4- Differentiation :

Def. (1) If $y = f(x)$ is a continous fun. then we define the derivative of a fun. as a limit as :

$$\bar{y} = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = \bar{f}(x)$$

Ex. Find the derivative of the fun. $y = x^2$

$$y = x^2 = f(x) \Rightarrow y + \Delta y = (x+\Delta x)^2 = f(x+\Delta x)$$
$$= x^2 + 2x\Delta x + \Delta^2 x \Rightarrow$$

$$\Delta y = x^2 + 2x\Delta x + \Delta^2 x - f(x) = \cancel{x^2} + 2x\Delta x + \Delta^2 x - \cancel{x^2}$$
$$= \Delta x[2x + \Delta x]$$

$$\therefore \frac{\Delta y}{\Delta x} = 2x + \Delta x$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} 2x + \Delta x = 2x .$$

Derivative of algebric functions

If $y_1 = f(x)$, $y_2 = g(x)$ then

1- $$\frac{d(f \mp g)(x)}{dx} = \frac{df(x)}{dx} \mp \frac{dg(x)}{dx}$$

2- $$\frac{d(f \cdot g)(x)}{dx} = f(x)\frac{dg(x)}{dx} + g(x)\frac{df(x)}{dx}$$

3- $$\frac{d(f/g)(x)}{dx} = \frac{g(x)\frac{df(x)}{dx} - f(x)\frac{dg(x)}{dx}}{[g(x)]^2}$$

4- $$\frac{d[kf(x)]}{dx} = k\frac{df(x)}{dx}$$ k is any number

5- $$\frac{d\,[f(x)]^n}{dx} = n\,[f(x)]^{n-1}\,\frac{df(x)}{dx}$$

6- $$\frac{d\,(x^n)}{dx} = n\,x^{n-1}$$

Chainrule

Suppose that $h = g \circ f$ is the composite of the differentiable functions $y = g(x)$ and $x = f(t)$. Then h is a differentiable function whose derivative at each value of t is

$$h'(t) = g'(f(t)) \cdot f'(t) \rightarrow \frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Ex. Find $\frac{dy}{dt}$ at $t = -1$, if $y = x^3 + 5x - 4 = g(x)$ and $x = t^2 + t = f(t)$

Solution

$$\left.\frac{dy}{dt}\right|_{\text{at } t=-1} = \frac{dy}{dx} \cdot \frac{dx}{dt} \underset{t=-1}{=} (3x^2+5)_{x=0} \cdot (2t+1)_{t=-1}$$

$$= (5)(-1) = -5.$$

L-Hopital

Suppose that $f(x_0) = g(x_0) = 0$ and $f'(x_0)$ and $g'(x_0)$ exist, and that $g'(x_0) \neq 0$. Then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

Ex. Evaluate $\lim_{x \to \infty} \frac{x^2+1}{x^3-3} = \frac{\infty}{\infty}$

$$= \lim_{x \to \infty} \frac{2x}{3x^2} = \frac{2}{\infty} = 0$$

Def. (2) Higher order derivative: if $y = f(x)$ a continuous fun. then the first derivative of it is $\bar{y} = \frac{dy}{dx} = \frac{df(x)}{dx} = \bar{f}(x)$, the second order derivative is $\bar{\bar{y}} = \frac{d^2y}{dx^2} = \frac{d^2f(x)}{dx^2} = \bar{\bar{f}}(x)$ ..., the nth order derivative is $y^n = \frac{d^ny}{dx^n} = \frac{d^nf(x)}{dx^n} = f^n(x)$.

Ex. If $y = 4x^3 + 2x + 1$, find $\frac{d^2y}{dx^2}$.

