Chapter Eleven/Normal shock in converging–diverging nozzles

We have discussed the isentropic operations of a converging–diverging nozzle. This type of nozzle is physically distinguished by its area ratio, the ratio of the exit area to the throat area. Furthermore, its flow conditions are determined by the operating pressure ratio, the ratio of the receiver (back) pressure to the inlet stagnation (reservoir) pressure ($p_b/p_\text{reservoir}$). From figure (11.1) we identified two significant critical pressure ratios.

With $p_b = p_T$, there is no flow in the nozzle (curve 1) from figure (11.1a). As $p_b$ is reduced below $p_T$, subsonic flow is induced through the nozzle, with pressure decreasing to the throat, and then increasing in the diverging portion of the nozzle (curve 2 and 3). For any pressure ratio above $p_{b,a}/p_T$, for curve (a), the nozzle is not choked and has subsonic flow throughout (typical venturi operation). When the back pressure is lowered to that of curve a, sonic flow occurs at the nozzle throat. Pressure ratio $p_{b,a}/p_T$ is called the \textit{first critical point} which represents flow that is subsonic in both the convergent and divergent sections but is choked with a Mach number of 1.0 in the throat. (\textit{chocked means flow maximum and fixed})

When the back pressure is lowered to that of curve f, subsonic flow exits in the converging section, and sonic flow exits in the throat and it is choked where $M = 1.0$. A supersonic flow exists in the entire diverging section. This is the \textit{third critical point} which represents the design operation condition.

The first and third critical points are the only operating points that have:

1. Isentropic flow throughout the nozzle, and
2. A Mach number of 1 at the throat, and
3. Exit pressure equal to receiver (surrounding) pressure.

Remember that with subsonic flow at the exit, $p_e = p_b$, and $p_b$ is back or receiver pressure.

Imposing a pressure ratio slightly below that of the first critical point presents a problem in that there is no way that isentropic flow can meet the boundary condition of pressure equilibrium at the exit. However, there is nothing to prevent a non-isentropic flow adjustment from occurring within the nozzle. This internal adjustment takes the form of a standing \textit{normal shock}, which we now know involves an entropy change (losses).
As the pressure ratio is lowered below the first critical point, a normal shock forms just downstream of the throat. The remainder of the nozzle is now acting as a diffuser since after the shock the flow is subsonic and the area is increasing. The shock will locate itself in a position such that the pressure changes that occur ahead of the shock, across the shock, and downstream of the shock will produce a pressure that exactly matches the outlet pressure. In other words, the operating pressure ratio determines the location and strength of the shock. An example of this mode of operation is shown in Figure 11.1b.

As the pressure ratio is lowered further, the shock continues to move toward the exit. When the shock is located at the exit plane (curve d), this condition is referred to as the second critical point.

When the operating pressure ratio is between the second and third critical points, a compression takes place outside the nozzle. This is called over-expansion (i.e., the flow has been expanded too far within the nozzle). As the back pressure is lowered below that of curve d, a shock wave inclined at an angle to the flow appears at the exit plane of the nozzle (Figure 11.2a). This shock wave, weaker than a normal shock, is called an oblique shock. Further reductions in back pressure cause the angle between the shock and the flow to decrease, thus...
decreasing the shock strength (Figure 11.2b), until eventually the isentropic case, curve f, is reached.

If the receiver pressure is below the third critical point, an expansion takes place outside the nozzle. This condition is called under-expansion. A pressure decrease occurs outside the nozzle in the form of expansion waves (Figure 11.2c). Oblique shock waves and expansion waves represent flows that are not one dimensional flow and will be treated later.

Illustrative example:

For the present we proceed to investigate the operational regime between the first and second critical points. For the nozzle and inlet conditions illustrated in figure (11.3), the nozzle has area ratio to be $A_3/A_2 = 2.494$ and is fed by air at 6.0 bar and 60 °C from a large tank.

Solution

The inlet conditions are essentially stagnation. For these fixed inlet conditions we find that a receiver pressure of 5.7642 bar (for operating pressure ratio of 0.9607) identifies the first critical point and a receiver pressure of 0.3858 bar (for operating pressure ratio of 0.06426) identifies the third critical point.

What receiver pressure do we need to operate at the second critical point? Figure 11.4 shows such a condition and you should recognize that the entire nozzle up to the shock is operating at its design or third critical condition.

From the isentropic table at $A/A* = 2.494$, $M_3 = 2.44$ and $p_3/p_o3 = 0.06426$

From the normal-shock table for $M_3 = 2.44$, $M_4 = 0.5189$ and $\frac{p_4}{p_3} = 6.7792$

and the operating pressure ratio will be

$$\frac{p_{rec}}{p_o1} = \frac{p_4}{p_o3} \frac{p_4}{p_3} \frac{p_3}{p_o3} = 6.7792 \times 0.06426 = 0.436$$

$p_1 = p_{reservoir} = p_o1 = 6.0$ bar

$p_4 = p_{receiver} = 6.0 \times 0.436 = 2.616$ bar

Thus for our converging–diverging nozzle with an area ratio of 2.494, any operating pressure ratio between 0.9607 and 0.436 will cause a normal shock to be located someplace in the diverging portion of the nozzle starting...
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from the throat and ending at exit plane.

Suppose that we are given an operating pressure ratio of 0.60. The logical question to ask is: Where is the shock? This situation is shown in Figure 11.5. We must take advantage of the only two available pieces of information and from these construct a solution. We know that

\[
\frac{A_5}{A_2} = 2.494 \quad \text{and} \quad \frac{p_5}{p_{01}} = 0.60
\]

We assume that all losses occur across the shock and we know that \( M_2 = 1.0 \). Since there are no losses up to the shock, the flow is isentropic and we know that

\[ A_2 = A_1^* \]

Thus

\[
\frac{A_5}{A_2} \cdot \frac{p_5}{p_{01}} = \frac{A_5}{A_1^*} \cdot \frac{p_5}{p_{01}}
\]

We know also across the normal shock \( p_{05} A_5^* = p_{01} A_1^* \), i.e.

\[
\frac{p_{05}}{p_{01}} = \frac{A_1^*}{A_5^*}
\]

So

\[
\frac{A_5}{A_1^*} \cdot \frac{p_5}{p_{01}} = \frac{A_5}{A_5^*} \cdot \frac{p_5}{p_{05}}
\]

The following data is known, \( A_5/A_2 = 2.494, p_5/p_{01} = 0.60 \) then;

\[
\frac{A_5}{A_5^*} \frac{p_5}{p_{05}} = 2.494 \cdot 0.60 = 1.4964
\]

And from isentropic table at \( A_5^* \), \( p_5/A_5^*, p_{05} = 1.4964 \)

\( M_5 \approx 0.38 \) and \( p_5/p_{05} = 0.9052 \)

To locate shock position, we seek the ratio \( p_{04}/p_{03} \).

We have \( p_{05} = p_{04} \), isentropic after the shock, and \( p_{03} = p_{01} \), isentropic before the shock. Then

\[
\frac{p_{04}}{p_{03}} = \frac{p_{05}}{p_{01}} = \frac{p_{05}}{p_{01}} \cdot \frac{p_5}{p_{01}} = \frac{1}{0.902} \cdot 0.60 = 0.664
\]

Then from normal shock table at \( p_{04}/p_{03} = 0.664 \)

\( M_3 = 2.12 \quad \text{and} \quad M_4 = 0.5583 \)

And then from the isentropic table that this Mach number, \( M_3 = 2.12 \), will occur at an area ratio of about \( A_3/A^* = A_3/A_2 = 1.869.. \)
We see that if we are given a physical converging–diverging nozzle (area ratio is known) and an operating pressure ratio between the first and second critical points, it is a simple matter to determine the position and strength of the normal shock in the diverging section.

**Example 11.1** A converging–diverging nozzle has an area ratio of 3.50. At off-design conditions, the exit Mach number is observed to be 0.3. What operating pressure ratio would cause this situation?

**Solution**

We have the nozzle area ratio \(A_5/A_2 = 3.5\).

Using the section numbering system of Figure 10.6, for \(M_5 = 0.3\), We have

\[
\frac{A_5}{A_5} = 1.9119, \quad \frac{A_5}{A_5} = 2.03507
\]

\[
p_{05}A_5^* = p_{01}A_1^*
\]

\[
\frac{p_5}{p_{01}} = \left( \frac{p_5}{p_{05}A_5^*} \right) \left( \frac{p_{05}A_5^*}{p_{01}A_1^*} \right) \left( \frac{A_1^*}{A_2} \right) A_5 = 1.9119 \cdot 1 \cdot 1 \cdot \frac{1}{3.5} = 0.546
\]

Could you now find the shock location and Mach number?

\[
\frac{p_{05}}{p_{01}} = \frac{A_1^*}{A_5} = \frac{A_1^*}{A_5} = \frac{1}{3.5} \cdot 2.03507 = 0.58145 = \frac{p_{04}}{p_{03}}
\]

From shock table at \(p_{04}/p_{03} = 0.58145\) gives \(M_3 = \)

From isentropic table at \(M_3 = \) gives \(A_3/A_3^* = A_3/A_2 = \)

**Example 11.2** Air enters a converging–diverging nozzle that has an overall area ratio of 1.76. A normal shock occurs at a section where the area is 1.19 times that of the throat. Neglect all friction losses and find the operating pressure ratio. Again, we use the numbering system shown in Figure 11.6.

**Solution**

From the isentropic table at \(A_3/A_2 = 1.19, \rightarrow M_3 = 1.52\).

From the shock table at \(M_3 = 1.52, \rightarrow M_4 = 0.6941\) and \(p_{04}/p_{03} = 0.9233\).

From isentropic table at \(M_4 = 0.6941\) gives \(A_4/A_4^* = 1.0988\. Then
Example 11.3 A converging-diverging nozzle is designed to operate with an exit Mach number of 1.75. The nozzle is supplied from an air reservoir at 5 MPa. Assuming one-dimensional flow, calculate the following:

a) Maximum back pressure to choke the nozzle.

b) Range of back pressures over which a normal shock will appear in the nozzle.

c) Back pressure for the nozzle to be perfectly expanded to the design Mach number.

d) Range of back pressures for supersonic flow at the nozzle exit plane.

Solution

The nozzle is designed for \( M_{exit} = 1.75 \). From Appendix A. at \( M_{exit} = 1.75 \), \( A_{exit}/A^* = 1.386 \) and \( p_{exit}/p_o = 0.1878 \).

a) The nozzle is choked with \( M = 1 \) at the throat, followed by subsonic flow in the diverging portion of the nozzle. From Appendix A. at \( A_{exit}/A^* = 1.386 \). \( M_{exit} = 0.477 \) and \( p_{exit}/p_o = 0.8558 \).

\[ p_{exit} = p_{exit}/p_o \times p_o = 0.8558 \times 5 = 4.279 \text{ MPa} \]

Therefore the nozzle is choked for all back pressures below 4.279 MPa.

b) Or a normal shock at the nozzle exit plane (Figure 11.7b). \( M_1 = 1.75 \) and \( p_1 = 0.1878 \times 5 = 0.939 \text{ MPa} \).

From normal shock, at \( M_1 = 1.75 \), \( p_2/p_1 = 3.406 \).

For a normal shock at the nozzle exit, the back pressure is

\[ p_b = 3.406(0.939) = 3.198 \text{ MPa} \]

For a shock just downstream of the nozzle throat, the back pressure is \( p_b = 4.279 \text{ MPa} \), i.e. the flow downstream the throat in the divergent part is subsonic. So a normal shock will appear in the nozzle over the range of back pressures from 3.198 to 4.279 MPa.
c) From isentropic table, at $M_{exit} = 1.75$, $p_{exit}/p_o = 0.1878$. For a perfectly expanded, supersonic nozzle, the back pressure is 0.939 MPa.

d) Referring again to Figure 11.7a supersonic flow will exist at the nozzle exit plane for all back pressures less than 3.198 MPa.
Chapter Twelve/Converging–Diverging Supersonic Diffusers

12.1 Converging-Diverging Supersonic Diffuser

With the jet engine, the inlet (diffuser) takes the incoming air, traveling at high velocity with respect to the engine, and slows it down and then delivers it to the axial compressor of the turbojet or the combustion zone of the ramjet engine. The amount of static pressure rise achieved during deceleration of the flow in the diffuser is very important to the operation of the jet engine, since the pressure of the air entering the nozzle affects the nozzle exhaust velocity.

