Partial differential equations of the first order

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2.1 Partial differential equations

Def

Any equation of the form

$$F\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \dots, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \dots, \frac{\partial^2 z}{\partial x \partial t}, \dots\right) = 0$$

is called paratial differential equation.

Def

The order of a partial differential equations is the highest order occurring in the equation.

Ex

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial t}$ second degree in two variable

$\left(\frac{\partial z}{\partial x}\right)^3 + \frac{\partial z}{\partial t} = 0$ first order in two variable

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + \frac{\partial z}{\partial t} = 0$ first order in three variable

Def

The equation of the form $F\left(z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \dots\right) = 0$ is called partial differential equation of first order.

Note

If $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$ then the P.D.E can be written as

$$f(x, y, z, p, q) = 0$$

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2.2 Linear equation of the first order

Def

The equation of the form $Pp + Qq = R$ where P, Q and R are given function of x, y, z (which do not involve p or q) is called linear equation of the first order (lagrange equation).

In general

Linear P.D.E of first order in n-independent variable

$$X_1 P_1 + X_2 P_2 + \dots + X_n P_n = Y$$

where $X_1, X_2, \dots, X_n, Y$ [illegible]

varaible $x_1, x_2, \ldots, x_n$ and dependent variable f ;
$\frac{\partial f}{\partial x_i} = P_i \quad (i = 1, 2, \ldots n)$

Ex

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z^2 + x^2$ L.P.D.E. of first order.

$x\frac{\partial z}{\partial x} = z^2 + x^2$ is not L.P.D.E.

Theorem

The general solution of the L.P.D.E $Pp + Qq = R$ is $F(u, v) = 0$ where F is an arbitrary function and $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ form a solution of the equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Example ① ② ③

Find the general solution of D.E.

$$x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = (x+y)z$$

Solution

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$$

$$\frac{dx}{dx^2} = \frac{dy}{y^2} \Rightarrow \frac{1}{x} - \frac{1}{y} = c_1 \Rightarrow \frac{y-x}{xy} = c_1 \quad \text{—— ①}$$

$$\frac{dx - dy}{x^2 - y^2} = \frac{dz}{(x+y)z} \Rightarrow \frac{dx - dy}{(x-y)} = \frac{dz}{z}$$

$$\Rightarrow \ln(x-y) = \ln z + c \Rightarrow \ln\left(\frac{x-y}{z}\right) = \ln c$$

$$\Rightarrow \frac{x-y}{z} = c_2 \quad \text{—— ②}$$

from ① & ② $\frac{xy}{z} = c_3$

The solution is $F\left(\frac{xy}{z}, \frac{x-y}{z}\right) = 0$

Example

Solve $(x^2 - y^2 - z^2)\frac{\partial z}{\partial x} + 2xy\frac{\partial z}{\partial y} = 2xz$.

Solution

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

$$\frac{dy}{2xy} = \frac{dz}{2xz} \Rightarrow \frac{dy}{y} = \frac{dz}{z}$$

Fourier Transform

Def:

Fourier transform of $f(t)$ is defined as:-

$$\boxed{F(p) = \mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-ipt}\, dt}$$

Def:

The inverse Fourier transform of $F(p)$ is defined as:-

$$\boxed{f(t) = \mathcal{F}^{-1}[F(p)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(p)\, e^{ipt}\, dp}$$

Example

Find fourier transform of $f(t)$ if:-

$$f(t) = \begin{cases} 1-t^2 & |t|<1 \\ 0 & |t|>1 \end{cases}$$

Then prove that $\int_0^{\infty} \left(\frac{\sin p - p\cos p}{p^3} \right) \cos \frac{p}{2}\, dp = \frac{3\pi}{16}$