Solution

$$\bar{y} = 12x^2 + 2 \Rightarrow \bar{\bar{y}} = 24x$$

Derivative of parameteric function

If $y = f(t)$, $x = g(t)$ then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \Rightarrow \bar{\bar{y}} = \frac{d^2y}{dx^2} = \frac{d\bar{y}}{dt} \Big/ \frac{dx}{dt}.$$

Ex. find $\frac{d^2y}{dx^2}$, if $x = t - t^2$ and $y = t - t^3$

Solution

$$\bar{y} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t}$$

$$\bar{\bar{y}} = \frac{d^2y}{dx^2} = \frac{d\bar{y}/dt}{dx/dt} = \frac{d\left[\frac{1-3t^2}{1-2t}\right]}{dx} \Big/ (1-2t)$$

$$= \frac{2-6t+6t^2}{(1-2t)^3}$$

Implicit differentiation

If we have implicit fun. for example $x^2 + 2xy + y^4 = 3$, then if we differention w.r.T x then we have $\frac{dy}{dx}$ as; $2x + 2\left[x\frac{dy}{dx} + y\right] + 4y^3\frac{dy}{dx} = 0 \Rightarrow$

$$2x + 2y = -2x\frac{dy}{dx} - 4y^3\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{2x+2y}{-2x-4y^3}$$

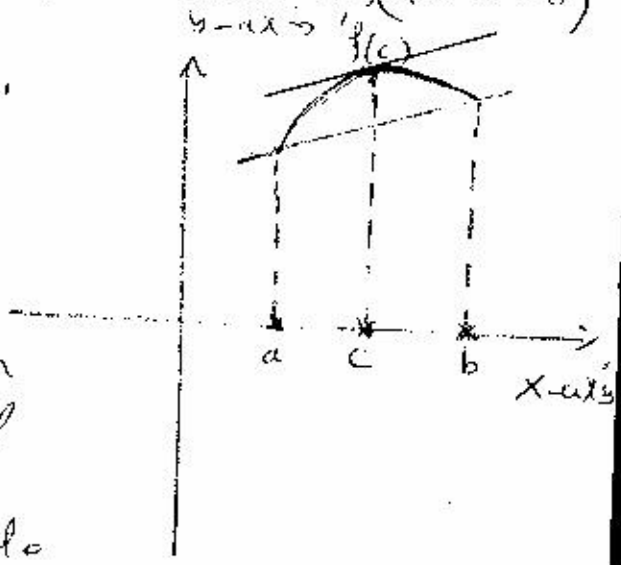
or we differention w.r.T y then we have $\frac{dx}{dy}$ as;

$$2x\frac{dx}{dy} + 2\left[x + y\frac{dx}{dy}\right] + 4y^3 = 0 \Rightarrow 2x + 4y^3 = -2x\frac{dx}{dy} - 2y\frac{dx}{dy}$$

$$\Rightarrow \frac{dx}{dy} = \frac{2x+4y^3}{-2x-2y} = \frac{1}{\frac{dy}{dx}}.$$

Mean value theorem

If the fun. f is defined and continous on closed interval $[a,b]$ and f is differentiable at (a,b). Then there exist at least a numbe c, $(a<c<b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.



Ex. If $f(x) = \frac{x^2}{6}$ on $[2,6]$ then

$f(x)$ is continous o $[2,6]$ and

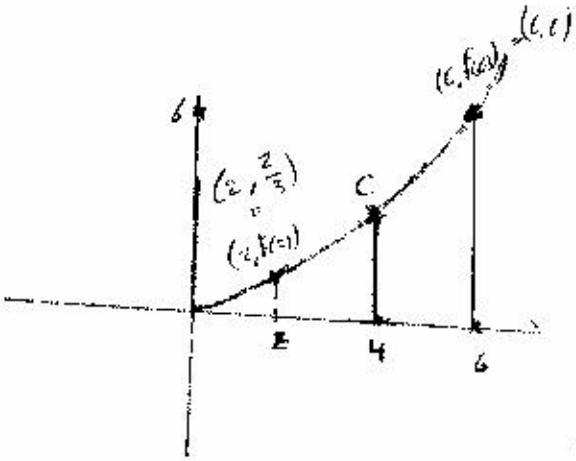
$f'(x) = \frac{x}{3} \Rightarrow f(x)$ differentiable

