Shifted Chybeshev Polynomials for a Certain System of Fractional Order Integro-Differential Equations

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Received on:29/3/2009
Accepted on:2/7/2009

Abstract
The main goal of this paper lies briefly in submitting and modifying some numerical methods for solving system of linear Fractional order Integro-Differential Equations of Fredholm type (L.FFIDE's). In this method four kinds of shifted Chybeshev polynomials (*T, *U, *V, and *W) are used as bases of independent polynomials approximation f_n(x). The general fractional derivatives of these polynomials are formulated (D^α T_n, D^α U_n, D^α V_n, and D^α W_n) in the framework of the Riemann-liouville definition. Some numerical examples are solving to show that the different between these polynomials, Furthermore Algorithms and programs by using MATLAB program are given.

Keywords: Shifted Chybeshev Polynomials, System of fractional integro-differential equations.

1-Introduction
Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order (real and complex numbers). The term fractional is a misnomer, but it is retained following the prevailing use.

System of fractional integro-differential equations are equations having unknown function together with both fractional differential and integral operations and has the form:

\[ D^α f(x) = g(x) + \sum_{j=1}^{b} k_j(x,t) f^{(j)}(t) dt \]

\[ i=1,2,....,m ; a \leq x \leq b \]

The theory and application of fractional integral and derivatives can be fund in many fields of science and engineering, such as Viscoelasticity, fractional differential which have been used to describe material's
constitutive equations. In fact a well-known equation which contains a fractional integral operator is the Abel integral equation.[1,2,4,5,9].

2-Basic Definitions: In this section we give definitions:

2-1 Definition: the fractional derivative by Reimann-Lovill (R-L) has the form:

$$D^\alpha_x f(x) = \frac{d^m}{d\alpha^m} \left[ \frac{1}{\Gamma(m)\alpha} \int_0^x (x-t)^{m-\alpha} f(t) dt \right]$$

Where $m$ is an integer number less than $\alpha$. [3, 5, 7]

Note: In this paper, we use R-L definition and its properties to find fractional derivatives of polynomials.

2-2 Some important properties of operator $(D^\alpha_x)$ [7,8]:

a- $D^\alpha_x \sum_{i=1}^n c_i f_i(x) = \sum_{i=1}^n c_i D^\alpha_x f_i(x)$

the linearity property.

b- $D^\alpha_x c = \frac{c}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \left( x^{n-\alpha} \right)$

where $c$ is constant.

c- $D^\alpha_x x^m = \frac{m!}{\Gamma(m-\alpha+1)} x^{m-\alpha}$

where $m=0, 1, 2, \ldots$, $\alpha > 0$.

In a special case, $\alpha = 0.5$ we have:

da- $D^{0.5} c = \frac{c}{\sqrt{\pi}}$

e- $D^{0.5} x^m = \frac{(m!)^2 (4x)^m}{(2m)! \sqrt{\pi} x}$

The general form given by:

$$U_n(x) = \sum_{r=0}^{[n/2]} (-1)^r \frac{(n-r)!}{r!(n-2r)!} (2x)^{(n-2r)} ; U_0(x)=1 ; n > 1$$

3- Chebyshev polynomials [3]:

Chebyshev polynomials are orthogonal functions, and every where dense in numerical analysis.

These polynomials have four kinds and have the forms:

3-1 First kind $T_n(x)$ is a polynomial in $x$ of degree $n$. Defined by the relation:

$$T_n(x) = \cos (n\theta)$$

where $x = \cos \theta$ …(1)

The range of the variable $x$ is the interval [-1,1], and the range of $\theta$ can be taken as [0,\pi].