The maximum pressure that can be achieved in the diffuser is the isentropic stagnation pressure. Any loss in available energy (or stagnation pressure) in the diffuser, or for that matter in any other component of the engine, will have a harmful effect on the operation of the engine as a whole. For a supersonic diffuser, it would be highly desirable to provide shock free isentropic flow.

A first approach is to operate a converging-diverging nozzle in reverse (see Figure 12.1.) At the design Mach number, $M_D$, for such a diffuser, there is no loss in stagnation pressure (neglecting friction). However, off-design performance has to be considered, since the external flow must be accelerated to the design condition. For example, if a supersonic converging-diverging diffuser is to be designed for a flight $M_D = 2.0$, the ratio $A_{inlet}/A_{throat}$ is $1.688$ (see isentropic flow table).

However, for a supersonic flight Mach number less than design Mach number, $M < M_D$, the area ratio $A/A^*$ is less than $1.668$, i.e. required throat area should be larger. This indicates that the actual throat area is not large enough to handle this flow. Under these conditions, flow must be bypassed around the diffuser. A normal shock stands in front of the diffuser with subsonic flow after the shock able to sense the presence of the inlet and an appropriate amount of the flow "spills over" or bypasses the inlet (see Figure 12.2).

As the Might Mach number is increased, the normal shock moves toward the inlet lip. When the design Mach number is reached during start-up, however, with a normal shock in front of the diffuser, some of the flow must still be bypassed, since the throat area of less than $A'_2$ is still not able to handle the entire subsonic flow after the shock.

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As the flight Mach number is increased above $M_D$, the shock moves eventually to the inlet lip. A further increase in $M$ causes the shock to reach a new equilibrium position in the diverging portion of the diffuser, in other words, the shock is "swallowed." Once the shock has been swallowed, a decrease in flight Mach number causes the shock to move back toward the throat, where it reaches an equilibrium position for $M$ equal to $M_D$.

At this position, the shock is of vanishing strength, at $M_t = 1.0$, so no loss in stagnation pressure occurs at the design condition. In actual operation, it is desirable to operate with the shock slightly past the throat; since operation at the design condition is unstable in that a slight decrease in Mach number results in the shock's moving back out in front of the inlet. In this case, the operation of over speeding to swallow the shock would have to be repeated (see Figure 12.3).

Another method for swallowing the shock is to use a variable throat area. With a shock in front of the diffuser, the throat area should be increased, which would allow more flow to pass through the inlet and consequently bring the shock closer to the inlet lip. To swallow the shock, the throat area would have to be slightly larger than that required to accept the flow with a shock at $M_D$ at the inlet lip, that is, slightly larger than $A_2^*$ with a normal shock at the design Mach number.

For $M_D = 2.0$, $A_1^*/A_2^* = 0.7209$, so an increase in area of greater than $(1 - 0.7209)/0.7209 = 39\%$ is required to swallow the shock. Once the shock is swallowed, the throat area must be decreased to reach the design condition.

Although the converging-diverging diffuser has favorable operating characteristics at the design condition, it involves severe losses at off-design operation. Operation with a normal shock in front of an inlet causes losses in the stagnation pressure.

To swallow this shock, the inlet must be accelerated beyond its design speed, or a variable throat area must be provided. Except for very low supersonic Mach numbers, the amount of over speeding required to swallow the shock during start-up becomes large enough to be totally impractical.
Furthermore, the incorporation of a variable throat area into a diffuser presents many mechanical difficulties. For these reasons, the converging-diverging diffuser is not commonly used; most engines utilize the oblique-shock type diffuser to be described later.

**Example 12.1.** A supersonic converging-diverging diffuser is designed to operate at a Mach number of 1.7 with design back pressure. To what Mach number would the inlet have to be accelerated in order to swallow the shock during stand-up?

**Solution**
From isentropic table at $M_{inlet} = 1.7 \Rightarrow A/A^* = 1.338$
So the diffuser is designed with $A_{inlet}/A_{throat} = 1.338$
The inlet must be accelerated to a Mach number slightly greater than that required to position the shock at the inlet lip (see Figure 12.4).
Assume a normal shock stands at diffuser lips as shown. For $M = 1.0$ at the diffuser throat and subsonic flow after a shock at the inlet lip, we have:
From isentropic table at $A/A^* = 1.338 \Rightarrow M_2 = 0.501$.
From normal shock table at $M_2 = 0.501 \Rightarrow M_1 = 2.63$.
If the back pressure conditions imposed on the diffuser are such that a Mach number of 1.0 cannot be achieved at the throat, then $M_2$ will be less than 0.501, and a value of $M_1$ greater than 2.63 will be required. However, with $M = 1.0$ at the diffuser throat, the diffuser must be accelerated to a Mach number slightly greater than 2.63 to swallow the initial shock during start-up.

6.7 Supersonic Wind Tunnel

To provide a test section with supersonic flow requires a converging–diverging nozzle. To operate economically, the nozzle–test-section combination must be followed by a diffusing section which also must be converging–diverging.
Starting up such a wind tunnel is another example of nozzle operation at pressure ratios above the second critical point. Figure 12.5 shows a typical tunnel in its *most unfavorable, off design*, operating condition, which occurs at startup.
Figure 12.5, which shows the shock located in the test section. The variation of Mach number throughout the flow system is also shown for this case. This is called the most unfavorable condition because the shock occurs at the highest possible Mach number and thus the losses are greatest. We might also point out that the diffuser throat (section 5) must be sized (adjusting area) for this condition.

As the exhauster fan is started, this reduces the pressure $p_{out} = p_6$ and produces flow through the tunnel. At first the flow is subsonic throughout, but at increased power settings the exhauster fan reduces pressures still further and causes increased flow rates until the nozzle throat (section 2) becomes choked. At this point the nozzle is operating at its first critical condition. As power is increased further, i.e the ratio $p_{out}/p_{in}$ is lowered further, a normal shock is formed just downstream of the throat, and if the tunnel pressure is decreased continuously, the shock will move down the diverging portion of the nozzle and pass rapidly through the test section and into the diffuser. If the ratio $p_{out}/p_{in}$ is lowered further then the diffuser swallows the normal shock to the diverging part.
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of diffuser. Increasing this pressure ratio a little will move the normal shock upstream to the
diffuser throat, the position at which the shock strength is a minimum. Figure 12.6 shows this
general running condition, which is called the most favorable condition.

Across the shock of figure 12.5

\[ p_{o2} A_2^* = p_{o5} A_5^* \]

At throat section 2 & 5 during start-up \( M = 1 \), then

\[ p_{o2} A_2 = p_{o5} A_5 \]

Due to the shock losses (and other friction losses) \( p_{o5} < p_{o2} \) and then \( A_5 > A_2 \)

For example if the test section Mach number is 2 then from normal shock table

\[ \frac{p_{o5}}{p_{o2}} = 0.7209 = \frac{A_2}{A_5} \]

And \( A_5 = 1/0.7209 \quad A_2 = 1.3872 \quad A_2 \)

Knowing the test-section-design Mach number fixes the shock strength in this unfavorable
condition and \( A_5 \) is easily determined. Keep in mind that this represents a minimum area for the
diffuser throat. If it is made any smaller than this, the tunnel could never be started (i.e., we
could never get the shock into and through the test section). In fact, if \( A_5 \) is made too small, the
flow will choke first in this throat and never get a chance to reach sonic conditions in section 2.

Once the shock has passed into the diffuser throat, knowing that \( A_5 > A_2 \) we realize that the
tunnel can never run with sonic velocity at section 5. Thus, to operate as a diffuser, there must be
a shock at this point, as shown in Figure 12.6. We have also shown the pressure variation through
the tunnel for this running condition.

To keep the losses during running at a minimum, the shock in the diffuser should occur at the
lowest possible Mach number, which means a small throat. However, we have seen that it is
necessary to have a large diffuser throat in order to start the tunnel. A solution to this dilemma
would be to construct a diffuser with a variable area throat. After startup, \( A_5 \) could be decreased,
with a corresponding decrease in shock strength and operating power. However, the
power required for any installation must always be computed on the basis of the unfavorable startup
condition.

**Figure 12.7** Continuous Closed-Circuit Supersonic Wind Tunnel

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Example:

A continuous supersonic wind tunnel is designed to operate at a test section Mach number of 2.0, with static conditions duplicating those at an altitude of 20 km where \( p = 5.5 \text{ kPa} \) and \( T = 216.7 \text{ K} \). Take \( \gamma = 1.4 \) and \( c_p = 1.004 \text{ kJ/kg.K} \). The test section is to be circular, 25 cm in diameter, with a fixed geometry and with a supersonic diffuser downstream of the test section. Neglecting friction and boundary-layer effects, determine the power requirements of the compressor during startup and during steady-state operation, [See Figure 12.8(a)]. Assume an isentropic compressor, with a cooler located between compressor and nozzle (after the compressor), so the compressor inlet static temperature can be assumed equal to the test section stagnation temperature.

solution

During startup, the worst possible case [see Figure 12.8(c)] is that of a shock in the test section with \( M_1 = 2.0 \). For this situation, which fixes the ratio of the two throat areas, we have

\[
\frac{p_{o2}}{p_{o1}} = 0.7209 = \frac{A_1^*}{A_2^*} = \frac{A_{t1}}{A_{t2}}
\]

To fix the size of the diffuser throat area, we first use the design Mach number to find \((A/A^*)_{test}\). From isentropic table, \((A/A^*)_{test} = 1.6875\)

\[
A_{test} = \pi \frac{D^2}{4} = \pi \frac{0.25^2}{4} = 0.04909 \text{ m}^2
\]

\[
\frac{T}{T_o}_{test} = 0.5556
\]

\[
T_{o1} = \frac{T_1}{\frac{T}{T_o}_{test}} = \frac{216.7}{0.5556} = 390.03 \text{ K}
\]

The throat area is then:

\[
A_1^* = A_{t1} = \frac{A_{test}}{(A/A^*)_{test}} = \frac{0.04909}{1.6875} = 0.02909 \text{ m}^2
\]

\[
A_2^* = A_{t2} = \frac{A_1^*}{A_1^*/A_2^*} = \frac{0.02909}{0.7209} = 0.04035 \text{ m}^2
\]
During steady-state operation [see Figure 12.8(a)], the mass flow through the test section is given by

$$\dot{m} = \rho AV = \frac{p_{\text{test}}}{RT_{\text{test}}} AM_{\text{test}} \sqrt{\gamma RT_{\text{test}}}$$

$$= \frac{5.5}{0.287 \times 216.7} (0.04909) 2\sqrt{1.4 \times 287 \times 216.7} = 2.5619 \text{ kg/s}$$

For this fixed geometry (i.e., $A_{t2}/A_{t1} = 1/0.7209 = 1.3872$), the optimum condition for steady-state operation is a normal shock at the diffuser throat. This means that the nozzle, test section and the converging part of the diffuser act as a duct of variable area with isentropic flow, where $M_{t1} = 1$ and $A_{t1} = A^* = 0.02909 \text{ m}^2$.