$f'(2) = \frac{2}{3}$, $f(6) = 6$

$f'(c) = \frac{c}{3}$

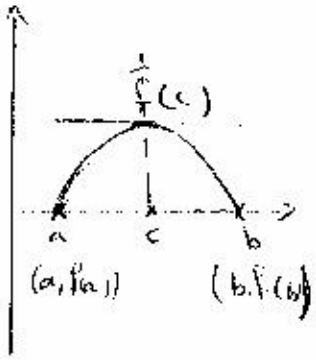
$\therefore \frac{c}{3} = \frac{6 - \frac{2}{3}}{6-2} \Rightarrow \frac{c}{3} = \frac{\frac{16}{3}}{4}$

$\therefore 4c = 3 \cdot \frac{16}{3} \Rightarrow c = 4$



Rolles theorem : If the function f is define and Continouse on closed interval $[a,b]$ and differentiable at (a,b) and $f(a)=f(b)=0$ Then there exist at least one number c $(a<c<b)$ such that $f'(c)=0$ and

$$f'(c)=\frac{f(b)-f(a)}{b-a}=\frac{0-0}{b-a}=0.$$



Ex. If $f(x)=6x^2-x^3$ is a continouse fun. on $[0,6]$ and differentiable fun. on $(0,6)$ and $f'(x)=12x-3x^2$. Hence

$f(0)=f(6)=0 \Rightarrow f'(c)=0$

$\therefore f'(c)=12c-3c^2=0 \Rightarrow 3c(4-c)=0$

$\Rightarrow c=0$ or $c=4$

Derivative of trigonomatric functions : we have the laws :

1- If $y=\cos\theta$ then $y'=-\sin\theta\frac{d\theta}{dx}$

2- If $y=\sin\theta$ then $y'=\cos\theta\frac{d\theta}{dx}$

3- If $y=\tan\theta$ then $y'=\sec^2\theta\frac{d\theta}{dx}$

4- If $y=\cot\theta$ then $y'=-\csc^2\theta\frac{d\theta}{dx}$

5- If $y=\sec\theta$ then $y'=\sec\theta\tan\theta\frac{d\theta}{dx}$

6- If $y=\csc\theta$ then $y'=-\csc\theta\cot\theta\frac{d\theta}{dx}$

Ex If $y=\cos x^2$ then $y'=-\sin x^2\cdot 2x$

Ex. If $y = \tan^2\sqrt{x}$ then $y' = 2\tan\sqrt{x}\ \sec^2\sqrt{x}\ \frac{1}{2\sqrt{x}}$

Ex. If $\tan y + 2\sin x\cos x = \sec y + 1$ then

the derivative is $\sec^2 y\ y' + 2\sin x[-\sin x] + 2\cos x(\cos x) = \sec y\tan y\ y'$

$\Rightarrow \sec^2 y\ y' - \sec y\tan y\ y' = 2\sin^2 x - 2\cos^2 x$

$\therefore\ y' = \dfrac{2[\sin^2 x - \cos^2 x]}{\sec^2 y - \sec y\tan y} = \dfrac{-2\cos(2x)}{\sec^2 y - \sec y\tan y}$

5- Integration :-

Def. ① (Indefinite integral) Let $F(x) = y$ be a fun. whose derivative is given by the equation $\frac{dy}{dx} = f(x)$. We called $F(x)$ the an antiderivative of $f(x)$, also $F(x)+C$ is and antiderivative of $f(x)$. The set of all antiderivative of $f(x)$ is called the indefinite integral of f w.r.T x and indenated by the symbol $\int f(x)\,dx$.

