A new Harmonic Elimination (HE) PWM method with fast recursive algorithm is used that provide the exact on-line solution to the optimal PWM problem. The proposed algorithm optimization technique is applied to a 3-level unipolar single-phase inverter to determine optimum switching angles for eliminating low order harmonics while maintaining the required fundamental voltage to drive single-phase induction motor with high quality.
The proposed HE method contributes to the existing methods because it not only generates the desired fundamental frequency voltage, but also completely eliminates any number of harmonics. It provides high quality sine-wave output voltage on the induction motor terminals with very low THD.

The high quality sinusoidal output voltage produced by the inverter at different number of switching angles is presented. The complete solutions for 3-level unipolar switching patterns to eliminate the 3rd and 5th harmonics are given. Finally, the unipolar case is again considered where the first 14 harmonics are eliminated.

Keywords: Harmonic Elimination, SHEPWM, THD.
Introduction

Harmonic elimination (HE) control has been a widely researched alternative to traditional PWM techniques. The Selective Harmonic Elimination (SHE) method, which is widely used for efficient inverter control, forms a basis of off-line digital PWM modulation techniques in the power electronics field [1].

A Unipolar PWM waveform consists of a series of positive and negative pulses of constant amplitude but with variable switching instances as depicted in Fig. 1 (as in a power electronic PWM full-bridge inverter). A typical goal is to generate a train of pulses such that the fundamental component of the resulting waveform has a specified frequency and amplitude (e.g., for a constant \( V/f \) speed control of an induction motor).

Some of the proposed methods for PWM waveform design are: modulating-function techniques, space-vector techniques, and feedback methods. These methods suffer, however, from high residual harmonics that are difficult to control and from limitations in their applicability.

PWM signals are used in power electronics, motor control and solid-state electric energy conversion. The best voltage signal for these purposes is one with a periodic time variation in which amplitudes of selected non-fundamental components of the signal have been controlled to increase efficiency and reduce damaging vibrations.

A sinusoidal PWM inverter, which is a DC-AC power inverter, is used for a wide variety of applications because of its flexibility in driving frequency and voltage in the power electronic field. If the switching frequency is not high, but the control accuracy is good, off-line PWM control is efficient because it optimizes PWM waveforms for harmonic elimination and total distortion.

The Selective Harmonic Elimination (SHE) technique (or Optimal HE technique) forms a basis of the harmonic reduction techniques. This technique consists of synthesizing a PWM waveform by setting its pulse-pattern properly to eliminate selected orders of harmonics. This is accomplished by solving HE equations which are nonlinear [1]. This technique theoretically offers the highest quality of the output waveform [2].

The optimal HEPWM offers several advantages compared to traditional modulation methods including acceptable performance with low switching frequency to fundamental frequency ratios, direct control over output waveform harmonics, and the ability to leave triplen harmonics uncontrolled in three-phase systems. These key advantages make the optimal HEPWM a viable alternative to other methods of modulation in applications such as ground power units, variable speed drives, or dual-frequency induction heating [3].

A. Optimal HEPWM

At a high power level, to limit switching losses, power switches (e.g., GTO’s) can only be switched at low frequencies (typically several hundred hertz). This implies that only a few switching actions may take place within each fundamental period, as far as the fundamental frequency is not very low.

In this case, optimizing the waveform based on specifying an optimal value for each switching instant is necessary for achieving the best modulation result. The high power and cost of the whole system also justify the use of such optimized modulation methods that, in principle, require more advanced (thus more expensive) implementation hardware and software than that of simple carrier-based PWM. The benefits of optimization are also more remarkable, considering the total power of the system. As the total energy of harmonics contained in a PWM waveform is constant that depends only on the fundamental amplitude, regardless of the actual waveform structure, optimizing the waveform implies not
eliminating or reducing the total harmonic energy, but altering its distribution among different frequency components. Considering that low-order harmonics are usually considered to be more harmful than high-order ones, thus they need to be controlled at smaller magnitudes. Hence, roughly speaking, the objective of optimal PWM is to push most harmonic energy into high-frequency regions such that low-frequency harmonics are well attenuated [4].