The recurrence relation:

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) ; n=2, 3, \ldots$$

where $T_0(x)=1$ and $T_1(x)=x$ …(2)

3-2 Second kind polynomial $U_n(x)$:

It is a polynomial of degree $n$ in $x$ defined by:

$$U_n(x) = \frac{\sin (n+1)\theta}{\sin \theta}$$

where $x = \cos \theta$ …(4)

The recurrence relation:

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) ; n=2, 3, \ldots$$

where $U_0(x)=1$ ; $U_1(x)=2x$ …(5)

3-3 Third kind polynomial $V_n(x)$:

Are polynomials of degree $n$ in $x$ defined by:

$$V_n(x) = \cos \left( \frac{n+1}{2} \theta \right)$$

where $x = \cos \theta$ …(7)

The recurrence relation:

$$V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x) ; n=2, 3, \ldots$$

where $V_0(x)=1$ ; $V_1(x)=2x-1$ …(8)

3-4 Fourth kind polynomial $W_n(x)$:

Are polynomials of degree $n$ in $x$ defined by:

$$W_n(x) = \sin \left( \frac{n+1}{2} \theta \right)$$

where $x = \cos \theta$ …(9)

The recurrence relation:
When $\sin (n+1)\theta - \sin (n-1)\theta = 2\sin \theta \cos n\theta$
we have:

\[ T_n(s) = \frac{1}{2} \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \frac{(n-r)!}{r!(n-2r)!} (2s)^{(n-2r)} ; \]

\[ T_0(s) = 1 ; n > 1 \quad \ldots (16) \]

\[ U_n(s) = \frac{1}{2} \left[ V_n(s) + W_n(s) \right] \]

\[ U_0(s) = 1 ; n > 1 \quad \ldots (17) \]

\[ V_n(s) = U_n(s) - U_{n-1}(s) \]

\[ V_0(s) = 2s \quad \ldots (18) \]

\[ W_n(s) = U_n(s) + U_{n-1}(s) \]

\[ W_0(s) = 1 ; U_1(s) = 2s \quad \ldots (19) \]

6- Fractional Derivatives of shifted Chebyshev polynomials:
Consider the properties of the operator \( (D_x^\alpha) \) eq's (1-2 a, b, and c)
putting in eq's (16,17,18 and 19) yield:

\[ D_x^\alpha T_n^*(x) = \frac{n}{2} \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \frac{\Gamma(n-r)(2^{(n-2r)})^{(n-2r-\alpha)}}{r!\Gamma(n-2r+\alpha)} \]

\[ ; n > 1 \quad \ldots (20) \]

where \( D_x^\alpha T_0^*(x) = \)

\[ = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \left( \frac{s^{n-\alpha}}{n-\alpha} \right) ; \]

\[ s = 2x-1 \quad \ldots (20a) \]

\[ D_x^\alpha U_n^*(x) = \]

\[ \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \frac{\Gamma(n-r)(2^{(n-2r)})}{r!\Gamma(n-2r+\alpha)} \left( \frac{s^{(n-2r-\alpha)}}{n-2r+\alpha} \right) ; n > 1 \quad \ldots (21) \]

where \( D_x^\alpha U_0^*(x) = \)

\[ = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \left( \frac{s^{n-\alpha}}{n-\alpha} \right) ; \]

\[ s = 2x-1 \quad \ldots (21a) \]
\[ D_x^{\alpha} V_n^*(x) = D_x^{\alpha} U_{n}(s) - D_x^{\alpha} U_{n-1}(s) \]
where \( s = 2x - 1 \) \( \ldots (22) \)

\[ D_x^{\alpha} W_n^*(x) = D_x^{\alpha} U_{n}(s) + D_x^{\alpha} U_{n-1}(s) \]
where \( s = 2x - 1 \) \( \ldots (23) \)

Where \( U_{0}(s) \) and \( U_{1}(s) \) from eq’s(17) respectively.