Hence the law of indefinite integral is

$$\int f(x)\,dx = F(x) + C \quad \text{where } C \text{ is constant}$$

Ex. Solve the differential equation

$$\frac{dy}{dx} = 3x^2$$

Solution

We have

$$\frac{d(x^3)}{dx} = 3x^2$$

Therefore $y = \int 3x^2\,dx = x^3 + C$.

properties of indefinite integrals

1- $\int [f(x) \mp g(x)]dx = \int f(x)dx \mp \int g(x)dx$

2- $\int k\, f(x)dx = k\int f(x)dx$ k is any Constant.

3- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ if $n \neq -1$

4- $\int (f(x))^n df(x) = \frac{f(x)^{(n+1)}}{n+1} + C$ $n \neq -1$

5- $\int dx = x + C$

6- $\int \sin(u)du = -\cos u + C$

7- $\int \cos(u)du = \sin u + C$

8- $\int \sec^2(u)du = \tan u + C$

9- $\int \csc^2(u)du = -\cot u + C$

10- $\int \csc(u)\cot u\, du = -\csc|u| + C$

11- $\int \sec u \tan u\, du = \sec|u| + C$

Ex.

1- $\int (x^3+2)^2 dx = \int (x^6 + 4x^3 + 4)dx = \frac{x^7}{7} + \frac{4x^4}{4} + 4x + C$

2- $\int (x+1)^3 dx = \frac{(x+1)^4}{4} + C$

3- $2\int \frac{\sin\sqrt{x}}{2\sqrt{x}} dx = -2\cos\sqrt{x} + C$

4- $\int \sec^2(x+1)dx = \tan(x+1) + C$

Definite integrals

If f is a continous fun. on $[a,b]$ and F is any antiderivative of f on $[a,b]$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Ex. $\int_1^3 x\,dx = \frac{x^2}{2}\Big]_1^3 = \frac{9}{2} - \frac{1}{2} = 4$

properties of definite integral

If a, b, c and k are real numbers and f is a continuous function then

1- $\int_a^b f(x)dx = -\int_b^a f(x)\,dx$

2- $\int_a^a f(x)\,dx = 0$

3- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

4- $\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$

5- $\int_a^b (f(x) \mp g(x))\,dx = \int_a^b f(x)\,dx \mp \int_a^b g(x)\,dx$

6- If f is even fun. then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

7- if f is odd fun. then $\int_{-a}^{a} f(x)\,dx = 0$

Ex.① $\int_{-3}^{3} (x^7 - 2x^3)\,dx = 0$ the fun. is odd.

Ex② $\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos x}{\sqrt{1-\sin x}}\,dx = -\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} -(1-\sin x)^{-\frac{1}{2}} \cos x\,dx = -\frac{(1-\sin x)^{\frac{1}{2}}}{\frac{1}{2}}\Big]_{-\frac{\pi}{2}}^{\frac{3\pi}{2}}$

$= 0$

Ex.(3) $\int_1^2 (1+u)\sqrt{u}\,du = \int_1^4 \sqrt{u}\,du + \int_1^4 u\sqrt{u}\,du = \int_1^4 u^{\frac{1}{2}}\,du + \int_1^4 u^{\frac{3}{2}}\,du$

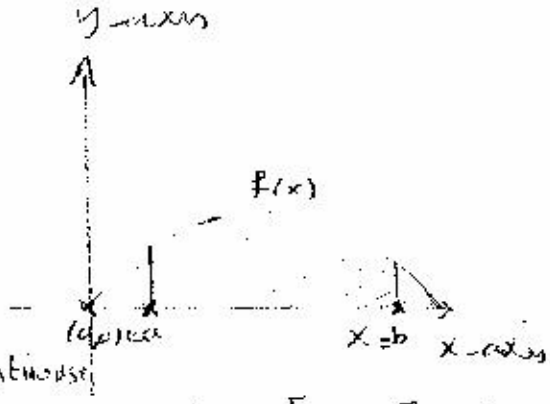
$= \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Big]_1^4 - \frac{u^{\frac{5}{2}}}{\frac{5}{2}}\Big]_1^4 = \left[\frac{2}{3}2^3 - \frac{2}{3}\right] + \left[\frac{2}{5}2^5 - \frac{2}{5}\right]$

$= \frac{14}{3} + 34 = \frac{116}{3}$

Definite integral -- An integral of $f(x)$ on a closed interval $[a,b]$ is called a definite integral of $f(x)$ from $x=a$ to $x=b$ and denoted by $\int_{x=a}^{x=b} f(x)dx$ the numbers a and b are called the lower and upper limits of integration respectively.