From isentropic table at $A_1/A^* = A_{t2}/A_{t1} = 1/0.7209 = 1.3872$

$$M_1 = 1.75 + 0.01 \left(\frac{1.38720 - 1.38649}{1.39670 - 1.38649}\right) = 1.7507$$

From normal shock table at $M_1 = 1.7507$

$$\frac{p_{o2}}{p_{o1}} = 0.83457 + (0.83024 - 0.83457) \left(\frac{1.7507 - 1.7500}{1.7600 - 1.7500}\right) = 0.8343$$

The loss in stagnation pressure must be compensated for by the compressor. For isentropic compressor, [see Figure 12.7(b)], the energy balance is

$$w = h_{o,\text{exit}} - h_{o,\text{inlet}} = c_p(T_{o,\text{exit}} - T_{o,\text{inlet}})$$

At design stage, i.e. steady state operation

$$\frac{T_{o,\text{exit}}}{T_{o,\text{inlet}}} = \left(\frac{p_{o1}}{p_{o2}}\right)^{\frac{\gamma - 1}{\gamma}} = \left(\frac{1}{0.8343}\right)^{0.4 \frac{14}{14}} = 1.0531$$

$$T_{o,\text{exit}} - T_{o,\text{inlet}} = T_{o,\text{inlet}}(1.0531 - 1) = 390.03 \times 0.0531 = 20.72 \text{ K}$$

$$w = 1.004(20.72) = 20.8029 \text{ kJ/kg}$$

$$Power = \dot{m}w = 2.5619 \times 20.8029 = 53.2949 \text{ kW}$$

At off-design stage, i.e. during startup

$$\frac{T_{o,\text{exit}}}{T_{o,\text{inlet}}} = \left(\frac{p_{o1}}{p_{o2}}\right)^{\frac{\gamma - 1}{\gamma}} = \left(\frac{1}{0.7209}\right)^{0.4 \frac{14}{14}} = 1.0980$$

$$T_{o,\text{exit}} - T_{o,\text{inlet}} = T_{o,\text{inlet}}(1.0980 - 1) = 390.03 \times 0.0980 = 38.223 \text{ K}$$

$$w = 1.004(38.223) = 38.376 \text{ kJ/kg}$$

$$Power = \dot{m}w = 2.5619 \times 38.376 = 98.3155 \text{ kW}$$

A more power is needed during startup by

$$\frac{98.3155 - 53.2949}{53.2949} = 84.47 \%$$
12.1 Moving Normal Shock Waves

Previous sections have dealt with the fixed normal shock wave. However, many physical situations arise in which a normal shock is moving. When an explosion occurs, a shock wave propagates through the atmosphere from the point of the explosion. As a blunt body reenters the atmosphere from space, a shock travels a short distance ahead of the body. When a valve in a gas line is suddenly closed, a shock propagates back through the gas. To treat these cases, it is necessary to extend the procedures already developed for the fixed normal shock wave.

Consider a normal shock moving at constant velocity into still air, \( T_{oa} = T_a \), and \( p_{oa} = p_a \), (Figure 13.1a). Let \( V_s \) = absolute shock velocity and \( V_g \) = velocity of gases behind the wave; both velocities are measured with respect to a fixed observer. For a fixed observer, the flow is not steady, since conditions at a point are dependent on whether or not the shock has passed over that point.

Now consider the same physical situation with an observer moving at the shock-wave velocity, a situation, for instance, with the observer "sitting on the shock wave." The shock is now fixed with respect to the observer (Figure 13.1b). But this is the same case already covered in previously. Relations have been derived and results tabulated for the fixed normal shock. To apply these results to the moving shock, consideration must be given to the effect of observer velocity on static and stagnation properties.

Static properties are defined as those measured with an instrument moving at the absolute flow velocity. Thus static properties are independent of the observer velocity, so

\[
\frac{p_2}{p_1} = \frac{p_b}{p_a} \quad \text{and} \quad \frac{T_2}{T_1} = \frac{T_b}{T_a}
\]

Stagnation properties are measured by bringing the flow to rest. Comparing the situations shown in Figure 13.1, if \( T_1 = T_a \) and \( p_1 = p_a \), it is evident that \( T_{o1} > T_a \) and \( p_{o1} > p_a \) since the gas at state 1 has velocity \( V_g \), and the gas at state a has zero velocity, \( T_a = T_{oa} \) and \( p_a = p_{oa} \). Thus stagnation properties are dependent on the observer velocity. To calculate the variation of stagnation properties across a moving shock wave, static conditions and velocities must first be determined.
Transformation of a stationary coordinate system to a coordinate system that moves with the shock makes analysis of the moving normal shock as of the steady-flow situation shown in Figure 13.1(b). The relations for stationary normal shock is now prevail.

\[ V_1 = V_s \quad V_2 = V_s - V_g \]

From continuity eq.:

\[ \frac{\rho_2 (V_s - V_g)}{\rho_1} = \frac{\rho_1 V_s}{V_s} \quad ... (13.1a) \]

\[ \frac{\rho_1}{\rho_2} = 1 - \frac{V_g}{V_s} = \frac{V_s - V_g}{V_s} \quad ... (13.1b) \]

From momentum eq.:

\[ p_2 + \rho_2 (V_s - V_g)^2 = p_1 + \rho_1 V_s^2 \quad ... (13.2) \]

\[ \frac{p_2}{p_1} = \frac{2\gamma}{\gamma + 1} M_1^2 - \frac{\gamma - 1}{\gamma + 1} \quad ... (10.1) \]

From energy eq.:

\[ h_2 + \frac{(V_s - V_g)^2}{2} = h_1 + \frac{V_s^2}{2} \quad ... (13.4a) \]

\[ T_2 + \frac{(V_s - V_g)^2}{2c_p} = T_1 + \frac{V_s^2}{2c_p} \quad ... (13.4b) \]

\[ \frac{T_2}{T_1} = \frac{1 + [(\gamma - 1)/2]M_1^2}{} \left[ \left\{ \frac{2\gamma}{(\gamma - 1)} \right\} \right] \quad ... (10.2) \]

And from eq.10.3 for velocity ratio:

\[ \frac{V_1}{V_2} = \frac{\rho_2}{\rho_1} = \frac{T_1 + p_2}{T_2 + p_1} = \frac{(\gamma + 1)M_1^2}{(\gamma - 1)M_1^2 + 2} \quad ... (10.3) \]

\[ \frac{V_s}{V_s - V_g} = \frac{(\gamma + 1) V_s^2 / \gamma R T_1}{(\gamma - 1) V_s^2 / \gamma R T_1 + 2} \quad ... (13.5) \]

**First Case:**

Either the shock velocity is known or the gas velocity behind the wave is known. When the shock velocity is known the gas velocity and other properties behind the moving wave are required. But when the velocity of the gas behind the shock is known, then shock velocity and other properties are required.

**Example 13.1** A normal shock moves at a constant velocity of 500 m/s into still air (100 kPa, 0°C). Determine the static and stagnation conditions present in the air after passage of the wave, as well as the gas velocity behind the wave.
Gas Dynamics  
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Solution

For a fixed observer, the physical situation is shown in Figure 13.3a. With respect to an observer moving with the wave, the situation transforms to that shown in Figure 13.3b.

\[
M_1 = \frac{V_s}{\sqrt{\gamma RT_1}} = \frac{500}{\sqrt{1.4 \times 287 \times 273}} = 1.510
\]

From normal shock table

\[
\frac{T_2}{T_1} = 1.327 \rightarrow T_2 = T_1 \times \frac{T_2}{T_1} = 273 \times 1.327 = 362.3 \text{ K}
\]

\[
\frac{p_2}{p_1} = 2.493 \rightarrow p_2 = p_1 \times \frac{p_2}{p_1} = 100 \times 2.493 = 249.3 \text{ K}
\]

From continuity equation

\[
\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = 1.879
\]

\[
\frac{V_1}{V_2} = \frac{V_1}{500 - V_g} = 1.879
\]

\[
V_g = 233.9 \text{ m/s}
\]

Since the velocity of the observer does not affect the static properties,

\[
p_b = 249.3 \text{ kPa}
\]

\[
T_b = 362.3 \text{ K}
\]

The Mach number of the gas flow behind the wave is given by

\[
M_g = \frac{V_g}{\sqrt{\gamma RT_b}} = \frac{233.9}{\sqrt{1.4 \times 287 \times 362.3}} = 0.613
\]

With the Mach number and static properties determined, the stagnation properties of the gas stream can be found from isentropic table at \( M = 0.613 \),

\[
\frac{T}{T_o} = 0.9301 \quad \text{and} \quad \frac{p}{p_o} = 0.7759
\]

After passage of the wave, the stagnation pressure is

\[
\frac{T_{ob}}{T_b} = \frac{T_b}{T_{ob}} \times \frac{T_b}{T_o} = \frac{362.3}{0.9301} = 389.5 \text{ K}
\]

\[
\frac{p_{ob}}{p_b} = \frac{p_b}{p_{ob}} \times \frac{p_b}{p_o} = \frac{249.3}{0.7759} = 321.3 \text{ kPa}
\]

Note that for a fixed observer the stagnation temperature after passage of the wave is greater than that before passage of the wave. For an observer "sitting on the wave," however, there is no change of stagnation temperature across the wave.
Example 13.2 An explosion occurs which produces a normal shockwave that propagates at a speed of 600 m/s into still air. The pressure and temperature of the motionless air in front of the shock are 101.3 kPa and 20 °C, respectively. Determine the velocity, static pressure, and static temperature of the air following the shock, i.e. \((V_2, p_2, and T_2)\).

Solution

\[
M_1 = \frac{V_s}{\sqrt{\gamma RT_1}} = \frac{600}{\sqrt{1.4 \times 287 \times 293}} = 1.749
\]

From isentropic table at \(M_1 = 1.749\) gives

\[
p_1/p_{o1} = 0.1882, T_1/T_{o1} = 0.6205
\]

And from normal table at \(M_1 = 1.749\) gives

\[
p_2/p_1 = 3.4009, T_2/T_1 = 1.4936, p_{o2}/p_{o1} = 0.8351 \text{ and } M_2 = 0.6284.
\]

So;

\[
T_{o1} = \frac{T_1}{(p_1/p_{o1})} = \frac{293}{0.6205} = 472.2 K = T_{o2}
\]

\[
p_{o1} = \frac{p_1}{(T_1/T_{o1})} = \frac{101.3}{0.1882} = 538.2572 \text{ kPa}
\]

\[
p_{o2} = \left(\frac{p_{o2}}{p_{o1}}\right)p_{o1} = 0.8351 \times 538.2572 = 449.4986
\]

\[
p_2 = \left(\frac{p_2}{p_1}\right)p_1 = 3.4009 \times 101.3 = 344.5112 \text{ kPa}
\]

\[
T_2 = \left(\frac{T_2}{T_1}\right)T_1 = 1.4936 \times 293 = 437.6248 \text{ K} = T_b
\]

\[
a_2 = a_b = \sqrt{\gamma RT_2} = \sqrt{1.4 \times 287 \times 437.6248} = 419.33 \text{ m/s}
\]

\[
(V_s - V_g) = a_2 M_2 = 419.33 \times 0.6284 = 263.507 \text{ m/s}
\]

\[
V_g = V - (V_s - V_g) = 600 - 263.507 = 336.493 \text{ m/s}
\]

\[
M_b = \frac{V_g}{a_b} = \frac{336.493}{419.33} = 0.8025
\]

From isentropic table at \(M_b = 0.8025\), gives;

\[
p_b/p_{ob} = 0.6544 \text{ and } T_b/T_{ob} = 0.8859, \text{ then}
\]

\[
p_{ob} = \frac{p_b}{p_b/p_{ob}} = \frac{344.5112}{0.6544} = 526.4535 \text{ kPa}
\]

\[
T_{ob} = \frac{T_b}{T_b/T_{ob}} = \frac{437.6248}{0.8859} = 493.9889 \text{ K}
\]

Example 13.3 The shock was given as moving at 548.64 m/s into air at 101.353 Pa and 289 K. Solve the problem represented in Figure 13.4.
Solution

We solve for fixed normal shock, i.e. moving coordinate system, (figure 13.4b).

\[ a_1 = \sqrt{\gamma RT_1} = \sqrt{1.4 \times 287 \times 289} = \]
\[ = 340.76 \text{ m/s} \]

\[ M_1 = \frac{V_1}{a_1} = \frac{548.64}{340.76} = 1.61 \]

From isentropic, at \( M_1 = 1.61 \),

\[ p_1/p_{o1} = 0.2318, \text{ then } \]

\[ p_{o1} = \frac{p_1}{p_{o1}} = \frac{101.353}{0.2318} = 437.243 \text{ kPa} \]

From normal shock table, at \( M_1 = 1.61 \)

\[ M_2 = 0.6655, \quad \frac{p_2}{p_1} = 2.8575, \quad \frac{T_2}{T_1} = 1.3949 \]

Thus

\[ p_2 = p_1 \cdot \frac{p_2}{p_1} = 101.353 \times 2.8575 = 289.616 \text{ kPa} \]

\[ T_2 = T_1 \cdot \frac{T_2}{T_1} = 289 \times 1.3949 = 403.13 \text{ K} \]

\[ a_2 = \sqrt{\gamma RT_2} = \sqrt{1.4 \times 287 \times 403.76} = 402.78 \text{ m/s} \]

\[ V_2 = a_2 M_2 = 402.78 \times 0.6655 = 268.1 \text{ m/s} \]

And from isentropic table at \( M_2 = 0.6655 \), \( p_2/p_{o2} = 0.7430 \) and \( T_2/T_{o2} = 0.9188 \), then

\[ p_{o2} = \frac{p_2}{p_2/p_{o2}} = \frac{289.616}{0.7430} = 389.8 \text{ kPa} \]

\[ T_{o2} = \frac{T_2}{T_2/T_{o2}} = \frac{403.13}{0.9188} = 438.76 \text{ K} \]

\[ V_g = V_s - V_2 = 548.64 - 268.1 = 280.54 \text{ m/s} \]

It is apparent that \( p_{o2} < p_{o1} \) as expected.