**Remark 6-1:** If \( \alpha = 0.5 \) and using properties of operator \( (D_x^\alpha) \) eq’s(c and d) then the general forms of fractional derivative of shifted Chebyshev polynomials yield:

\[ D_{x}^{0.5} T_0^*(x) = \frac{1}{\sqrt{\pi s}}; \quad \text{where} \quad s = 2x - 1 \quad \ldots (24) \]

\[ D_{x}^{0.5} U_0^*(x) = \frac{1}{\sqrt{\pi s}}; \quad \text{where} \quad s = 2x - 1 \quad \ldots (25) \]

**7- System of Linear Fractional Integro-Differential Equation of Fredholm type (S.LFFIDE’s):** This system is given by the form \([6]\): 

\[
\sum_{j=1}^{m} \int_{a}^{b} k_{ij}(x,t)f^{i}(t)dt
\]

\( i=1,2,\ldots,m; \quad a \leq x \leq b \quad \ldots (26) \)

The system has \( (m) \) independed equations. Any equation has \( (m) \) unknown functions \( f^{i}(x) \). For example let \( m=2 \) the system eq(26) gives two the following equations:

\[
D=\frac{d}{dx}f^{i}(x) = g^{i}(x) + \sum_{j=1}^{m} \int_{a}^{b} k_{ij}(x,t)f^{j}(t)dt
\]

\( i=1,2,\ldots,m; \quad a \leq x \leq b \quad \ldots (27) \)

**Step1:** In this step we choose \( (N) \) known functions \( (\Phi_r) \) by the form:

\[
f^{i}(x) \approx f^{i}_n(x) = \sum_{r=0}^{N} c_r^{i}\Phi_r
\]

**Step2:** In this step we choose \( (N) \) points \( (x_k) \) in the interval \([a,b]\) i.e \( a \leq x_0 < x_1 < \ldots < x_N \leq b \). Putting these points in the system equations (28) we have system eq(29) which gives \( (m(N+1)) \) independed equations with \( (m(N+1)) \) unknown parameters \( (c^{i}_r) \), we must found them to find solution of the system.

\[
\sum_{r=0}^{N} c_r^{i}\Phi_r(x_k) = g^{i}(x_k) + \sum_{j=1}^{m} \int_{a}^{b} k_{ij}(x,t)\sum_{r=0}^{N} c_r^{j}\Phi_r(t)dt
\]

\( i=1,2,\ldots,m; \quad k=0,1,\ldots,N \quad \ldots (29) \)

**Remark8-1:** We can write the system of eq(29) by a matrix from as:

\[
A C = G \quad \ldots (30)
\]

where \( A \) is \( (m(N+1)) \) by \( (m(N+1)) \) matrix contains \( (m) \ by \ m \) sub-
matrices \( A_{ij}^{rk} \) where \( i,j=1,2,...,m \); \( r,k=0,1,2,...,N \) we can write as:

\[
A = \begin{bmatrix}
A_{11}^{r} & A_{12}^{r} & \cdots & A_{1m}^{r} \\
A_{21}^{r} & A_{22}^{r} & \cdots & A_{2m}^{r} \\
\vdots & \vdots & \ddots & \vdots \\
A_{m1}^{r} & A_{m2}^{r} & \cdots & A_{mm}^{r}
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
C_{r1} \\
C_{r2} \\
\vdots \\
C_{rn}
\end{bmatrix}
\]

where:

\[
A_{ij}^{r,k,i} = \begin{bmatrix}
a_{ij}^{00} & a_{ij}^{01} & \cdots & a_{ij}^{0N} \\
a_{ij}^{10} & a_{ij}^{11} & \cdots & a_{ij}^{1N} \\
\vdots & \vdots & \ddots & \vdots \\
a_{ij}^{m0} & a_{ij}^{m1} & \cdots & a_{ij}^{mN}
\end{bmatrix}
\]

where:

\[
C_{r,i} = \begin{bmatrix}
c_{0,i} \\
c_{1,i} \\
\vdots \\
c_{n,i}
\end{bmatrix}
\]

Similarly, we can write the \( G \) column by \((N+1)\) sub-columns as:

\[
G = \begin{bmatrix}
g_{1,k} \\
g_{2,k} \\
\vdots \\
g_{m,k}
\end{bmatrix}
\]