B. Harmonic Elimination (HE)

One frequently studied optimal PWM method is the HE technique, which aims at the complete elimination of some low-order harmonics [2,5,6]. The underlying principle of HE is that the fundamental and harmonic amplitudes of a symmetrical PWM waveform are nonlinear functions of the \( n \) switching angles in the first quarter fundamental period. Setting of the fundamental amplitude to a pre-specified value and other \( n-1 \) low-order harmonics to zero, results in a system of \( n \) nonlinear equations. The desired optimal pulse patterns can thus be determined by solving these equations. Two major advantages of applying this technique are:

1) If the inverter is used to supply AC power of constant frequency to general AC loads, a filter is usually installed at its output. In this case, when low-order harmonics are eliminated through the modulation of the inverter, only high-order harmonics will appear at the output and need to be attenuated by the filter. The cut-off frequency of the filter can thus be increased, leading to a significant reduction of the filter size and cost. System efficiency also tends to increase.

2) When used in an AC drive system, eliminating the low-order harmonic voltages leads to great reduction of low-order harmonic torques generated by the motor. Although harmonic torque is the results of interacting between stator and rotor harmonic currents of different orders, higher-order harmonic currents have smaller magnitudes due to the larger impedance that the motor presents to higher-order harmonic voltages. Their contributions to lower-order harmonic torques are thus less significant. Lower-order harmonic torques generated by the motor are thus greatly reduced [4].

The problem of eliminating harmonics in switching inverters has been the focus of research for many years. If the switching losses in an inverter are not a concern (i.e., switching on the order of a few kHz is acceptable), then the sine-triangle PWM method and its variants are very effective for controlling the inverter [7]. On the other hand, for systems where high switching efficiency is of utmost importance, it is desirable to keep the switching frequency much lower. In this case, another approach is to choose the switching times (angles) such that a desired fundamental output is generated and specifically chosen harmonics of the fundamental are suppressed [2,4-6]. This is referred to as Selective Harmonic Elimination (SHE) or Programmed Harmonic Elimination (PHE) as the switching angles are chosen (programmed) to eliminate specific harmonics.

Specifically, in [2,5,6] the HE problem was formulated as a set of transcendental equations that must be solved to determine the times (angles) in an electrical cycle for turning the switches on and off in a full bridge inverter so as to produce a desired fundamental amplitude while eliminating, for example, the 3\(^{rd}\) and 5\(^{th}\) harmonics. These transcendental equations are then solved using iterative numerical techniques to compute the switching angles [8]. The Walsh function method [9] also had been proposed to simplify the process. Recently, on-line computation methods have been proposed to make the technique a more flexible and interactive one, by using
Genetic algorithms [10] and a DSP [4]. The complete solution to the HE problem can be found using the theory of Resultants from Elimination theory. The solution is complete in the sense that any and all solutions were found [11].

Until now, the transcendental equations characterizing the harmonic content have been converted into polynomial equations, and elimination theory (using resultants) has been employed to determine the switching angles to eliminate specified harmonics, such as 3rd, 5th, ..., 13th for 3-level unipolar inverters [11]. However, as the number of eliminated harmonics increases, the degree of the polynomials in these equations are large and one reaches the limitations of the capability of contemporary computer algebra software tools (e.g., Mathematica or Maple) to solve the system of polynomial equations by using elimination theory.

To conquer this problem, the switching angles computation with general number \( n \) of the 3-level unipolar switching scheme is solved by using the proposed fast recursive on-line algorithm.

The present work would be to extend the SHEPWM switching scheme in [11] to include more than 7-switching angles per quarter cycle \( (n > 7) \) or eliminating harmonics more than six.