Now we solve for moving shock, i.e. fixed coordinate system (figure 13.4a).

Remembering that pressure, temperature and sonic velocity values after the shock wave are not changed due to shock wave movement.

\[ p_2 = 289.616 \text{ kPa} \]

\[ T_2 = 403.13 \text{ K} \]

\[ a_2 = 402.78 \text{ m/s} \]

\[ V_g = 280.54 \text{ m/s} \]
And from isentropic table, at \( M_g = 0.697 \), \( p_2/p_{o2} = 0.7220 \) and \( T_2/T_{o2} = 0.9095 \), then:

\[
p_{o2} = \frac{p_2}{p_{2}/p_{o2}} = \frac{289.616}{0.7220} = 401.130 \text{ kPa}
\]

\[
T_{o2} = \frac{T_2}{T_{2}/T_{o2}} = \frac{403.13}{0.9095} = 443.2 \text{ K}
\]

Therefore, after the shock passes (referring now to Figure 13.4a), the pressure and temperature will be 289.616 kPa and 403.13 K, respectively, and the air will have acquired a velocity of 280.54 m/s to the left. It will be interesting to compute and compare the stagnation pressures in each case. Notice that they are completely different because of the change in reference that has taken place.

**Second case**

Developing an expressions for the case of a normal shock traveling at a constant speed \( V_s \) into a gas that is moving with a speed \( V \). The shock induces a speed \( V_g \) of the gas it passes over, as shown in Figure 13.6. Here simply replace each \( V_s \) & \( V_g \) in eqs. 13.1 to 13.5 by \( V_s - V \) & \( V_g - V \).

**Example 13.4** A piston in a tube is suddenly accelerated to a velocity of 50 m/s, which causes a normal shock to move into the air at rest in the tube. Several seconds later, the piston is suddenly accelerated from 50 to 100 m/s, which, causes a second shock to move down the tube. Calculate the velocities of the two shock waves for an initial air temperature of 300 K.

**Solution**

The air next to the piston must move at the same velocity as the piston, since it can neither move through the face of the piston nor move away from the piston and leave a vacuum behind. Therefore, for a fixed observer, the air velocities are as shown in Figure (13.7).

\[
a_1 = \sqrt{\gamma R T_1} = \sqrt{1.4 \times 287 \times 300} = 347.2 \text{ m/s}
\]

From eq. 13.5
\[ V_s = \frac{(\gamma + 1) V_0}{4} \pm \sqrt{\left(\frac{(\gamma + 1) V_0}{4}\right)^2 + a_1^2} \]

\[ V_{s1} = \frac{(1.4 + 1)50}{4} \pm \sqrt{\left(\frac{(1.4 + 1)50}{4}\right)^2 + 347.2^2} \]

\[ V_{s1} = 30 + 348.5 = 378.5 \text{ m/s} \]

\[ M_{s1} = \frac{V_{s1}}{a_1} = \frac{378.5}{347.2} = 1.090 \]

From normal shock table, at \( M_1 = 1.090 \rightarrow T_2/T_1 = 1.059 \), so;
\[ T_2 = 300 \times 1.059 = 317.7 \text{ K} \]

For the second shock, the situation is shown in Figure (13.8a). Figure (13.8b) shows an observer “sitting on the second wave”. Using eq. (10.5), we obtain

\[ \frac{V_1}{V_2} = \frac{(\gamma + 1) M_1^2}{(\gamma - 1) M_1^2 + 2} \]

Where

\[ V_1 = V_{s2} - 50, \quad V_2 = V_{s2} - 100 \]

\[ M_1^2 = \frac{(V_{s2} - 50)^2}{\gamma R T_1} \]

Substituting yields

\[ \frac{V_{s2} - 50}{V_{s2} - 100} = \frac{2.4 \cdot (V_{s2} - 50)^2}{1.4 \cdot 287 \cdot 317.7} \left[ \frac{0.4 \cdot (V_{s2} - 50)^2}{1.4 \cdot 287 \cdot 317.7} + 2 \right] \]

\[ = \frac{2.4(V_{s2} - 50)^2}{0.4(V_{s2} - 50)^2 + 2 \cdot 127651.86} \]

To solving this quadratic equation, Let \( x = (V_{s2} - 50) \)

\[ \frac{2.4x^2}{0.4x^2 + 255303.72} \]

\[ 0.4x^2 + 255303.72x = 2.4x^2 - 120x \]

\[ 2x^2 - 120x - 255303.72 = 0 \]

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{120 \pm \sqrt{120^2 + 4 \cdot 2 \cdot 255303.72}}{2 \cdot 2} \]

\[ x = \frac{120 \pm \sqrt{120^2 + 4 \cdot 2 \cdot 255303.72}}{2 \cdot 2} = \frac{120 + 1434.165}{4} \]

\[ V_{s2} - 50 = 388.543 \]

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Thus, the second wave travels at a greater velocity than the first and eventually overtakes it. This result is a demonstration of the principles formation of normal shock. Compression waves are able to overtake and reinforce one another. In this example problem, the second wave travels at a greater velocity because it is both moving into the compressed, higher-temperature gas behind the first wave and also moving into a gas stream already traveling in the same direction with a velocity of 50 m/s. A new set of gas properties now can be computed before and after the second shock.

12.2 Reflected Waves.

When a wave impinging on the end of a tube, two cases should be studied, a closed tube and a tube open to the atmosphere. The reflected wave in closed end tube is treated as a reflected normal shock while for open end tube is treated as reflected expansion waves.

To complete this study of moving normal shock waves, consider the result of a wave impinging on the end of a tube. Two cases will be studied; a closed tube and a tube open to the atmosphere. In both cases it is desired to determine whether the reflected wave is a compression shock wave or a series of weak expansion waves. For reflected wave in closed tube, (see Figure 13.9), the gas next to the fixed end of the tube must be at rest, with the gas behind the incident shock moving to the right with velocity \( V_g \). For an observer moving with the reflected wave, the physical indicates that a decrease in velocity and a corresponding increase in static pressure across the reflected wave, which is physically the situation for a normal shock. Therefore, a normal shock reflects from a closed tube as a normal shock.

For reflected in open tube to atmosphere, the boundary condition imposed on the system is the static pressure at the end of the tube. Because the flow in front of the moving shock is subsonic, the back pressure and the exit pressure must be the same, see figure 13.10. there will be a decrease in pressure across the reflected wave and a normal shock reflects from an open end of a tube as a series of expansion waves.
Example 13.4 A normal shock wave with pressure ratio of 4.5 impinges on a plane wall (see Figure 13.11a). Determine the static pressure ratio for the reflected normal shock wave. The air temperature in front of the incident wave is 20°C.

Solution

Solution for incident wave:

To determine the velocity $V_g$ of the gas behind the incident wave, utilize a reference system moving with the wave, as shown in Figure 13.11b.

From normal shock table $p_2/p_1 = 4.5$, gives:

$M_1 = 2.0$, $p_2/p_1 = 2.667$ and $T_2/T_1 = 1.688$

$V_{si} = M_1 \sqrt{\gamma \frac{RT_1}{M}} = 2.0 \sqrt{1.4 * 287 * 293} = 686.2 \text{ m/s}$

$\frac{V_{si}}{V_g} = \frac{V_1}{V_2} = \frac{p_2}{p_1} = 2.667$

$(686.2 - V_g) = 686.2 \div 2.667$

$V_g = 428.9 \text{ m/s}$

$T_2 = T_1 \times \frac{T_2}{T_1} = 293 * 1.688 = 494.6 K$

Solution for reflected wave:

To find the reflected shock velocity, fix the reflected shock by using (see Figure 13.11c)

$\frac{V_2}{V_3} = \frac{(\gamma + 1)M_2^2}{(\gamma - 1)M_3^2 + 2}$ \hspace{1cm} (8.16)

For this case

$V_2 = 428.9 + V_{sr}$

$V_3 = V_{sr} = V_2 - 428.9$

$T_2 = 494.6 K$

$M_2^2 = \frac{V_2^2}{\gamma RT_2} = \frac{V_2^2}{1.4 * 287 * 494.6} = \frac{V_2^2}{198730.28}$

$V_2 = \frac{2.4 V_2^2}{198730.28} = \frac{2.4V_2^2}{198730.28 + 0.4V_2^2 + 397460.56}$

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For the fixed shock, back to fig. 13.10a

From normal shock table, at \(\frac{V_2}{V_1} = 1029.36 \pm \sqrt{1029.36^2 + 4 \times 2 \times 397460.56}\)

\[ V_2 = \frac{1029 - 2058.948}{4} = -257.487 \text{ m/s} \text{ ignored} \]

\[ V_2 = \frac{1029 + 2058.948}{4} = 771.987 \text{ m/s} \]

\( V_{sr} = 771.987 - 428.9 = 343.1 \text{ m/s} \)

For the fixed shock, back to fig. 13.10a

\[ \frac{V_2}{V_3} = \frac{428.9 + V_{sr}}{771.987} = \frac{343.1}{2.250} = \frac{\rho_3}{\rho_2} \]

From normal shock table, at \(\frac{\rho_3}{\rho_2} = 2.250\), gives

\( \frac{p_3}{p_2} = 3.333 \) static pressure ratio for reflected normal shock.

\( \frac{p_3}{p_1} = \frac{p_3}{p_2} \times \frac{p_2}{p_1} = 3.333 \times 4.5 \approx 15 \)

That means the in zone 3 after reflection becomes fifteen times the pressure in zone 1 before incident.

Another type of moving shock is occurred when air is flowing through a duct under known conditions and a valve is suddenly closed, as shown in fig. 13.12. The fluid is compressed as it is quickly brought to rest. This results in a shock wave propagating back through the duct. In this case the problem is not only to determine the conditions that exist after passage of the shock but also to predict the speed of the shock wave. This can also be viewed as the reflection of a shock wave, similar to what happens at the end of a shock tube. We transfer the fixed coordinate into a moving coordinate system by riding the shock wave and superimpose the reflected wave velocity \( V_{sr} \) on the entire flow field. With this new frame of reference we have the standing normal-shock problem shown in Figure 13.12.
Example 13.5 Air of speed of 240 m/s is flowing through a duct where its pressure and temperature are 2 bar and 300 K respectively. Then a valve exists in the duct is suddenly closed. Find fluid properties next to the valve after it closed and shock velocity, as show in figure 13.13.

Answer

\[
V_2 = V_1 - 240
\]
\[
\frac{V_1}{V_2} = \frac{(\gamma + 1)M_1^2}{(\gamma - 1)M_1^2 + 2}
\]
\[
\frac{V_1}{V_2} = \frac{2.4V_1^2/120540}{0.4V_1^2/120540 + 2}
\]

\[
0.4V_1^3 + 2 * 120540V_1 = 2.4V_1^3 - 576V_1^2
\]
\[
2V_1^2 - 576V_1 - 241080 = 0
\]

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
\[
V_1 = \frac{576 + \sqrt{576^2 - 4 * 2 * 241080}}{2 * 2} = 519.867 \text{ m/s}
\]
\[
V_2 = 519.867 - 240 = 279.867 \text{ m/s}
\]
\[
a_1 = \sqrt{\gamma RT_1} = \sqrt{1.4 * 287 * 300} = 347.189 \text{ m/s}
\]
\[
M_1 = \frac{V_1}{a_1} = \frac{519.867}{347.2} = 1.497
\]

From normal shock table at \( M_1 = 1.5 \) gives

\[
M_2 = 0.7011, \quad \frac{p_2}{p_1} = 2.458 \quad \text{and} \quad \frac{T_2}{T_1} = 1.320
\]
\[
p_2 = 2.458 * 2 = 4.916 \text{ bar}
\]
\[
T_2 = 1.320 * 300 = 396 \text{ K}
\]

12.3 Shock Tube

The shock tube is a device in which normal shockwaves are generated by the rupture of a diaphragm separating a high-pressure gas from a gas at low pressure. As such, the shock tube is a useful research tool for investigating not only shock phenomena, but also the behavior of materials and objects when subjected to the extreme conditions of pressure and temperature prevalent in the gas flow behind the wave. Thus, the kinetics of a chemical reaction taking place at high temperature can be studied, as well as the performance, for example, of a body during reentry from space back into the earth’s atmosphere.
Chapter Fourteen/Oblique Shock Waves

14.1 Introduction.

An oblique shock wave, a compression shock wave that is inclined at an angle to the flow, either straight or curved, can occur in such varied examples as supersonic flow over a thin airfoil or in supersonic flow through an over-expanded nozzle.