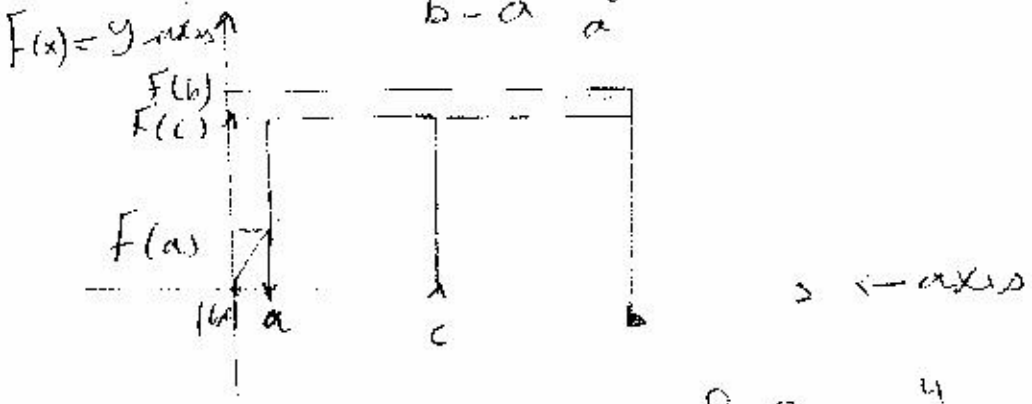
Theorem -- if $f(x)$ is non negative and continuous on the interval $[a,b]$ then a define integral of $f(x)$ from $x=a$ to $x=b$ exists and equal the **Area** bounded by $f(x)$, x-axis, the lines $x=a$ and $x=b$. In other words the definite integral $\int_{x=a}^{x=b} f(x)dx$ represents the area under the curve $f(x)$ on the interval.

$$\text{Area} = A = \int_a^b f(x)dx$$



Mean value theorem for integrals -- If $f(x)$ is continuous on the interval $[a,b]$, then there exists a number c in $[a,b]$ at which

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$



Ex. Evaluate the mean value theorem of $f(x)=\sin^4 x\cos x$ on the interval $[0,\frac{\pi}{2}]$. Solution

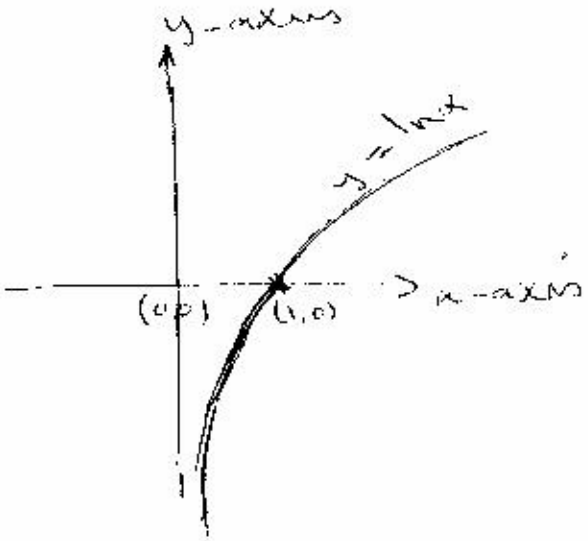
$$f(c) = \frac{1}{\frac{\pi}{2}-0}\int_0^{\frac{\pi}{2}} \sin^4 x\cos x\,dx = \frac{1}{\frac{\pi}{2}-0}\left[\frac{\sin^5 x}{5}\right]_0^{\frac{\pi}{2}} = \frac{2}{\pi}\cdot\frac{1}{5} = \frac{2}{5\pi}$$