The problem of the optimal design of PWM waveforms for single-phase inverters [2,5] is examined in this paper.

**Optimized PWM Switching Angles**

The quarter-wave symmetry assumption in Fig. 1, guarantees that the even harmonics will be zero and that all harmonics will be either in phase or anti-phase with the fundamental. Only the odd harmonics exist [3]. Assuming that the PWM waveform is chopped times per half a cycle, the Fourier coefficients of odd harmonics are given by:

\[
a_k = \frac{4V_{dc}}{k\pi} \sum_{n=1}^{\infty} (-1)^{(n-1)} \cos(k\alpha_n)
\]

where \( k = 1, 3, 5, ... \), \( V_{dc} \) is the amplitude of the square wave and \( \alpha_n \) are the optimized switching angles. Amplitudes of any harmonics can be set by solving a system of nonlinear equations obtained from setting (1) equal to pre-specified values.

In the optimal HEPWM method, the fundamental component is set to required amplitude and \( n-1 \) low-order harmonics are set to zero. This is the most common approach in electric drives since low-order harmonics are the most detrimental to motor performance. In other applications, like active harmonic filters or control of electromechanical systems, harmonics are set to nonzero values. This task of designing a PWM waveform, the first \( n \) Fourier series coefficients of which match those of a desired waveform has been the subject of many papers [2,4,5,9]. Often, the Newton iteration method [8] is used to solve the system of nonlinear Eqs. 1. Those methods are computationally intensive for on-line calculations and the storage of off-line calculations leads to high memory requirements. Another approach is to simplify the nonlinear HE equations in order to obtain real-time approximate solutions using modern DSPs [4].

The current paper uses the recursive algorithm [12] with some developments and modifications to reduce the computational complexity for online calculation for solving the PWM harmonic elimination problem without any approximations in the problem statement. Since many PWM applications allow for a computational time frame of a few milliseconds, the developed algorithm will allow for real-time generation of switching patterns with high order.

The optimized unipolar waveform shown in Fig. 1 is assumed to be the quarter-wave symmetric. The Fourier series of the general quarter-wave symmetric H-bridge inverter output waveform is written as follows:
where \( \alpha_n \) is the optimized switching angles, which must satisfy the following condition:

\( a_1 < a_2 < \ldots < a_n < \pi/2 \).

The optimal PWM problem, as it is considered here, is the design of a PWM waveform \( u(t) \) so that its first Fourier coefficients \( h_k \) are equal to prescribed values (1). Therefore, the optimal PWM problem gives rise to the following design equations [2]:

\[
\begin{align*}
\cos \alpha_1 - \cos \alpha_2 + \cos \alpha_3 - \ldots - \cos \alpha_n &= h_1 \\
\cos 3 \alpha_1 - \cos 3 \alpha_2 + \cos 3 \alpha_3 - \ldots - \cos 3 \alpha_n &= h_3 \\
\text{M} &
\end{align*}
\]

(2)

\[
\begin{align*}
\cos (2n-1) \alpha_1 - \cos (2n-1) \alpha_2 + \cos (2n-1) \alpha_3 - \ldots - \cos (2n-1) \alpha_n &= h_{2n-1} \\
\text{M} &
\end{align*}
\]

(3)

Given the \( n \) values \( h_k = k \pi a_k / 4 V_{dc} \), we have \( n \) equations and \( n \) unknowns; we would like to find the \( n \) unknowns \( \{ a_1, a_2, \ldots, a_n \} \), with \( 0 < a_1 < a_2 < \ldots < a_n < \pi/2 \). The symmetry with respect to \( \alpha_i \) holds for odd \( h_k \), for even \( i \), and \( \alpha_i = \pi - \alpha_i \), for odd \( i \), and \( \alpha_i = \pi - \alpha_i \), for even \( i \). Due to the symmetry with respect to \( x_n \), any permutation of a solution set \( \{ x_i \} \) is also a solution set; likewise for \( \beta_i \). Let it be necessary to order \( \beta_i \) appropriately such that \( 0 < a_1 < a_2 < \ldots < a_n < \pi/2 \). Note that for odd \( i \), \( 0 < a_i < \pi/2 \), gives \( 0 < \beta_i < \pi/2 \). For even \( i \), \( 0 < a_i < \pi/2 \), gives \( 0 < \pi - \beta_i < \pi/2 \) or \( \pi/2 < \beta_i < \pi \). This indicates how to obtain \( a_i \) with the desired ordering from \( \beta_i \) for those values of