The oblique shock wave is a two-dimension problem. The method of handling the oblique shock is alike that of handling the normal shock. Even though inclined to the flow direction, the oblique shock still represents a sudden, almost discontinuous change in fluid properties, with the shock process itself being adiabatic. Attention will be focused on the two-dimensional straight oblique shock wave, a type that might occur during the presence of a wedge in a supersonic stream (Figure 14.1a) or during a supersonic compression in a corner (Figure 14.1b). As with the normal shock wave, the equations of continuity, momentum, and energy will first be derived. An additional variable is introduced because of the change in flow direction across the wave. However, momentum is a vector quantity, so two momentum equations are derivable for this two-dimensional flow.

With the additional variable and equation, the analysis of two-dimensional shock flow is somewhat more complex than that for normal shock flow. However, as with the normal shock wave, solutions to the equations of motion will be presented in a form suitable for the working of practical engineering problems.

14.2 Equations of Motion for a Straight Oblique Shock Wave

When a uniform supersonic stream is forced to undergo a finite change in direction due to the presence of a body in the flow, the stream cannot adjust gradually to the presence of the body; rather, a shock wave or sudden change in flow properties must occur. A simple case is that of supersonic flow about a two-dimensional wedge with axis aligned parallel to the flow direction.

For small wedge angles, the flow adjusts by means of an oblique shock wave, attached to the apex of the wedge. Flow after the shock is uniform, parallel to the wedge surface (as shown in Figure 14.2), with the entire flow having been turned through the wedge half-angle $\delta$. 
The equations of continuity, momentum, and energy will now be written for uniform, supersonic flow over a fixed wedge. If one selects the control volume indicated in Figure 14.2. The continuity equation for steady flow is

$$\int_{cs} \rho \left( \mathbf{V} \cdot \hat{n} \right) dA = 0$$

For the case under steady, it simplifies to

$$\rho_1 V_{1n} A = \rho_2 V_{2n} A$$

$$\rho_1 V_{1n} = \rho_2 V_{2n}$$  \hspace{1cm} (14.1)

Where $V_{1n}$ and $V_{2n}$ are the velocity components normal to the wave. $A$ is the control volume surface and it is the same for both sides. The momentum equation for steady flow is;

$$\sum F = \int_{cs} \mathbf{V} \rho \left( \mathbf{V} \cdot \hat{n} \right) dA = 0$$

Momentum is a vector quantity, so momentum balance equations can be written both in the direction normal to the wave and in the direction tangential to the wave. The normal momentum equation yields;

$$p_1 A_1 - p_2 A_2 = \rho_2 A_2 V_{2n}^2 - \rho_1 A_1 V_{1n}^2$$

The shock is very thin so as we assume that $A_2 = A_1$. Thus;

$$p_1 - p_2 = \rho_2 V_{2n}^2 - \rho_1 V_{1n}^2$$  \hspace{1cm} (14.2)

In the tangential direction there is no change in pressure so;

$$0 = \int_{cs} V_t \rho \left( \mathbf{V} \cdot \hat{n} \right) dA = 0$$

$$(\rho_1 V_{1n} A_1) V_{1t} = (\rho_2 V_{2n} A_2) V_{2t}$$

Cancelling, we obtain;

$$V_{1t} = V_{2t}$$  \hspace{1cm} (14.3)

where $V_{1t}$ & $V_{2t}$ are the velocity components tangential to the wave. The energy equation for adiabatic, no work steady flow simplifies to;

$$\left( h_1 + \frac{V_1^2}{2} + gz_1 \right) = \left( h_2 + \frac{V_2^2}{2} + gz_2 \right)$$
Expanding this equation and ignoring potation term for gas and remembering that a velocity is a vector \( \vec{V} = V_n + V_t \), we get;
\[
\left( h_1 + \frac{V_{1n}^2}{2} + \frac{V_{1t}^2}{2} \right) = \left( h_2 + \frac{V_{2n}^2}{2} + \frac{V_{2t}^2}{2} \right)
\]
Since \( V_{1t} = V_{2t} \) then;
\[
\left( h_1 + \frac{V_{1n}^2}{2} \right) = \left( h_2 + \frac{V_{2n}^2}{2} \right) \quad \ldots (14.4a)
\]
\[
T_{o1} = T_{o2} \quad \ldots (14.4b)
\]
\[
M_{1n} = M_1 \sin \theta \quad \ldots (14.5a)
\]
\[
M_{1t} = M_1 \cos \theta \quad \ldots (14.5b)
\]
\[
M_{2n} = M_2 \sin (\theta - \delta) \quad \ldots (14.6a)
\]
\[
M_{2t} = M_2 \cos (\theta - \delta) \quad \ldots (14.6b)
\]

From the geometry of the oblique wave;

It can be seen that eqs. (14.1), (14.2), and (14.4) contain only the normal velocity components, and as such are the same as eqs. (9.1), (9.2), and (9.4) for the normal shock wave. In other words, an oblique shock acts as a normal shock for the component normal to the wave, while the tangential velocity component remains unchanged. The pressure ratio, temperature ratio, and so on, across an oblique shock can be determined by first calculating the component of \( M_n \), normal to the wave and then referring this value to the normal shock tables.

Note that the Mach number after an oblique shock wave can be greater than 1 without violating the second law of thermodynamics. The normal component of \( M_2 \) however, must still be less than 1. In most cases, the shock wave angle \( \theta \) is not known, but rather incoming Mach number \( M_1 \) and deflection angle \( \delta \) appear as the independent variables. Therefore, it is more convenient to express the wave angle \( \theta \) and \( M_2 \) in terms of \( M_1 \) and \( \delta \). From eq. 14.1
\[
\rho_1 V_{1n} = \rho_2 V_{2n}
\]
\[
\frac{\rho_2}{\rho_1} = \frac{V_{1n}}{V_{2n}} = \frac{V_{1t} \tan \theta}{V_{2t} \tan(\theta - \delta)} = \frac{\tan \theta}{\tan(\theta - \delta)} \quad \ldots (14.7)
\]
A cross the normal shock
\[
\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)M_{1n}^2}{(\gamma - 1)M_{1n}^2 + 2} \quad \ldots (10.3)
\]
\[
\frac{\tan \theta}{\tan(\theta - \delta)} = \frac{(\gamma + 1)M_{1n}^2}{(\gamma - 1)M_{1n}^2 + 2} \quad \ldots (14.8a)
\]
\[
\frac{\tan \theta}{\tan(\theta - \delta)} = \frac{(\gamma + 1)M_{1n}^2 \sin^2 \theta}{(\gamma - 1)M_{1n}^2 \sin^2 \theta + 2} \quad \ldots (14.8b)
\]

Eq. 14.8 relates deflection angle \( \delta \) incoming Mach number \( M_1 \) and shock wave angle \( \theta \).
Now $\theta$ can be plotted versus $\delta$ for a given value of $M_1$. Also $M_2$ can be plotted versus $\delta$ for given $M_1$. For $M_1 = 2.0$, the results appear as shown in Figures 14.4a and 14.4b.

Detailed oblique shock charts are provided in charts C1 and C2 for $\gamma = 1.4$. But chart C2 is not accurate and it will not recommended. Several characteristics of the solution to the oblique shock equations can be seen from these charts. For a given $M_1$ and $\delta$, either two solutions are possible or none at all. For supersonic flow in varying area channels, it is the pressure boundary conditions imposed on the channel that determines the type of solution.

If a solution exists, there may be
1. A weak oblique shock, with $M_2$ either supersonic or slightly less than 1.
2. A strong oblique shock, with $M_2$ subsonic.

Both oblique shocks have different characteristics, see figure 14.5, such as:

a. For the strong oblique shock:
   - The wave makes a large angle $\theta$ (close to 90°) with the approach flow.
   - It accompanied by a relatively large pressure ratio
b. For the weak oblique shock,
   - The wave makes a much less angle $\theta$ with the approach flow.
   - It accompanied by a relatively small pressure ratio
c. The supersonic flow is turned through the same angle in both cases.

A strong oblique shock with ($\delta = 0$), gives a normal shock. A weak oblique shock with ($\delta = 0$) gives an isentropic flow (no shock). Therefore, the normal wave can be generalized to the oblique shock. The strong oblique shock occurs when a large back pressure is imposed on a supersonic flow, as might possibly take place during flow through a duct or intake.

When a wedge or airfoil travels through the atmosphere at supersonic velocities with an oblique shock attached to the body only a weak shock solution is found to occur, since, with a uniform pressure after the shock, large pressure differences cannot be exist. This is identical to determine whether isentropic flow or a normal shock will occur in a supersonic flow for flow through converging-diverging nozzles, we know that for low enough back pressures, isentropic
flow occurs in the nozzle; for higher back pressures, a normal shock takes place in the diverging section of the nozzle.

14.3 Detached shock Wave

Another characteristic of the oblique shock equations is that, for a great enough turning angle \( \delta > \delta_{max} \), no solution is possible. Under these conditions it is observed that the shock is no longer attached to the wedge, but stands detached, in front of the body (see Figure 14.6).

The detached shock is curved, as shown, with the shock strength decreasing progressively from that of a normal shock at the apex of the wedge to that of a Mach wave far from the body. Thus, with a detached shock, the entire range of oblique shock solutions is obtained for the given Mach number \( M_1 \).

The shape of the wave and the shock-detachment distance are dependent on the Mach number and the body shape. Flow over the body is subsonic in the vicinity of the wedge apex, where the strong oblique shocks occur, and it is supersonic farther back along the wedge, where the weak oblique shocks are present.

A detached oblique shock can also occur with supersonic flow in a concave corner. Again, if the turning angle is too great, a solution cannot be found in Charts C1 and C2, so a detached shock forms ahead of the corner (see Figure 14.7). The characteristics of this shock are exactly the same as those of the upper half of the detached shock shown in Figure 14.6. Thus flow after the shock is subsonic near the wall and supersonic farther out in the flow and it is treated as a stationary normal shock near the wall.

**Example 14.1** A uniform supersonic airflow traveling at Mach 2.0 passes over a wedge (Figure 14.4). An oblique shock, making an angle of 40° with the flow direction, is attached to the wedge under these flow conditions. If the static pressure and temperature in the uniform flow are, respectively, 20 kPa and \(-10 \, ^\circ\text{C}\), determine the static pressure and temperature behind the wave, the Mach number of the flow passing over the wedge, and the wedge half-angle.