The Logarithmic and exponential functions

Def. (1) A natural logarithms is a function from the domain R^+ to the range $\{x \in R : x > 0\}$ and denoted by $y = \ln x$

Graph of $y = \ln x$

x	$y = \ln x$
∞	$\lim_{x \to \infty} \ln x = \infty$
1	$y = \ln 1 = 0 \Rightarrow (1,0)$
$-\infty$	$\lim_{x \to -\infty} \ln x = 0$



properties

1- $\ln u.v = \ln u + \ln v$

2. $\ln\left(\frac{u}{v}\right) = \ln u - \ln v$

3. $\ln u^v = v \ln u$

Ex. $\ln \sqrt[3]{25} = \frac{1}{3} \ln 25 = \frac{1}{3} \ln 5^2 = \frac{2}{3} \ln 5$

Def. (2) [Exponential fun.] : The function $y = f(x) = \ln x$ have inverse function called exponential fun. denoted by $x = e^{y} = \ln^{-1} y$ with domain $\{x : x > 0\}$ and Range $\mathbb{R}$.

For Ex. $y = e^{x^2}$, $y = e^{\sin x}$ . . .

properties

1- $e^{x} \cdot e^{y} = e^{x+y}$

2- $e^{x} / e^{y} = e^{x-y}$

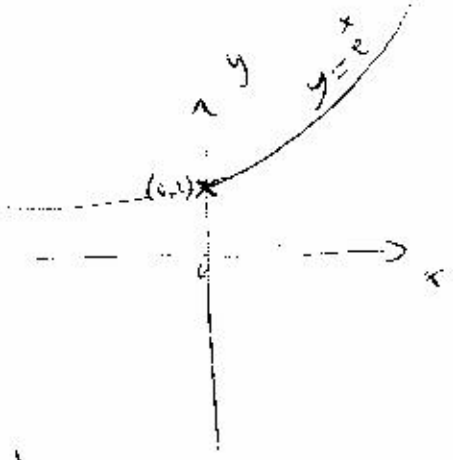
3- $[e^{x}]^{y} = e^{xy}$

4- $e^{-x} = \frac{1}{e^{x}}$

5- $e^{\ln x} = \ln e^{x} = x$

Graph of $y = e^{x}$

x	$y = e^{x}$
∞	$\lim_{x \to \infty} e^{x} = e^{\infty} = \infty$
0	$e^{0} = 1$
$-\infty$	$\lim_{x \to -\infty} e^{x} = e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$



laws of derivative for $y = e^{x}$, $y = \ln x$

1- If $y = e^{u}$ then $\dot{y} = e^{u} \frac{du}{dx}$

2- If $y = \ln u$ then $\dot{y} = \frac{1}{u} \frac{du}{dx}$

Ex. Find $\frac{dy}{dx}$ for the following :

1- $y = e^{\sin x} \Rightarrow y' = e^{\sin x} \cos x$

2- $y = \ln\sqrt{x} \Rightarrow y' = \frac{1}{\sqrt{x}} \frac{1}{2\sqrt{x}}$

3- $y = e^{2\ln x} \Rightarrow y = \cancel{e}^{\cancel{\ln} x^2} = x^2 \Rightarrow y' = 2x$

4- $y = \frac{x(x+1)}{(x-1)^2(x^2+1)} \Rightarrow \ln y = \ln \frac{x(x+1)}{(x-1)^2(x^2+1)}$

$\Rightarrow \ln y = \ln x(x+1) - \ln(x-1)^2(x^2+1)$

$\ln y = \ln x + \ln(x+1) - 2\ln(x-1) - \ln(x^2+1)$

$\frac{1}{y} y' = \frac{1}{x} + \frac{1}{x+1} - 2\frac{1}{x-1} - \frac{1}{x^2+1} 2x$