\[
\begin{align*}
T_i(x_1) + T_i(x_2) + \ldots + T_i(x_n) &= h_i \\
\text{M} &
\end{align*}
\]

(5)

As the odd-indexed Chebyshev polynomials are odd polynomials, the PWM equations can be writing:

\[
\begin{align*}
\sum_{j=1}^{n} T_{2k-1}(x) = \sum_{j=1}^{n} \sum_{j=1}^{k} c_{k,j} \cdot x^{2j-1} = h_{2k-1} (1 \leq k \leq n) \\
\text{or} &
\end{align*}
\]

(6)

where \( s_j = \sum_{i=1}^{n} x_i^j \) are the sums of powers of \( \{ x_i \} \). Eq. 6 forms a set of \( n \) linear equations for \( s_{2j-1}, 1 \leq j \leq n \). Once the values \( s_{2j-1} \) are obtained by solving the linear system (6), one has the following problem. Given \( \{ s_1, s_2, \ldots, s_{2n-1} \} \), find the solution \( \{ x_1, x_2, \ldots, x_n \} \) to the following system of nonlinear equations:

\[
\begin{align*}
x_1 + x_2 + \ldots + x_n &= s_1 \\
x_1^3 + x_2^3 + \ldots + x_n^3 &= s_3 \\
\text{M} &
\end{align*}
\]

(7)

Once \( x_i \) are obtained, the original variables \( a_i \) can be found by letting \( \beta_i = \arccos x_i \), \( a_i = \beta_i \), for odd \( i \), and \( a_i = \pi - \beta_i \), for even \( i \). Due to the symmetry with respect to \( x_n \), any permutation of a solution set \( \{ x_i \} \) is also a solution set; likewise for \( \beta_i \). Let it be necessary to order \( \beta_i \) appropriately such that \( 0 < a_1 < a_2 < \ldots < a_n < \pi/2 \). Note that for odd \( i \), \( 0 < a_i < \pi/2 \), gives \( 0 < \beta_i < \pi/2 \). For even \( i \), \( 0 < a_i < \pi/2 \), gives \( 0 < \pi - \beta_i < \pi/2 \) or \( \pi/2 < \beta_i < \pi \). This indicates how to obtain \( a_i \) with the desired ordering from \( \beta_i \) for those values of

\[
\begin{align*}
\sum_{k=1}^{n} T_k(x) &= \sum_{k=1}^{n} \sum_{j=0}^{k} c_{k,j} \cdot x^j = h_k \\
\text{M} &
\end{align*}
\]
is the monic polynomial

However, the design Eqs. 7 are nonlinear, so obtaining the desired solution \( \{x_i\} \) is not so straightforward. In the following sections, this nonlinear system of equations will be closely examined and in Section IV, a systematic procedure is given to obtain the solutions.