**Solution**
From Figure 14.4,

\[ M_{1n} = M_1 \sin 40^\circ = 2.0 \sin 40^\circ = 1.286. \]
\[ M_{1t} = M_1 \cos 40^\circ = 2.0 \cos 40^\circ = 1.532. \]

Therefore, from normal shock table at \( M_{1n} = 1.286 \)

\[ M_{2n} = 0.794, \quad \frac{p_2}{p_1} = 1.763, \quad \frac{T_2}{T_1} = 1.182 \]

\[ p_2 = p_1 \times \frac{p_2}{p_1} = 20 \times 1.763 = 35.26 \text{ kPa} \]

\[ T_2 = T_1 \times \frac{T_2}{T_1} = 263 \times 1.1814 = 310.7 \text{ K} \]

For the adiabatic shock process, \( T_{o1} = T_{o2} \). From isentropic table at \( M_1 = 2.0 \),

\[ \frac{T_1}{T_{o1}} = 0.5556, \quad \text{Then} \]

\[ T_{o1} = T_{o2} = \frac{T_1}{T_{o1}} = \frac{263}{0.5556} = 473.4 \text{ K} \]

Now

\[ \frac{T_2}{T_{o2}} = 310.7/473.4 = 0.6563 \]

From isentropic table A at \( \frac{T_2}{T_{o2}} = 0.6563; \rightarrow M_2 = 1.617 \)

\[ \sin(\theta - \delta) = \frac{V_2}{V_2} = \frac{M_{2n} a_{2n}}{M_2 a_2} = \frac{0.794}{1.617} = 0.491 \]

\[ a_{2n} = a_2 \text{ scaler} \]

\[ \theta - \delta = 29.4^\circ \]

\[ \delta = 40 - 29.4 = 10.6^\circ \text{ end of the solution.} \]

Solving graphically;

From Chart C1 at \( M_1 = 2.0 \) & \( \theta = 40^\circ \) gives \( \delta = 10.6^\circ \)

From Chart C2 at \( M_1 = 2.0 \) & \( \delta = 10.6^\circ \) gives \( M_2 = 1.62 \)

Solving by the exact equations;

\[ \tan \delta = (\cot \theta) \left( \frac{M_1^2 \sin^2 \theta - 1}{\gamma + 1 \over 2} M_1^2 - (M_1^2 \sin^2 \theta - 1) \right) \]

\[ \tan \delta = (\cot 40) \left( \frac{2.0^2 \sin^2 40 - 1}{\gamma + 1 \over 2} M_1^2 - (2.0^2 \sin^2 40 - 1) \right) \]

\[ = (1.19175) \left( \frac{0.6527}{4.8 - 0.6527} \right) = 0.1756 \]
\[ \delta = \tan^{-1} 0.1756 = 10.6^\circ \]

\[ M_2 = \sqrt{\frac{1 + \frac{\gamma - 1}{2} M_1^2}{\gamma M_1^2 \sin^2 \theta - \frac{\gamma - 1}{2}} + \frac{M_1^2 \cos^2 \theta}{1 + \frac{\gamma - 1}{2} M_1^2 \sin^2 \theta}} \]

\[ M_2 = \sqrt{\frac{1 + \frac{1.4-1}{2} 2^2}{1.4 \times 2^2 \sin^2 40 - \frac{1.4-1}{2}} + \frac{2^2 \cos^2 40}{1 + \frac{1.4-1}{2} 2^2 \sin^2 40}} \]

\[ M_2 = \sqrt{\frac{1.8}{2.1138} + \frac{2.3473}{1.3305}} = 1.617 \]

**Example 14.2** Uniform flow at \( M = 2.0 \) passes over a wedge of 10° half-angle., find \( M_2, p_2/p_1, T_2/T_1 \) and \( p_{o2}/p_{o1} \), and also the half-angle above which the shock will become detached.

**Solution**

From Chart Cl at \( M = 2.0 \) and \( = 10^\circ \), the weak solution yields \( \theta = 39.3^\circ \)

\( M_{1n} = M_1 \sin \theta = 2.0 \sin 39.3 = 1.267 \)

\( M_{1e} = M_1 \cos \theta = 2.0 \cos 39.3 = 1.548 \)

From the normal shock tables at \( M_{1n} \approx 1.27 \)

\( p_2/p_1 = 1.71505 \); \( T_2/T_1 = 1.17195 \); \( p_{o2}/p_{o1} = 0.98422 \) and \( M_{zn} = 0.80164 \)

From Chart Cl it can be seen that \( \delta_{max} \) for \( M = 2.0 \) is 23°.

**Example 14.3** A supersonic two-dimensional inlet is to be designed to operate at \( M = 3.0 \). Two possibilities will be considered, as shown in Figure 14.8. In one, the compression and slowing down of the flow take place through one normal shock; in the other, a wedge-shaped diffuser, the deceleration occurs through two weak oblique shocks, followed by a normal shock. The wedge turning angles are each 8°. Compare the loss in stagnation pressure for the two cases shown.

**Solution**

For the normal shock diffuser, the ratio \( p_{2o}/p_{1o} \) can be found from normal shock table at \( M_1 = 3.0 \): so

\( p_{o2}/p_{o1} = 0.328 \).

For the wedge-shaped diffuser, \( M_2 \) and \( M_3 \), as well as the wave angles,
can be found from Charts C1 and C2. Thus

\[ M_2 = 2.60 \text{ and } M_3 = 2.255. \]

The wave angles are, respectively, 25.6° and 29.0°.

\[ M_{1n} = M_1 \sin \theta_1 = 3.0 \sin 25.6 = 1.3 \]

From normal shock table at \( M_{1n} = 1.30 \), \( p_{o2}/p_{o1} = 0.979 \)

\[ M_{2n} = M_2 \sin \theta_2 = 2.60 \sin 29.0 = 1.26 \]

From normal shock table at \( M_{2n} = 1.26 \), \( p_{o3}/p_{o2} = 0.986 \).

From normal shock table at \( M_3 = 2.255 \), \( p_{o4}/p_{o3} = 0.603 \), so that;

\[ \frac{p_{o4}}{p_{o1}} = \frac{p_{o4}}{p_{o3}} \cdot \frac{p_{o3}}{p_{o2}} \cdot \frac{p_{o2}}{p_{o1}} = 0.603 \cdot 0.986 \cdot 0.979 = 0.582 \]

Note; Solve the same example without using chart C2.

Therefore, the overall stagnation pressure ratio is 0.582. The advantage of diffusing through several oblique shocks rather than one normal shock can be seen. The greater the number of oblique shocks, the less the overall loss in stagnation pressure. Theoretically, if the flow is allowed to pass through an extremely large number of oblique shocks, each turning the flow through a very small angle, the inlet flow should approach that of an isentropic compression. The oblique shock diffuser will be discussed in detail in later.

### 14.4 Oblique Shock Reflections

When a weak, two-dimensional oblique shock impinges on a plane wall, the presence of a reflected wave is required to straighten the flow, since there can be no flow across the wall surface (see Figure 14.11).

Flow after the incident wave is deflected toward the wall. Hence, a reflected oblique shock wave must be present to deflect the flow back through the same angle and restore the flow direction parallel to the wall. The reflected shock is weaker than the incident shock, since \( M_2 < M_1 \).

**Example 14.4** For \( M_1 = 2.0 \), and \( \theta_i = 40^\circ \), determine \( \theta_r, M_2 \) and \( M_3 \). Refer to Figure 14.11.

**Solution**

From Chart C1, for \( M_1 = 2.0 \) and \( \theta_i = 40^\circ \), the deflection angle \( \delta \) is equal to 10.6°. This corresponds to the angle through which the flow is turned after the incident wave and also the angle through which the flow is turned back after the reflected wave.
From Chart C2, for $M_1 = 2.0$ and $\delta = 10.6^\circ$, $M_2$ is equal to 1.62.
From the same chart, for $M_2 = 1.62$ and $\delta = 10.6^\circ$, $M_3$ is equal to 1.24.
From Chart C1, for $M_2 = 1.6$ and $\delta = 10.6^\circ$, the shock wave angle is 51.2°, which is the angle between the flow direction in region 2 and the reflected wave. From geometrical consideration, $\theta_r = 51.2^\circ - 10.6^\circ = 40.6^\circ$.

If $M_2$ is low enough, a simple shock reflection may be impossible. That is, for a given $M_2$, the required turning angle may be great enough so that no solution exists from Charts C1 and C2.

In a real fluid, the problem of oblique shock reflections is complicated by the presence of a boundary layer on the wall. The analysis presented here of oblique shock reflections is an approximate one, which neglects real fluid effects.

### 14.5 Conical Shock Waves

Supersonic flow about a right circular cone is considerably more complex than that about a wedge. But it has many similarities to wedge flow. For a cone at zero angle of attack with the oncoming stream, a conical shock is attached to the apex of the cone for small cone angles. (see Figure 14.12.)

It is interesting to compare the resultant wedge and cone flows (see Figure 14.13.) For a wedge, straight parallel flow exists before the oblique shock and after the shock.

For the three-dimensional semi-infinite cone, this is no longer possible. Streamlines after the conical shock must be curved in order that the three-dimensional continuity equation be satisfied. For axisymmetric flow about a semi-infinite cone, with no characteristic length along the cone surface, conditions after the shock are dependent only on the conical coordinate $\omega$. That is, along each line of constant $\omega$, the flow pressure, velocity, and so on, are constant. This indicates that the pressure on the surface of the cone after the shock is constant, independent of distance from the cone apex.

At each point on the conical wave, the oblique shock equations already presented are valid. Conical flow behind the wave is isentropic, with the static pressure increasing to the cone surface pressure. A solution for the conical shock thus requires fitting the isentropic compression behind the shock to the shock equations already derived. Results are shown
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in Charts C3, C4, and C5, which show the variation of shock wave angle, surface pressure coefficient, and surface Mach number with cone semi-vertex angle and Mach number.

Whereas the conical flow equations yield two shock solutions, the only one observed on an isolated conical body is the weak shock. As with wedge flow, for large enough cone angles there is no solution; the shock stands detached from the cone.

If we compare again the wedge and cone solutions, it can be seen from Charts C3, C4, and C5 that, for a given body half-angle and $M_1$ the shock on the wedge is inclined at a greater angle to the flow direction than the shock on the cone; this indicates that a stronger compression takes place across the wedge oblique shock. In other words, the wedge presents a greater flow disturbance than the cone. Again, this results from three-dimensional effects.

From a physical standpoint, the flow is unable to pass around the side of the two-dimensional wedge since it extends to infinity in the third dimension. Flow can pass around the sides of the three-dimensional cone, however, so the cone presents less overall disruption to the supersonic flow.

**Example 14.5** Uniform supersonic flow at Mach 2.0 and $p = 20 \text{ kPa}$ passes over a cone of semi-vertex angle of 10° aligned parallel to the flow direction. Determine the shock wave angle, Mach number of the flow along the cone surface, and the surface pressure coefficient.

**Solution**
From Chart C3, the shock wave angle is 31.2°.
From Chart C4, the Mach number along the cone surface is 1.85.
From Chart C5, the surface pressure ratio is 1.29

$p_c = 20 * 1.29 = 25.8 \text{ kPa}$

\[ C_p = \frac{25.8 - 20}{0.5 * 1.4 * 20 * 2^2} = 0.104 \]
14.6 Supersonic oblique Shock Diffuser.

For a turbojet or ramjet traveling at high velocity, it is necessary to provide an inlet, or diffuser, that will perform the function of slowing down the incoming air with a loss of stagnation pressure. The use of a converging-diverging passage as an inlet for supersonic flow was studied in Chapter 4. Because such an internal deceleration device can operate isentropically only at the design speed, this type of diffuser has been found to be impractical during startup and when operating in an off-design condition. In fact without provisions for either varying the throat area or over speeding, the design condition could not be attained.

To eliminate the starting problem involved with the converging-diverging passage, the internal throat must be removed. Thus, a possible design is the normal-shock diffuser, where the deceleration takes place through a normal shock followed by subsonic diffusion in a diverging passage. (See Figure 14.14.) The disadvantage of this setup is the large loss in stagnation pressure incurred by the normal shock. Only at Mach numbers close to unity would this design be practicable.

The advantage of decelerating through several oblique shocks rather than one normal shock was shown. The oblique-shock spike-type diffuser takes advantage of this condition and hence represents a practical device for decelerating a supersonic flow. The operation of a single oblique-shock inlet at design speed is depicted in Figure 14.15. External deceleration is accomplished through an oblique shock attached to the spike. Further deceleration takes place through a normal shock at the engine cowl inlet, with subsonic deceleration occurring internally. Even though a normal shock occurs in this system, the flight Mach number $M$ has been reduced by the oblique shock, thus reducing the normal-shock strength and resultant stagnation pressure loss.

Theoretically, the greater the number of oblique shocks, the less the resultant total loss in stagnation pressure becomes. For example, a two-shock inlet is shown in Figure 14.16. Note, however, that along the surface of the spike, the boundary layer increases in thickness. The adverse pressure gradient created by the second shock may be sufficient to cause flow separation, with resultant loss of available energy. The greater the number of shocks, then, the
greater the tendency toward flow separation is.

It is necessary to affect a compromise in supersonic diffuser design between the increased total-pressure recovery achieved by increasing the number of oblique shocks through which the flow must be diffused and the increased tendency toward separation brought about by the shocks. For this reason, with flight Mach numbers up to 2.0, a single-shock diffuser is generally employed, whereas multiple-shock inlets are required for higher flight Mach numbers.

Several different modes of operation of the spike diffuser may occur, depending on the downstream engine conditions such as nozzle opening, turbine speed, and fuel flow rate. This situation is in contrast to the converging-diverging inlet, where operation was dependent on the inlet’s geometry. The spike diffuser’s modes of operation are termed \textit{subcritical}, \textit{critical}, and \textit{supercritical}, depending on the location of the normal shock.