Solution

$$F(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-ipt}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-t^2)\, e^{-ipt}\, dt$$

u		dv
$1-t^2$	$\oplus$	e^{-ipt}
$-2t$	$\ominus$	$-\frac{1}{ip} e^{-ipt}$
-2	$\oplus$	$-\frac{1}{p^2} e^{-ipt}$
0		$\frac{1}{ip^3} e^{-ipt}$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1-t^2}{ip} e^{-ipt} - \frac{2t}{p^2} e^{-ipt} - \frac{2}{ip^3} e^{-ipt} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\frac{2}{p^2} e^{-ip} - \frac{2}{ip^3} e^{-ip} - \frac{2}{p^2} e^{ip} + \frac{2}{ip^3} e^{ip} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{4}{p^3} \frac{e^{ip} - e^{-ip}}{2i} - \frac{4}{p^2} \frac{e^{ip} + e^{-ip}}{2} \right]$$

$$= 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin p}{p^3} - \frac{\cos p}{p^2} \right] = 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin p - p\cos p}{p^3} \right]$$

From inverse fourier transform of $F(p)$ we have:

$$f(t) = \mathcal{F}^{-1}[F(p)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin p - p\cos p}{p^3} \right] e^{ipt}\, dp$$

$$= \begin{cases} 1-t^2 & |t|<1 \\ 0 & |t|>1 \end{cases}$$

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$e^{i\theta} = \cos\theta + i\sin\theta$

$$\therefore \frac{2}{\pi}\int_{-\infty}^{\infty}\left[\frac{\sin p - p\cos p}{p^3}\right] e^{ipt}\,dp = \begin{cases} 1-t^2 & |t|<1 \\ 0 & |t|>1 \end{cases}$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin p - p\cos p}{p^3}\cos pt\,dp + \frac{2i}{\pi}\int_{-\infty}^{\infty}\underbrace{\frac{\sin p - p\cos p}{p^3}}_{\text{odd}}\underbrace{\sin pt}_{\text{odd}}\,dp = \begin{cases} 1-t^2 \\ 0 \end{cases}$$

odd odd even odd

Since the function in second integral is odd then it value is zero.

$$\frac{4}{\pi}\int_0^{\infty}\left[\frac{\sin p - p\cos p}{p^3}\right]\cos pt\,dp = \begin{cases} 1-t^2 & |t|<1 \\ 0 & |t|>1 \end{cases}$$

if $t = \frac{1}{2}$

$$\frac{4}{\pi}\int_0^{\infty}\left(\frac{\sin p - p\cos p}{p^3}\right)\cos\frac{p}{2}\,dp = \frac{3}{4}$$

$$\int_0^{\infty}\left(\frac{\sin p - p\cos p}{p^3}\right)\cos\frac{p}{2}\,dp = \frac{3\pi}{16}$$

Fourier Cosine transform

suppose $f(t)$ is an even function (i.e. $f(-t) = f(t)$)

Then Fourier transform of $f(t)$ is:-

$$\mathcal{F}[f(t)] = F(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{-ipt}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} f(t)\,e^{-ipt}\,dt + \frac{1}{\sqrt{2\pi}}\int_0^{\infty} f(t)\,e^{-ipt}\,dt$$

put $t = -T$ in first integral:-

$$\mathcal{F}[f(t)] = -\frac{1}{\sqrt{2\pi}}\int_{+\infty}^{0} f(-T)\,e^{ipT}\,dT + \frac{1}{\sqrt{2\pi}}\int_0^{\infty} f(t)\,e^{-ipt}\,dt$$

$t = -T$

t	T
0	0
$-\infty$	∞

$$= \frac{1}{\sqrt{2\pi}}\int_0^{\infty} f(T)\,e^{ipT}\,dT + \frac{1}{\sqrt{2\pi}}\int_0^{\infty} f(t)\,e^{-ipt}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^{\infty} f(t)\left[e^{ipt} + e^{-ipt}\right]dt$$

$$= \frac{2}{\sqrt{2\pi}}\int_0^{\infty} f(t)\cos pt\,dt$$

So Fourier transform to an even function:-

$$\mathcal{F}[f(t)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\cos pt\,dt$$

Solution of Non-Homogeneous Equations

Recall that:- A PDE is called homogeneous if all the partial deriva
have the same order

Ex $\frac{\partial^2 z}{\partial x^2} - x\frac{\partial^2 z}{\partial x \partial y} = 4x^2$

$x^2\frac{\partial^3 z}{\partial x^3} + 5xy\frac{\partial^3 z}{\partial x^2 \partial y} - \frac{\partial^3 z}{\partial x^3} = x^2+y^2$ $\Big\}$ → are homogeneous

$x^2\frac{\partial^3 z}{\partial x^3} + 5xy\frac{\partial^3 z}{\partial x^2 \partial y} + \frac{\partial^2 z}{\partial x^2} = x^2$ $\Big\}$ → are not homogeneous.