For the HEPWM problem, the Fourier coefficients of the PWM waveform \( u(ot) \) should match the Fourier coefficients of a pure sine wave. That is, the values \( h_{2i-1} \) appearing in (3) are given by \( h_{1=m} \), and \( h_{2i-1} = 0 \) for \( 2 \leq i \leq n \). For this case, the values \( s_{2i-1} \) depend on \( n \) only and are given by:

\[
s_{2i-1} = \frac{m}{4^i} \left( \frac{2i-1}{i-1} \right) \quad (1 \leq i \leq n) \quad \text{... (8)}
\]

**Solving the Optimal PWM Problem**

To solve the optimal PWM problem, we will first write the polynomial \( P(x) \) having roots \( \{x_1, x_2, \ldots, x_n\} \) as:

\[
P(x) = \prod_{i=1}^{n} (x - x_i)
\]

then the logarithmic derivative is given by:

\[
\frac{P'(x)}{P(x)} = \sum_{i=1}^{n} \frac{1}{x - x_i}
\]

Expanding each term in the sum, one gets:

\[
\frac{P'(x)}{P(x)} = \sum_{i=1}^{n} \sum_{j=0}^{\infty} \frac{x_j}{x^{j+i}} = \sum_{m=0}^{\infty} \frac{s_j}{x^{j+1}} \quad \text{... (9)}
\]

where \( s_j \) are the sums of the root powers and \( s_0 = n \). Integrating (9) gives:

\[
\ln P(x) = n \ln x - \sum_{j=1}^{m} \frac{s_j}{jx^j}
\]

Raising \( e \) to the power of \( \ln P(x) \) and using the last equation gives:

\[
P(x) = x^n \exp \left( -\sum_{j=1}^{m} \frac{s_j}{jx^j} \right)
\]

\[\text{......(10)}\]

In order to generalize procedure, we will obtain an expression similar to (10), but having only odd \( s_j \). To this end, note that:

\[
P(-x) = (-1)^x x^n \exp \left( -\sum_{j=1}^{m} \frac{s_j}{jx^j} \right)
\]

therefore:

\[
\frac{P(x)}{P(-x)} = (-1)^x \exp \left( -\sum_{j=1}^{m} \frac{s_j}{jx^j} \frac{(-1)^j}{j} \right)
\]

or

\[
\frac{P(x)}{P(-x)} = (-1)^x \exp \left( -2 \sum_{j=1, \text{odd}}^{m} \frac{s_j}{jx^j} \right)
\]

then:

\[
P(x) = (-1)^x P(-x) G(1/x)
\]

where

\[
G(x) := e^{V(x)}
\]

\[
V(x) := -2 \left( s_3 x^3 + \frac{s_5}{3} x^5 + \frac{s_7}{5} x^7 + \ldots \right)
\]

let

\[
\tilde{P}(x) = (-1)^x P(-x) \quad \text{...... (11)}
\]

then

\[
P(x) = \tilde{P}(x) G(1/x) \quad \text{...... (12)}
\]

where \( \tilde{P}(x) \) is the monic polynomial related to \( P(x) \) by negating the roots of \( P(x) \).

Eq. (12) is the counter-part to (10). Likewise, by setting like powers of \( x \) equal, we can obtain equations that relate \( p_k \) and \( s_i \), where \( p_k \) is the polynomial coefficients of \( P(x) \). However, in order to do this, we need to expand \( G(x) = e^{V(x)} \) into a power series of
x. Such a power series for \( e^{V(x)} \) can be obtained using the following algorithm. Let:

\[
V(x) = \sum_{j=0}^{\infty} v_j x^j
\]

and

\[
G(x) = e^{V(x)} = \sum_{j=0}^{\infty} g_j x^j
\]

If \( v_j \) for \( 0 \leq i \leq j \) are known, then the values of \( g_i \) for \( 0 \leq i \leq j \) are given by:

\[
g_0 = e^{v_0}
\]

\[
g_i = \frac{1}{i} \sum_{k=1}^{i} k v_k g_{i-k} \quad (1 \leq i \leq j)
\]

When the first \( n \) odd values of \( s_i \) are known, then \( v \) are known for \( 0 \leq i \leq 2n \). \( v_{2i} = 0 \) for \( 0 \leq i \leq n \) and \( v_{2i+1} = -2s_{2i+1} (2i-1) \) for \( 1 \leq i \leq n \). Therefore, using the relations (13) and (14), we obtain \( g_i \) for \( 0 \leq i \leq 2n \), and consequently, we can write out Eq. (12), matching like powers of \( x \), to obtain linear equations from which \( p_k \) can be obtained. For example, we write out the expressions for \( n = 3 \). That is, we are given \( s_1 \), \( s_2 \), and \( s_3 \), and our goal is to find the corresponding monic 3rd degree polynomial \( P(x) \), \( P(x) = x^3 + p_1 x^2 + p_2 x + p_3 \).