\textit{Critical operation} occurs with the normal shock at the cowl inlet, as shown in Figure 14.17(a), with the engine operating at design speed. If the flow resistance downstream of the inlet is increased, with the engine still at the design flight Mach number, the normal shock moves ahead of the inlet, with some of the subsonic flow after the shock able to spill over or bypass the inlet. [See Figure 14.17(b).] For this \textit{subcritical condition}, the inlet is not handling the maximum flow rate; furthermore, the pressure recovery is unfavorable, since at least some of the inlet air passes through a normal shock at the design Mach number.

If the downstream resistance is reduced below that for critical operation, the normal shock reaches an equilibrium position inside the diffuser. For this \textit{supercritical condition} [see Figure 8.4(c)], the inlet is still handling maximum mass flow, yet the pressure recovery is less than that for critical operation, since the normal shock occurs at a higher Mach number in the diverging passage.

A turbojet engine must be able to operate efficiently both at other-than-design speeds and at different angles of attack. An engine operating at the critical mode may be pushed over into...
the undesirable subcritical mode by a small change of speed or angle of attack. For this reason, in actual operation, it is more practical to operate in the supercritical mode. While not providing quite as good a pressure recovery as critical operation, the supercritical mode still yields maximum engine-mass flow and makes a safety margin so that a small decrease in engine speed will not cause a transition to the subcritical mode. Thus, the supercritical mode provides a more stable engine operation.

**Example 14.6.** Compute the pressure recovery in one- and two-shock spike inlets. Compare the loss in total pressure for a one-shock spike diffuser (two dimensional) with that for two-shock diffuser operating at Mach 2.0. Also repeat for inlet Mach 4.0. (See Figure 14.18.) Assume that each oblique shock turns the flow through an angle of $\delta = 10^\circ$. Take $\gamma = 1.4$.

**Solution**

From the charts C1 & C2 at $M_1 = 2.0$ and $\delta = 10^\circ$, the weak solution yields

$\theta_1 = 39.3^\circ$, and $M_2 = 1.65$.

$M_{1n} = M_1 \sin \theta_1 = 2.0 \sin 39.3 = 1.2668$

For one oblique shock spike diffuser

From normal shock wave table at $M_{1n} = 1.2668$

$M_{2n} = 0.80709 + (0.80164 - 0.80709) \times \frac{1.2668 - 1.2600}{1.2700 - 1.2600}

= 0.80344$

$\theta_2 = \sin^{-1}(M_{2n}/M_2) = \sin^{-1}(0.80344/1.65) = 29.14^\circ$

$p_{o2}/p_{o1} = 0.98568 + (0.98422 - 0.98568) \times \frac{1.2668 - 1.2600}{1.2700 - 1.2600} = 0.9847$

From normal shock wave table at $M_2 = 1.65$

$M_3 = 0.65396$ and $p_{o3}/p_{o2} = 0.87599$

$p_{o3}/p_{o1} = \frac{p_{o3}}{p_{o2}} \times \frac{p_{o2}}{p_{o1}} = 0.9847 \times 0.87599 = 0.8626$

For two oblique shock spike diffuser

From the charts C1 & C2 at $M_2 = 1.65$ and $\delta = 10^\circ$, the weak solution yields

$\theta_2 = 49.4^\circ$, and $M_3 = 1.28$.

$M_{2n} = M_2 \sin \theta_2 = 1.65 \sin 49.4 = 1.2524$

From normal shock wave table at $M_{2n} = 1.2524$
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\[ M_{3n} = 0.81264 + (0.80709 - 0.81264) \times \frac{1.2524 - 1.2500}{1.2600 - 1.2500} = 0.8113 \]

\[ \theta_3 = \sin^{-1}(M_{3n}/M_3) = \sin^{-1}(0.8113/1.28) = 39.33^\circ \]

\[ p_{o3}/p_{o2} = 0.98706 + (0.98568 - 0.98706) \times \frac{1.2668 - 1.2600}{1.2700 - 1.2600} = 0.9867 \]

From normal shock wave table at \( M_3 = 1.28 \)

\[ M_4 = 0.79631 \quad \text{and} \quad p_{o4}/p_{o3} = 0.98268 \]

\[ \frac{p_{o4}}{p_{o1}} = \frac{p_{o4}}{p_{o3}} \times \frac{p_{o2}}{p_{o2}} = 0.9847 \times 0.98679 \times 0.98268 = 0.9548 \]

\[ \text{improvement} = \frac{0.9548 - 0.8626}{0.9548} \times 100 = 9.66\% \]

When \( M_1 = 4.0 \):

\[ \frac{p_{o3}}{p_{o1}} = 0.2372 \quad \text{and} \quad \frac{p_{o4}}{p_{o1}} = 0.3629 \]

\[ \text{improvement} = \frac{0.3629 - 0.2372}{0.3629} \times 100 = 34.6\% \]

The improvement in total-pressure ratio gained by using a two-shock inlet over a one-shock inlet is (9.66%) when \( M_1 = 2.0 \) and (34.6%) when \( M_1 = 4.0 \). Thus, at flight Mach numbers of 2.0 and below, the use of an inlet with one oblique shock is satisfactory; at flight Mach numbers of 4.0, an inlet with two oblique shocks (or more) is necessary.

**Example 14.7** A two-dimensional, spike-type inlet is operating in the supercritical mode at a flight Mach number of 3.0. The local static pressure and temperature are 50 kPa and 260 K, respectively. The flow cross-sectional area at the cowl inlet \( A_2 = 0.1 \text{ m}^2 \); the cross-sectional area at the location where the normal shock occurs in the diverging passage \( A_3 = A_4 = 0.12 \text{ m}^2 \). (See Figure 14.19.) Calculate the mass-flow rate and total-pressure ratio \( p_{o4}/p_{o3} \). Neglect friction. The spike half-angle is 10°, and the ratio of specific heats is \( \gamma = 1.4 \).

**Solution**

From the oblique shock wave charts C1 and C2 \( M_1 = 3.0 \) and \( \delta = 10^\circ \), the weak solution yields \( \theta_1 = 27.4^\circ \) and \( M_2 = 2.5 \)

\[ M_{1n} = M_1 \sin \theta_1 = 3.0 \sin 27.4^\circ = 1.3806 \]

From normal shock wave table at \( M_{1n} = 1.3806 \)
The flow from region 2 to region 3 is assumed to be isentropic. Thus, from isentropic flow table at \( M_2 = 2.5 \) gives \( A_2/A_2^* = 2.63672 \), then:

\[
\frac{A_3}{A_2} = \frac{A_3^*}{A_2^*} = 0.12 \star 2.63672 = 3.178 \quad (A_3^* = A_2^* \text{ for isentropic flow})
\]

From isentropic at this value gives:

\[
M_3 = 2.69 + (2.70 - 2.69) \frac{3.178 - 3.15299}{3.18301 - 3.15299} = 2.6983
\]

\[
p_{o3}/p_{o2} = 1 \quad (\text{isentropic flow})
\]

From normal shock table at \( M_3 = 2.6983 \)

\[
p_{o4}/p_{o3} = 0.42714 + (0.42359 - 0.42714) \frac{2.6983 - 2.690}{2.7000 - 2.690} = 0.4242
\]

So the total pressure ratio is:

\[
\frac{p_{o4}}{p_{o1}} = \frac{p_{o4} \star p_{o3} \star p_{o2}}{p_{o1} \star p_{o3} \star p_{o2}} = 0.4242 \star 1.0 \star 0.9630 = 0.4085
\]

To calculate mass flow rate

\[
p_{o1} = p_1 \star \frac{p_{o1}}{p_1} = 50 \star \left(1 + \frac{1.4 - 1}{2} 3^2\right)^{14-1} = 50 \star 36733 = 1836.636 \text{ kN/m}^2
\]

\[
p_{o2} = p_{o1} \star \frac{p_{o2}}{p_{o1}} = 1836.636 \star 0.9630 = 1768.68 \text{ kN/m}^2
\]

\[
p_2 = p_{o2}/\left(1 + \frac{\gamma - 1}{2} M_2^2\right)^{14-1} = 1836.636/\left(1 + \frac{1.4 - 1}{2} 2.5^2\right)^{14-1} = 110.207 \text{ kN/m}^2
\]

\[
\frac{T_{o1}}{T_1} = \left(1 + \frac{\gamma - 1}{2} M_1^2\right) = \left(1 + \frac{\gamma - 1}{2} 3^2\right) = 2.8
\]

\[
\frac{T_{o2}}{T_2} = \left(1 + \frac{\gamma - 1}{2} M_2^2\right) = \left(1 + \frac{\gamma - 1}{2} 2.5^2\right) = 2.25
\]

\[
T_2 = \frac{T_2}{T_{o2}} \star \frac{T_{o1}}{T_1} = \frac{1}{2.25} \star 2.8 \star 260 = 323.6 \text{ K} \quad \text{stagnation temp is constant}
\]

\[
m = \rho_2 A_2 V_2 = \left(\frac{p_2}{RT_2}\right) A_2 M_2 \sqrt{\gamma RT_2}
\]

\[
m = \left(\frac{110.2071}{0.287 \star 323.6}\right) * 0.1 \star 2.5 \star \sqrt{1.4 \star 287 \star 323.6} = 106.971 \text{ kg/s}
\]
15.1 Introduction

When a supersonic compression takes place at a concave corner, an oblique shock has been shown to occur at the corner. When supersonic flow passes over a convex corner, it is evident that some sort of supersonic expansion must take place. Previous results indicate that an expansion shock is impossible. However, a means must be available for the supersonic flow of Figure (15.1) to negotiate the corner. Here will present an analysis of the mechanism of two-dimensional, supersonic expansion flow, as might occur, for example during supersonic flow over a convex corner or at the exit of an under-expanded supersonic nozzle.

15.2 Thermodynamic Considerations

Two-dimensional, supersonic flow is to be turned through a finite angle at a convex corner. The mechanism of the resultant flow is of interest. Consider first the possibility of an oblique adiabatic shock occurring at the corner. Figure 15.2 shows the velocity vectors normal and tangential to such a wave. For this two-dimensional flow, uniform conditions prevail upstream and downstream of the wave. The equations of motion are exactly the same as those presented for oblique shock compression shock. Again, with no pressure gradient in the direction tangential to the wave, the tangential momentum equation yields

\[ V_{1t} = V_{2t} \]  

(15.1)

From geometrical considerations, as \( V_2 > V_1 \), it follows that \( V_{2n} \) must be greater than \( V_{1n} \). The normal momentum equation, eq. (14.2), yields

\[ p_1 + \rho_1 V_{1n}^2 = p_2 + \rho_2 V_{2n}^2 \]

Combining this with the continuity equation, eq. (14.1), where \( A = constant \);

\[ \rho_1 V_{1n} A = \rho_2 V_{2n} A \]

We obtain,

\[ p_2 - p_1 = \rho_1 V_{1n} (V_{1n} - V_{2n}) \]  

(15.2)
Since $V_{2n} > V_{1n}$, see figure (15.2), it follows that $p_2 < p_1$, indicating that the resultant flow must be an expansion.

It has been shown that an oblique shock reduces to a normal shock for the velocity component normal to the wave, with the tangential component remaining unchanged. The ratios of pressure, temperature, and density across an oblique shock are functions of $M_{1n}$ alone. The entropy change across an oblique shock can be written, then, in terms of $M_{1n}$, the resultant variation of $\Delta s$ with $M_{1n}$ being exactly the same as that for the normal shock. Hence, an oblique expansion shock $V_{2n} > V_{1n}$, just as a normal expansion shock, would involve a decrease in entropy during an adiabatic process. This violates the second law of thermodynamics and is impossible since $\Delta s \geq 0$. Therefore, the expansion shock, with sudden changes in flow properties, cannot occur at the convex corner. Instead, a more gradual type of supersonic expansion must take place.

15.3 Gradual Compressions and Expansions

When a supersonic stream undergoes a compression due to a finite, sudden change of direction at a concave corner, an oblique shock occurs at the corner. However, if the flow is allowed to change direction in a more gradual fashion, the compression can approach an isentropic process. Allowing supersonic flow to pass through several weak oblique shocks rather than one strong shock has been shown to reduce the resultant loss in stagnation pressure (or entropy rise) for a given change in flow direction (see Figure 15.3). In the limit, as the number of oblique shocks gets larger and larger, with each shock turning the flow through a smaller and smaller angle, the oblique shocks approach the Mach waves. The Mach wave, brought about by the presence of an infinitesimal disturbance in a supersonic flow, here corresponds to an oblique shock of vanishing strength, with infinitesimally small changes of velocity, flow direction, entropy, and so on, taking place across the wave (see Figure 15.4).