$\frac{\partial z}{\partial x} + xz = y$ لأن $\frac{\partial z}{\partial x}$ درجتها واحد ، z واحد برضه

In this chapter we find the solution of non-homogeneous.

$$\boxed{f(D_x, D_y)\, z = f(x,y)}$$

this solution contains two parts:-

① General solution (z_1) of the equation $f(D_x, D_y)\, z = 0$

② special solution (z_2) which has the same formulas as the method in the perivous chapter.

first cases: PDE with constant coefficient

[1] when $f(D_x, D_y) = aD_x + bD_y + c$

$$\boxed{\text{The solution of } (aD_x + bD_y + c)\, z = 0 \text{ is } z = e^{-\frac{cy}{b}}\phi(ay - bx) \quad \text{where } a, b, c \text{ are constants}}$$

Ex solve $(2D_x - D_y + 3)\, z = 0$

$a = 2 \quad b = -1 \quad c = 3$

$z_1 = e^{3y}\phi(2y + x)$

Ex solve $(D_x - 3D_y + 2)\, z = 0$

$a = 1 \quad b = -3 \quad c = 2$

$z_1 = e^{\frac{2}{3}y}\phi(y + 3x)$

Ex $(D_y + 2)\, z = 0$

$a = 0 \quad b = 1 \quad c = 2$

$z_1 = e^{-2y}\phi(-x)$

when $f(D_x, D_y) = (aD_x + bD_y + c)^k$

The solution of $(aD_x + bD_y + c)^k z = 0$ is :-

$$z = e^{-\frac{cy}{b}}\left[\phi_1(ay-bx) + x\phi_2(ay-bx) + x^2\phi_3(ay-bx) + \cdots + x^{k-1}\phi_k(ay-bx)\right]$$

where $\phi_1, \phi_2, \ldots \phi_k$ are arbitrary function

Ex solve $(D_x - 2D_y + 1)^4 z = 0$

$a = 1,\ b = -2,\ c = 1 \qquad k = 4$

$$z_1 = e^{\frac{y}{2}}\cdot\left[\phi_1(y+2x) + x\phi_2(y+2x) + x^2\phi_3(y+2x) + x^3\phi_4(y+2x)\right]$$

Ex solve $(3D_x + 4D_y + 5)^3 z = 0$

$a = 3 \quad b = 4 \quad c = 5 \quad k = 3$

$$z_1 = e^{-\frac{5}{4}y}\left[\phi_1(3y-4x) + x\phi_2(3y-4x) + x^2\phi_2(3y-4x)\right]$$

3 $f(D_x, D_y)$ written as Linear factor of form ① & ②

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Ex $(2D_x - D_y + 1)^2(D_x + 3D_y + 4)z = 0$

$k = 2,\ a = 2,\ b = -1,\ c = 1 \qquad k = 1,\ a = 1,\ b = 3,\ c = 4$

$$z_1 = e^{y}\left[\phi_1(2y+x) + x\phi_2(2y+x)\right] + e^{-\frac{4}{3}y}\left[\phi_3(y-3x)\right]$$

Ex $(2D_x - D_y + 6)(D_x - 5D_y + 8)^2 z = 0$

$k = 1,\ a = 2,\ b = -1,\ c = 6 \qquad k = 2,\ a = 1,\ b = -5,\ c = 8$

$$z_1 = e^{6y}\left[\phi_1(2y+x)\right] + e^{\frac{8}{5}y}\left[\phi_2(y+5x) + x\phi_2(y+5x)\right]$$

H.W $(2D_x + 3D_y - 5)(D_y + 2D_y)(D_x + D_y)(D_y - 2)z = 0$

Formation of PDE

The PDE can be formed by one of:-

① elimination of arbitrary constant.

