, follow these steps:

1. **Compute the determinant of $A - \lambda I$.** With $\lambda$ subtracted along the diagonal, this determinant starts with $\lambda^n$ or $-\lambda^n$. It is a polynomial in $\lambda$ of degree $n$.

2. **Find the roots of this polynomial.** by solving $\det(A - \lambda I) = 0$. The $n$ roots are the $n$ eigenvalues of $A$. They make $A - \lambda I$ singular.

3. For each eigenvalue $\lambda$, solve $(A - \lambda I)x = 0$ to find an eigenvector $x$. \
Example:- Find the eigenvalues and eigenvectors of

\[ A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \]

\[ A = \begin{bmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{bmatrix} \]