Periodic Functions and Fourier Series

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• What is Fourier Series?
  – Representation of a periodic function with a weighted, infinite sum of sinusoids.

• Why Fourier Series?
  – Any arbitrary periodic signal, can be approximated by using some of the computed weights
  – These weights are generally easier to manipulate and analyze than the original signal

• What is a periodic Function?
  – A function which remains unchanged when time-shifted by one period
    
    \[ f(t) = f(t + T) \quad \text{or} \quad f(t) = f(t + 2p) \]

    Where \( T \) is the period of periodic function (\( T = 2p \))
Fourier Series

If \( f(t) \) is a periodic function with period \( 2p \),

The function can be represented by a trigonometric series as:

\[
f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n \pi t}{p} \right) + \sum_{n=1}^{\infty} b_n \sin \left( \frac{n \pi t}{p} \right)
\]  

(1)
We want to determine the coefficients, $a_n$ and $b_n$.

Let us first remember some useful integrations.

\[
\int_{-\pi}^{\pi} \cos n\theta \cos m\theta \, d\theta = 0 \quad n \neq m
\]

\[
\int_{-\pi}^{\pi} \cos n\theta \cos m\theta \, d\theta = \pi \quad n = m
\]
\begin{align*}
\int_{-\pi}^{\pi} \sin n\theta \cos m\theta \, d\theta &= 0 \\
\text{for all values of } m. \\
\int_{-\pi}^{\pi} \sin n\theta \sin m\theta \, d\theta &= 0 \quad n \neq m \\
\int_{-\pi}^{\pi} \sin n\theta \sin m\theta \, d\theta &= \pi \quad n = m
\end{align*}
Determination of $a_0$

Integrate both sides of Eq. (1)

$$\int_{d}^{d+2p} f(t)dt = \int_{d}^{d+2p} \left[ \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{p} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{p} \right]dt$$

$$\int_{d}^{d+2p} f(t)dt = \frac{1}{2} \int_{d}^{d+2p} a_0 dt + 0 + 0$$

$$a_0 = \frac{1}{p} \int_{d}^{d+2p} f(t)dt$$
Determine $a_n$

Multiply Eq. (1) by $\cos \frac{n \pi t}{p}$ and then integrate both sides from $d$ to $d+2p$

$$\int_{d}^{d+2p} f(t) \cos \frac{n \pi t}{p} \, dt$$

$$= \int_{d}^{d+2p} \left[ \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n \pi t}{p} + \sum_{n=1}^{\infty} b_n \sin \frac{n \pi t}{p} \right] \cos \frac{n \pi t}{p} \, dt$$

$$a_n = \frac{1}{p} \int_{d}^{d+2p} f(t) \cos \frac{n \pi t}{p} \, dt$$
Determine $b_n$

Multiply Eq.(1) by $\sin \frac{n\pi t}{p}$

and then Integrate both sides from $d$ to $d+2p$

\[
\int_d^{d+2p} f(t) \sin \frac{n\pi t}{p} \, dt
\]

\[
= \int_d^{d+2p} \left[ \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{p} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{p} \right] \sin \frac{n\pi t}{p} \, dt
\]

\[
b_n = \frac{1}{p} \int_d^{d+2p} f(t) \sin \frac{n\pi t}{p} \, dt
\]