$y' = y\left[\frac{1}{x} + \frac{1}{x+1} - \frac{2}{x-1} - \frac{2x}{x^2+1}\right]$

$= \frac{x(x+1)}{(x-1)^2(x^2+1)}\left[\frac{1}{x} + \frac{1}{x+1} - \frac{2}{x-1} - \frac{2x}{x^2+1}\right]$

5- $y = x^x \Rightarrow \ln y = \ln x^x = x \ln x$

$\Rightarrow \frac{1}{y} y' = x \frac{1}{x} + \ln x \cdot 1 = 1 + \ln x$

$y' = y[1 + \ln x] = x^x[1 + \ln x]$.

laws of integrations

1- $\int \frac{du}{u} = \ln|u| + C$

2- $\int e^u du = e^u + C$

Ex. Evaluate

1- $\int \frac{dx}{x+1} = \ln|x+1| + C$

2- $\int e^{x^2} x dx = \frac{1}{2}\int e^{x^2} 2x dx = \frac{1}{2} e^{x^2} + C$

3- $\int \tan x\,dx = -\int \frac{-\sin x}{[\cos x]}\,dx = -\ln|\cos x| + C$

4- $\int \frac{[\ln x]^1 dx}{x} = \frac{[\ln x]^2}{2} + C$

5- $\int \sec x\,dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\,dx = \int \frac{\sec^2 x + \sec x \tan x}{[\tan x + \sec x]^1}\,dx$

$= \ln|\tan x + \sec x| + C$

<u>The logarithm function and a^x function</u> we have the functions 1- $y = \log_a x = \frac{\ln x}{\ln a}$

2- $y = a^x = e^{x \ln a}$

<u>properties</u>

1- $\log_a(x \cdot y) = \log_a x + \log_a y$

2- $\log_a x^y = y \log_a x$

3- $\log_a(x/y) = \log_a x - \log_a y$

4- $a^x \cdot a^y = a^{x+y}$

5- $\frac{a^x}{a^y} = a^{x-y}$

6- $[a^x]^y = a^{xy}$

7- $a^1 = a \Rightarrow a^1 = e^{1 \ln a} = e^{\ln a} = a$, $a^1 \cdot a^1 = a \cdot a = a^2, \ldots$

<u>laws of derivative</u> 1- If $y = \log_a u \Rightarrow y' = \frac{1}{u} \frac{1}{\ln a} \frac{du}{dx}$

2- If $y = a^u \Rightarrow y' = a^u \frac{du}{dx} \ln a$

Ex. Find $\frac{dy}{dx}$ for the following :

1- $y = \log_2 x^2 \Rightarrow y' = \frac{1}{x^{\cancel{2}}} \cdot 2\cancel{x} \frac{1}{\ln 2} = \frac{2}{x \ln 2}$

2- $y = \log_3 \sin x \Rightarrow y' = \frac{1}{\sin x} \cdot \cos x \frac{1}{\ln 3}$

3- $y = 5^{\sin x} \Rightarrow y' = 5^{\sin x} \cos x \ln 5$

4- $y = 2^{\ln x} \Rightarrow y' = 2^{\ln x} \cdot \frac{1}{x} \ln 2$

5- $y = (10)^{\tan x} \Rightarrow y' = 10^{\tan x} \sec^2 x (\ln 10)$

Laws of integrations

1- $\int \frac{du}{u \ln a} = \log_a |u| + c$

2- $\int a^u du \ln a = a^u + C$

Ex. Evaluate

1- $\int \frac{dx}{x \ln 2} = \log_2 |x| + c$

2- $\int 3^{\sin x} \cos x dx \ln 3 = 3^{\sin x} + C$

limit of the funs

Ex. Evaluate

1- $\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = \frac{1}{\infty} = 0$

2- $\lim_{x \to \infty} 3^x = \lim_{x \to -\infty} 3^x = 3^{-\infty} = \frac{1}{3^{\infty}} = \frac{1}{\infty} = 0$