The \( C \) column has \( m \) sub-columns \((C_{r,i})\) and we can write them as:

\[
C_{r} = \begin{bmatrix}
c_{r1} \\
c_{r2} \\
\vdots \\
c_{rn}
\end{bmatrix}
\]

Remark 8-2: Since eq(33) and properties \((d,e)\) of operator \((\frac{D_{0.5}}{D_{X}})\) then all parts of matrix \((A)\) has the part \((\frac{1}{\sqrt{\pi}})\) so that we must choose points \((x_{k}, k=0,1,2,...,N)\) s.t. \(2x_{k} - 1 > 0\) or \((x_{k} > 0.5)\), in general \(x_{k} = 0.55 + k(0.45/N)\) …(36)
Algorithm: Steps to solve (S.LFFIDE's):

Step1: Choose points $x_k$ use eq(36) where $k=0,1,\ldots,N$.

Step2: Evaluate $(a_{ij}^k)$ use eq's(33) and eq's (16-19, 20-23) for all $i,j=1,2,\ldots,m$ ; $k,r=0,1,2,\ldots,N$.

Step3: Compute $G^k_i$ use eq(35) for all $i=1,2,\ldots,m$ ; $k=0,1,\ldots,N$.

Step4: Compute the general matrix $A$ use eq's(31-32) with step2 and column $G$ using eq(35) with step3.

Step5: Solve the system $AC = G$ use Gause's-Elimination to find column $C$ where

$C = A/G$ or $C = \text{inv}(A)*G$.

Step6: Find the approximate solutions $f_N^i(x)$ use eq(27) where the bases functions $\Phi_j$ are one of bases $(T^*_n, U^*_n, V^*_n, W^*_n)$.

Note: in this work we use programs in MATLAB with six steps above to find approximate solution where:

- $F_N^iT^*$ means programs use steps above to find approximate solution to $f^i(x)$.
- $F_N^iU^*$ means programs use steps above to find approximate solution to $f^i(x)$.
- $F_N^iV^*$ means programs use steps above to find approximate solution to $f^i(x)$.
- $F_N^iW^*$ means programs use steps above to find approximate solution to $f^i(x)$.

Example 1: Solve the linear system (FFIDE's):

$$D_s^{0.5}f^i(x) = g^i(x) + \int_0^1 k_{ij}(x,t)f^j(t)dt$$

where $k_{ij}(x,t)=1$; $i,j=1,2$ and $g^1(x) = \frac{2x}{\sqrt{\pi x}} - 1$.

$g^2(x) = \frac{4x}{\sqrt{\pi x}} - 0.5$ with exact solution $f^1(x) = x$; $f^2(x) = 2x$.

Solution: Choose $N=1$ and running four programs above the results of this example listed in tables (1-1), (1-2) which obtained by using bases $(T^*_n, U^*_n, V^*_n, W^*_n)$.

Example 2: Solve the linear system (FFIDE's):

$$D_s^{0.5}f^i(x) = g^i(x) + \int_0^1 k_{ij}(x,t)f^j(t)dt$$

where $k_{ij}(x,t)=1$; $i,j=1,2$ and $g^1(x) = \frac{2x}{\sqrt{\pi x}} - 3$.

$g^2(x) = 1 + \frac{4x}{\sqrt{\pi x}} + \frac{8x^2}{\sqrt{\pi x}} - 0.5$ with exact solution $f^1(x) = x$; $f^2(x) = 1 + 2x + 3x^3$.

Solution: Choose $N=2$ and running four programs above the results of this example listed in tables (2-1), (2-2) which obtained by using bases $(T^*_n, U^*_n, V^*_n, W^*_n)$.

Example 3: Solve the linear system (FFIDE's):

$$D_s^{0.5}f^1(x) = g^1(x) + \int_0^1 f^2(t)dt$$

$$D_s^{0.5}f^2(x) = g^2(x) + \int_0^1 f^1(t)dt$$

$g^1(x) = \frac{1}{\sqrt{\pi x}} + e^x erf(\sqrt{x}) - (e^3 - e^2)$.
Shifted Chybeshev Polynomials for a certain system of fractional order Integro-Differential Equations

\[ g^2(x) = \frac{1}{\sqrt{\pi x}} + e^x \text{erf} (\sqrt{x}) - (e - 1) \]

with exact solution \( f^1(x) = e^x \); 
\( f^2(x) = e^{x+2} \).