A Recurrence Algorithm for \( P(x) \)

The notation \( P_n(x) \) will be used to emphasize the dependence of \( P(x) \) on \( n \). Specifically, \( P_n(x) \) denotes the monic degree-\( n \) polynomial associated with the HE problem, with coefficients \( p_{n,k} \):

\[
P(x) = x^n + p_{n,1} x^{n-1} + \ldots + p_{n,n}
\]

With this notation, a recurrence relation:

\[
P_{n+1}(x) = x P_n(x) + C_n P_{n-1}(x) \quad \ldots (15)
\]

can be used to compute \( P_n(x) \). For the HE problem, the initial conditions can be taken to be \( P_0(x) = 1 \) and \( P_1(x) = x - m \). The coefficients \( C_n \) in the recursion can be computed using the following formula:

\[
C_n = -\sum_{i=0}^{n} (-1)^i g_{2n+1-k} P_{n,k}
\]

\[
\sum_{k=0}^{n} (-1)^k g_{2n-1-k} P_{n-1,k}
\]

The coefficients \( p_{n,k} \) are then determined recursively as [this implements (15)]:

\[
p_{n+1,k} = p_{n,k}, \quad k = 1
\]

\[
p_{n+1,k} = p_{n,k} + C_n \cdot p_{n-1,k-2}, \quad k = 2, \ldots, n
\]

\[
p_{n+1,k} = C_n \cdot p_{n-1,k-2} \quad k = n + 1
\]

The Recursive algorithm for computing can be summarized as follows:

Given \( m \) and \( n \), the polynomials \( P_n(x) \) for are recursively computed as follows:

1) Set \( g_0 = 1 \) and for \( k = 1 \) to \( 2n \), find \( g_i \) from Eq. 14.

2) Set \( P_0(x) = 1 \) and \( P_1(x) = x - m \) and for \( k = 1 \) to \( n+1 \) let:

\[
C_k = -\sum_{i=0}^{k} (-1)^i g_{2k+1-i} P_{k,i}
\]

\[
\sum_{i=0}^{k} (-1)^i g_{2k-1-i} P_{k-1,i}
\]

\[
P_{k+1}(x) = x P_k(x) + C_k P_{k-1}(x)
\]

Find the roots \( x_i \) of \( P_n(x) \).

3) Set \( \beta_i = \arccos x_i \), \( i = 1, \ldots, n \), with \( \beta_i \in (0, \pi) \). For \( \beta_i \in (0, \pi) \), set \( a_i = \beta_i \). For \( \beta_i \in (\pi/2, \pi) \), set \( a_i = \pi - \beta_i \). Sort the angles \( a_i \).

Fig. 2 illustrates the flowchart of the recursive algorithm for on-line calculation of the optimal switching angles. Using a computer algebra system, such as Maple or Mathematica, this recursive algorithm allows one to obtain \( P_n(x) \) as an explicit function of \( m \).

Simulation Results

The computer software package Maple was used to perform all of the above calculations as a first part. The second part of the theoretical calculations involved
organizing and analyzing all of the collected optimized switching angles (α). For this purpose, the software package MATLAB was utilized. Using MATLAB, the collected switching angles were organized into look-up tables to be used later in simulations [See Table 1]. Also, MATLAB was used to generate plots of α and Total Harmonic Distortion THD versus m. The THD mathematically calculated by:

$$\text{THD} = \sqrt{\frac{1}{m} \sum_{k=2}^{m} h_k^2}$$  \hspace{1cm} (18)

The computation was done as m increased between (0 and 1) resulting in α versus m; seem to be almost as straight lines as shown in Fig. 3.