② elimination of arbitrary function from a relation involving three or more variable.

Ex

Derive a PDE (by eliminating the constant) from the equation $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Sol

Differentiate with respect to x and y we get

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2} \Rightarrow \frac{1}{a^2} = \frac{1}{x}\frac{\partial z}{\partial x} = \frac{p}{x}$$

$$2\frac{\partial z}{\partial y} = \frac{2y}{b^2} \Rightarrow \frac{1}{b^2} = \frac{1}{y}\frac{\partial z}{\partial y} = \frac{q}{y}$$

$$2z = xp + yq.$$

Note

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad \frac{\partial^2 z}{\partial x^2} = r \quad \frac{\partial^2 z}{\partial x \partial y} = s \quad \frac{\partial^2 z}{\partial y^2} = t.$$

Ex

find a PDE from the relation $z = ax^3 + by^3$ where a and b are arbitrary constant.

Sol

$$\frac{\partial z}{\partial x} = 3ax^2 \Rightarrow a = \frac{1}{3x^2}\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial y} = 3by^2 \Rightarrow b = \frac{1}{3y^2}\frac{\partial z}{\partial y}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$$

Ex find a PDE from the relation $z = ax^2 + bxy + cy^2$ where $a, b,$ and c be a constant

Sol

$$\frac{\partial z}{\partial x} = 2ax + by \xrightarrow{\text{نضرب } x} x\frac{\partial z}{\partial x} = 2ax^2 + bxy$$

$$\frac{\partial z}{\partial y} = bx + 2cy \xrightarrow{\text{نضرب } y} y\frac{\partial z}{\partial y} = bxy + 2cy^2$$

$$\text{نجمع} \Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$$

(2)

Ex Derive a PDE (by eliminating the arbitrary funct
fro—

(1) $z = f(x^2+y^2)$

$$\left.\begin{aligned} p &= \frac{\partial z}{\partial x} = f'(x^2+y^2)(2x) \\ q &= \frac{\partial z}{\partial y} = f'(x^2+y^2)(2y) \end{aligned}\right\} \longrightarrow \frac{p}{q} = \frac{x}{y} \Rightarrow yp - xq = 0$$

(2) $z = f(x+it) + g(x-it)$

$$\frac{\partial z}{\partial x} = f'(x+it) + g'(x-it)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+it) + g''(x-it)$$

$$\frac{\partial z}{\partial t} = if'(x+it) - ig'(x-it)$$

$$\frac{\partial^2 z}{\partial t^2} = i^2 f''(x+it) + i^2 g''(x-it) = -f''(x+it) - g''(x-i$$

$$\therefore \frac{\partial^2 z}{\partial t^2} = -\frac{\partial^2 z}{\partial x^2}$$

$$\frac{\partial^2 z}{\partial t^2} + \frac{\partial^2 z}{\partial x^2} = 0.$$

(3) $z = f(x) + g(y)$

$$\frac{\partial z}{\partial y} = g'(y) \Rightarrow \frac{\partial^2 z}{\partial x \partial y} = 0$$

(4) $z = f(x+3y) + g(x-3y)$.

$$\frac{\partial z}{\partial x} = f'(x+3y) + g'(x-3y)$$

$$\left.\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= f''(x+3y) + g''(x-3y) \\ \frac{\partial z}{\partial y} &= 3f'(x+3y) - 3g'(x-3y) \\ \frac{\partial^2 z}{\partial y^2} &= 9f''(x+3y) + 9g''(x-3y) \end{aligned}\right] \Rightarrow \frac{\partial^2 z}{\partial y^2} = 9\frac{\partial^2 z}{\partial x^2}$$

$$\cancel{\frac{\partial^2 z}{\partial x \partial y} = 3f''(x+3y) - 3g''(x-3y)}$$

$$\cancel{\frac{\partial^2 z}{\partial y^2} = 3\frac{\partial^2 z}{\partial x \partial y}}$$