**Solution:** Choose \( N = 5, 8 \) and running four programs above the results of this example listed in tables (3-1), (3-2) which obtained by using bases \( (T^*_n, U^*_n, V^*_n \text{ and } W^*_n) \).

**Conclusions**
1. All kinds of shifted Chebyshev polynomials \((T^*_n, U^*_n, V^*_n \text{ and } W^*_n)\) as a basis function which are used in this paper have proved their effectiveness in solving linear system (FFIDE’s) numerically and finding accurate results, when unknown functions are algebraic functions, other functions the \( U^*_n \) gives better solution than other bases.
2. All Chebyshev polynomials depends on \( N \) as \( N \) increasing, the error term is decreased.
3. The results show a marked improvement in the least square errors from which we conclude that.

**References**
Shifted Chebyshev Polynomials for a certain system of fractional order Integro-Differential Equations

Ex1-(N1) \( \Omega_1=\chi \)

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Table (1-1)

\( \Omega_2=2\chi \)

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Table (1-2)
Shifted Chybeshev Polynomials for a certain system of fractional order Integro-Differential Equations

Ex2:

- \( f_1 = x \)

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<td>9.999999999977e+01</td>
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<td>9.999999999977e+01</td>
</tr>
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L.S. Error: 0.356225565570654e+00

\[ \text{Table (2-1)} \]

- \( f_2 = 1 + 2x + 3x^2 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f_{2,1} )</th>
<th>( f_{2,2} )</th>
<th>( f_{2,3} )</th>
<th>( f_{2,4} )</th>
<th>( f_{2,5} )</th>
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</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.52</td>
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<td>1.509999999999691e+00</td>
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<td>1.509999999999691e+00</td>
<td>1.509999999999691e+00</td>
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</tbody>
</table>

L.S. Error: 0.56583938348739e+00

\[ \text{Table (2-2)} \]
Shifted Chebyshev Polynomials for a certain system of fractional order Integro-Differential Equations

### Ex3-(N=5)

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$f_1 = e^x$</th>
<th>$f_2 = e^{x^2}$</th>
<th>$f_3 = \sin(x)$</th>
<th>$f_4 = \cos(x)$</th>
<th>$f_5 = \tan(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
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<td>0.123456</td>
<td>0.987654</td>
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<tr>
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<td>0.987654</td>
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<tr>
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<td>0.456789</td>
<td>0.123456</td>
<td>0.987654</td>
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<tr>
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<td>0.456789</td>
<td>0.123456</td>
<td>0.987654</td>
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<tr>
<td>0.05</td>
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<td>0.987654</td>
</tr>
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</table>

### Table (3-1)

Table (3-2)
**Shifted Chebyshev Polynomials for a certain system of fractional order Integro-Differential Equations**

Ex3-(N=8)

\[ f_2 = e^{x^2} \]

<table>
<thead>
<tr>
<th>(x_i)</th>
<th>(f_{2} = e^{x^2})</th>
<th>(f_{12I})</th>
<th>(f_{12U})</th>
<th>(f_{12V})</th>
<th>(f_{12W})</th>
</tr>
</thead>
<tbody>
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<tr>
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\[ \text{L.S. Error} = 4.466123592770e-007 \]

Table (4-2)

\[ f_1 = e^{x} \]

<table>
<thead>
<tr>
<th>(x_i)</th>
<th>(f_{1} = e^{x})</th>
<th>(f_{12I})</th>
<th>(f_{12U})</th>
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<th>(f_{12W})</th>
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<tbody>
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<tr>
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\[ \text{L.S. Error} = 4.277273208817e-007 \]

Table (4-1)