Fig. 3 represents the exact solution of the optimized Unipolar HEPWM switching angles with the variation of the number of switching angles (n). We can show that increasing n causes decreasing the m range. The m range for example decreases approximately by 67% when n = 15 with respect to n = 3. It can seen from the figure that the solution is not continuous for some values of m. Note that not all the range of m has a solution, for example, in the case of 3-switching scheme (n = 3), there are solutions in the interval m ∈ [0,0.83]. On the other hand, for m ∈ [0.83,1], there are no solutions that solve the Eqs. 1. Also it can be seen that for 15-switching scheme (n = 15), there are solutions to the Eqs. 1 in the interval m ∈ [0.72,0.78] only and there are no solutions for the rest interval. Interestingly, in the schemes with n = (8-14), there are (isolated solutions).

Fig. 4 shows the instantaneous unipolar voltage waveforms with optimal HEPWM technique for n = 3 (eliminating two harmonics with orders 3rd and 5th) and n=15(eliminating fourteen harmonics with orders 3rd, 5th... 29th) and with minimized THD where the number of harmonics to be eliminated = n – 1. Therefore, increasing n will cause increase in the number of low order eliminated harmonics, which causes to push more harmonic energy into high frequency regions, therefore low frequency harmonics are well attenuated. It can be seen, that the variation of n values affect the location of the harmonics in the spectrum, (i.e. the first significant component in the inverter output for n = 3 is equal to 7th or 350Hz, while it is equal to 31th or 1550 Hz for n = 15) [See also Table 2].

The inverter is loaded by single-phase capacitor-run induction motor with the following ratings: 175Watt, 220V, 1.22 A, and 1275Rpm. The behavior of the motor is explained by the simulation program. By using the equivalent circuit of the motor and the performance equations [14], it can be easy to analyze the performance of the motor operated on an optimal HEPWM 3-level inverter, and using Fourier series to calculate the harmonic currents amplitude to get the frequency spectra for motor current as shown in Fig. 4(d).

Increasing of n causes increase of the motor impedance with frequency ($X=2\pi fL$); therefore the harmonics currents decrease for constant harmonic voltage amplitude and as result, the ripple in the instantaneous motor current will decrease with increasing n as in Fig. 4(c and d).

Evaluation of the inverter performance can be calculated from the performance factor THD in Eq. 18. THD versus the modulation index m for all switching (2-15) is shown in Fig. 5. Fig. 6 (a and b) illustrates the relationship between this factor and n. We can see that increasing n causes increasing THDmin [approximately constant for n >7] and decreasing THDmax.

Decreasing the harmonic currents with increasing n causes decreasing of additional torque pulsations ($T_{pulsaddmin}$) and additional motor losses ($\Sigma P_{add}$), as illustrated in Fig. 5(c and d). Notice that, the torque pulsations amplitude decreases with
increasing $n$ until the motor will behave just like motor supplied directly by sinusoidal power supply [See Table 3], so that the motor will be more quite with higher $n$.

It also can be seen from the figure that, the motor performance (additional torque pulsation and additional motor losses and as a result the overall efficiency $\eta_{\text{max}}$) will be approximately constant for $n > 14$ [See Fig. 5(c, d and f) and Table 1]. So there is no need to solve the switching angles for $n > 15$.

It should be known that for $n = 3$, the switching frequency equal 7-times the base frequency and equal 31-times the base frequency for $n = 15$. Therefore the switching losses ($P_{sw_{\text{min}}}$) will increases with increasing $n$ as shown in Fig. 5(e).

**Conclusions**

1) However, increasing the number of $\alpha$ more than 7 per quarter cycle will lead to polynomial equations of higher degree. Therefore, Resultant theory will not be effective of solving these polynomials.

2) This paper presents a contribution to the theory of optimal PWM. It develops and uses a fast algorithm for efficient on-line calculation of PWM switching patterns for general $n$.

3) The proposed algorithm extends the traditional SHEPWM switching scheme to completely eliminates any number of harmonics to get highest quality motor drive with very low distortion AC waveform.

4) The proposed technique enables the motor to behave just like one supplied directly by sinusoidal power supply and to be more quite with higher switching.

5) The best compromise between high efficiency and high quality of the inverter operation is achieved by the optimal HEPWM technique.

**References**


Table (1) Highest-Quality of the 3-Level Unipolar Inverter and Motor fed from using the Fast Recursive Algorithm with the Optimal HEPWM Technique

<table>
<thead>
<tr>
<th>Switching No. (n)</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>α</strong></td>
<td>30.229</td>
<td>21.8958</td>
<td>22.925</td>
<td>18.8804</td>
<td>18.2243</td>
<td>16.3179</td>
<td>15.2280</td>
</tr>
<tr>
<td><strong>α</strong></td>
<td>89.7701</td>
<td>36.196</td>
<td>38.219</td>
<td>28.0493</td>
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<td>0.81</td>
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<td>0.8</td>
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Table (2) the Harmonics Amplitude ($h_k$) in p.u. with the Number of Switching Angles

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908
Table (3) Parameters and Specifications of the Proposed Motor

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<tr>
<th>Parameter</th>
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<td>Number of pole pair</td>
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<td>Main winding resistance</td>
<td>33.5 Ω</td>
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<tr>
<td>Main winding leakage reactance</td>
<td>27 Ω</td>
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<td>Auxiliary winding resistance</td>
<td>34.5 Ω</td>
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<tr>
<td>Auxiliary winding leakage reactance</td>
<td>28 Ω</td>
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<tr>
<td>Rotor resistance</td>
<td>20 Ω</td>
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<tr>
<td>Rotor leakage reactance</td>
<td>12.5 Ω</td>
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<td>Magnetization reactance</td>
<td>173 Ω</td>
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<td>Rated supply voltage</td>
<td>220 V</td>
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<td>Rated current</td>
<td>1.215 A</td>
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<td>Total Power losses</td>
<td>85 W</td>
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<tr>
<td>Output power</td>
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<td>Efficiency</td>
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<td>Capacitance</td>
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<td>Rated speed</td>
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</table>

Figure (1) Unipolar PWM Switching Scheme

Figure (2) Fast Recursive Algorithm for Optimal HEPWM
Figure (3) the Solutions of $\alpha$ vs. $m$ with Different Values of $n$ for 3-Level Unipolar Optimal HEPWM Inverter
Figure (4) Shows the Instantaneous Inverter output Voltage Waveform (a), Normalized Voltage Spectra (b), Instantaneous Motor Current (c), and Motor Current Spectra (d) for 3-Level Inverter with i) \( n = 3, m = 0.82 \), and THD\(_{\text{min}}\) = 43.6109 to Eliminate the 3\(^{\text{rd}}\) and 5\(^{\text{th}}\) harmonics and ii) \( n = 15, m = 0.79 \), and THD\(_{\text{min}}\) = 49.3008 to Eliminate the 3\(^{\text{rd}}\), 5\(^{\text{th}}\), ..., 29\(^{\text{th}}\) harmonics.
Figure (5) the Voltage THD vs. the modulation index \( m \) for all Switching \( (n=2-15) \) of SHPWM.

Figure (6) the Quality of the Inverter and Motor Fed from vs. No. of Switching Angles \( n \): (a) THDmin, (b) THDmax, (c) Minimum Motor Additional Losses, (d) Minimum Motor Additional Pulsating Torque, (e) Minimum Switching Losses, and (f) Overall Maximum Efficiency.