The wave angle is given by Equation $\mu = \sin^{-1}(1/M)$. Note that, from the oblique shock charts, Tables C, for an oblique shock of vanishing strength ($\delta = 0$), $\mu$ is evaluated from Mach number; for example, at $M_1 = 2.0$, $\delta = 0$ and $\mu = \theta = 30^\circ$. 

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So, by employing a smooth turn, with the resultant oblique shocks approaching Mach waves, a continuous compression is achieved in the vicinity of the wall with vanishingly small entropy rise (see Figure 15.5).

Away from the wall, however, the compression waves converge (Figure 15.6), coalescing to form a finite oblique shock wave. The characteristics of this shock are the same as those already discussed previously for an oblique shock wave of given $M_1$ and turning angle $\delta$. In fact, far enough away from the wall, flow about the smooth turn cannot be distinguished from the flow about a sharp, concave corner of angle $\delta$. It is important to note that here, again, the weak compression waves, each involving only an infinitesimal entropy rise, are able to reinforce one another to form a compression shock wave, with the resultant shock process involving a finite increase of entropy.

Now consider a supersonic expansion through a series of infinitesimally small convex turns (see Figure 15.7). Mach waves are generated at each corner, with each wave inclined at an angle to the flow direction. For this expansion flow, unlike the compressive flow discussed previously, waves do not coalesce but rather spread out. The divergent waves cannot reinforce one another; the oblique expansion shock is physically impossible.

Flow between each of the waves in Figure (15.7) is uniform, so the length of the wall between waves has no effect on the variation of flow properties. Thus the lengths of the wall segments can be made vanishingly small, without affecting the overall variation of flow properties across the expansion. By thus reducing the wall segments, the series of convex turns becomes a sharp corner (see Figure 15.8.) The resultant series of expansion waves, centered at the corner, is called a Prandtl Meyer expansion fan.
15.4 Flow Equations for a Prandtl Meyer Expansion Fan

It has been shown that supersonic expansion flow around a convex corner involves a smooth, gradual change in flow properties. The Prandtl Meyer fan consists of a series of Mach waves, centered at the convex corner. The initial wave is inclined to the approach flow at an angle \( \mu_1 = \sin^{-1}(1/M_1) \) the final wave is inclined to the downstream flow at an angle \( \mu_2 = \sin^{-1}(1/M_2) \). Flow conditions along each Mach wave are uniform; the variation of pressure, velocity and so on, through the expansion is only a function of angular position.

The equations for two-dimensional Prandtl Meyer flow will now be presented so that the variation of flow properties can be determined for a given flow turning angle. A perfect gas with constant specific heats will be assumed in the following analysis.

Consider first a single Mach wave, expanding the supersonic flow through an angle of magnitude \( dv \). With no pressure gradient in the tangential direction, there is no change of the tangential velocity component across the wave. Equating the expressions for \( V_t \) upstream and downstream of the Mach wave (see figure 15.9);

\[
V \cos \mu = (V + dV) \cos(\mu + dv)
\]

\[
= (V + dV)(\cos \mu \cos dv - \sin \mu \sin dv)
\]

Since \( dv \) is very small, then

\[
\cos dv = 1 \quad \text{and} \quad \sin dv = dv, \text{ therefore;}
\]

\[
V \cos \mu = (V + dV)(\cos \mu - dv \sin \mu)
\]

\[
V \cos \mu = V \cos \mu + dV \cos \mu - Vdv \sin \mu - dVdv \sin \mu
\]

(15.3)

The last term, containing the product of two differentials, can be dropped in comparison with the other terms of the equation. Simplifying, we obtain

\[
0 = dV \cos \mu - Vdv \sin \mu
\]
\[ \frac{dV}{V} = dv \tan \mu \]

Since \( \mu = \sin^{-1}(1/M) \), i.e. \( \sin \mu = 1/M \), it follows that

\[ \tan \mu = \frac{1}{\sqrt{M^2 - 1}} \]

\[ \frac{dV}{V} = \frac{1}{\sqrt{M^2 - 1}} dv \quad (15.4) \]

To solve for \( M \) as a function of \( \nu \), velocity \( V \) must be expressed in terms of \( M \). For a perfect gas with constant specific heats, we can write,

\[ V = M \sqrt{\gamma RT} \]

Taking log and differentiating, we obtain

\[ \log V = \log M + \log \sqrt{\gamma R} + \frac{1}{2} \log T \]

\[ \frac{dV}{V} = \frac{dM}{M} + \frac{dT}{T} \quad (15.5) \]

But, for this adiabatic flow, there is no change in stagnation temperature.

\[ T_0 = \text{constant} = T \left( 1 + \frac{(\gamma - 1)}{2} M^2 \right) \]

Taking logs and differentiating, we obtain

\[ 0 = \frac{dT}{T} + \frac{(\gamma - 1) M dM}{1 + \frac{(\gamma - 1)}{2} M^2} \quad (15.6) \]

Combining eqs. 5 & 6 gives

\[ \frac{dV}{V} = \frac{dM}{M} - \frac{(\gamma - 1) M dM}{2 \left( 1 + \frac{(\gamma - 1)}{2} M^2 \right)} \quad (15.7) \]

\[ \frac{dV}{V} = \frac{dM}{M} \left[ 1 - \frac{(\gamma - 1) M^2}{2 \left( 1 + \frac{(\gamma - 1)}{2} M^2 \right)} \right] \]

\[ \frac{dV}{V} = \frac{dM}{M} \left[ \frac{1}{1 + \frac{(\gamma - 1)}{2} M^2} \right] \quad (15.8) \]

Substitute eq. 8 into eq.4 gives

\[ \frac{dV}{V} = \frac{dM}{M} \left[ \sqrt{M^2 - 1} \right] \left( 1 + \frac{(\gamma - 1)}{2} M^2 \right) \quad (15.9) \]

To determine the change of Mach number associated with a finite turning angle, the above eq. (15.9) can be integrated
\[ \Delta v = (v_2 - v_1) = \int_{M_1}^{M_2} \frac{\sqrt{M^2 - 1}}{M \left( 1 + \frac{\gamma - 1}{2} M^2 \right)} \, dM \]

\[ \Delta v = (v_2 - v_1) = \left[ \frac{\gamma + 1}{\gamma - 1} \tan^{-1} \left( \frac{\gamma - 1}{\gamma + 1} (M^2 - 1) - \tan^{-1} \sqrt{M^2 - 1} \right) \right]_{M_1}^{M_2} \]  

(15.10)

For the purpose of tabulating this result, it is convenient to define a reference state 1, so that

\[ \Delta v = (v_2 - v_{\text{ref}}) = \left[ \frac{\gamma + 1}{\gamma - 1} \tan^{-1} \left( \frac{\gamma - 1}{\gamma + 1} (M^2 - 1) - \tan^{-1} \sqrt{M^2 - 1} \right) \right]_{M_{\text{ref}}}^{M_2} \]

Let the reference state be \( v = 0 \) at \( M = 1 \). Now

\[ v = \left[ \frac{\gamma + 1}{\gamma - 1} \tan^{-1} \left( \frac{\gamma - 1}{\gamma + 1} (M^2 - 1) - \tan^{-1} \sqrt{M^2 - 1} \right) \right] \]  

(15.11)

The symbol \( v \) represents the angle through which a stream, initially at \( M = 1 \), must be expanded to reach a supersonic Mach number \( M > 1 \). Values of \( v \) have been tabulated in isentropic table, for Mach numbers from 1.0 to 5.0 for \( \gamma = 1.4 \). Also presented are values of the wave angle \( \mu \), with both \( v \) and \( \mu \) expressed in degrees.

To determine the angle through which a flow would have to be turned to expand from \( M_1 \) to \( M_2 \) with \( M_1 \) not equal to 1, it is necessary only to subtract the value of \( v_1 \) at \( M_1 \) from the value of \( v_2 \) at \( M_2 \), where \( v_1 \) and \( v_2 \) are found in isentropic table (see Figure 15.10).

The variation of pressure, temperature, and other thermodynamic properties through the expansion can be found from the usual thermodynamic relations for isentropic flow, presented in Chapter 3. For this isentropic process, with no change in stagnation pressure;

\[ \frac{p_2}{p_1} = \left( 1 + \frac{\gamma - 1}{2} M_1^2 \right)^{\gamma/(\gamma - 1)} \]  

(15.12)

\[ \frac{T_2}{T_1} = 1 + \frac{\gamma - 1}{2} M_2^2 \]  

(15.13)
Example 15.1 A uniform supersonic flow at Mach 2.0, with static pressure of 75 kPa and a temperature of 250 K, expands around a 10° convex corner. Determine the downstream Mach number $M_2$, pressure $p_2$, temperature $T_2$, and the fan angle. See Figure (15.11).

Solution
From isentropic table, at $M_1 = 2.0 \rightarrow v_1 = 26.380°$ and $\mu_1 = 30.00°$
But $v_2 = v_1 + 10° = 36.38°$
Again from isentropic table at $v_2 = 36.38 \rightarrow M_2 = 2.385$ and $\mu_2 = 24.79°$
From isentropic table at $M_2 = 2.385 \rightarrow p_2/p_{2o} = 0.07003$, $T_2/T_{2o} = 0.4678$
From Table A at $M_1 = 2.000 \rightarrow p_1/p_{1o} = 0.12780$ and $T_1/T_{1o} = 0.5556$.
With no change in stagnation pressure $p_{1t} = p_{2t}$ and constant stagnation temperature
\[
\frac{p_2}{p_1} = \frac{p_2}{p_{2o}} \cdot \frac{p_{1o}}{p_1} = \frac{0.07003}{0.1278} = 0.548
\]
\[
p_2 = 75 \times 0.548 = 41.10 \text{ kPa}
\]
\[
\frac{T_2}{T_1} = \frac{T_2}{T_{2o}} \cdot \frac{T_{1o}}{T_1} = \frac{0.4678}{0.5556} = 0.842
\]
\[
T_2 = 250 \times 0.842 = 210 \text{ K}
\]
fan angle $= (\mu_1 + v_2 - v_1) - \mu_2$
$= 30.0 + 36.38 - 26.38 - 24.79 = 15.21°$

Example 15.2 Flow in Example 15.1 is expanded through a second convex turn of angle 10° (see Figure 15.12). Determine the downstream Mach number $M_3$ and the angle of the second fan.

Solution
The initial wave of the second fan must be parallel to the final wave of the first fan. Again, the distance between waves can have no effect on the resultant flow, since the flow between the waves is uniform. Therefore, the variation of properties is the same whether the flow is expanded through two 10° turns or one 20° turn.
\[
v_3 = v_2 + 10° = 36.38° + 10° = 46.38°
\]
From isentropic table at $v_3 = 46.38 \rightarrow M_3 = 2.831 \rightarrow \mu_3 = 20.68°$
fan angle$_{2nd} = v_3 - v_2 + \mu_2 - \mu_3$
$= 46.38 - 36.38 + 24.79 - 20.68 = 14.11°$
EXAMPLE 15.3 An under-expanded, two-dimensional, supersonic nozzle exhausts into a region where \( p_2 = 100 \text{ kPa} \) (Figure 15.13). Flow at the nozzle exit plane is uniform, with \( p_1 = 200 \text{ kPa} \) and \( M_1 = 2.0 \). Determine the flow direction and Mach number after the initial expansion.

Solution

From isentropic table at \( M_1 = 2.0 \rightarrow p_1/p_{1o} = 0.1278 \)

Since \( p_{1o} = p_{2o} \) for an isentropic expansion, then

\[
\frac{p_2}{p_{2t}} = \frac{p_2}{p_1} \cdot \frac{p_1}{p_{1o}} = \frac{100}{200} \cdot 0.1278 = 0.0639
\]

From isentropic table at \( p_2/p_{2o} = 0.0639 \rightarrow M_2 = 2.444 \)

From isentropic table, at \( M_1 = 2.000 \rightarrow v_1 = 36.830^\circ \)

\( M_2 = 2.444 \rightarrow v_2 = 37.803^\circ \)

So the flow is turned through \( v_2 - v_1 = 37.803^\circ - 26.830^\circ = 11.42^\circ \)

15.5 Prandtl Meyer Flow in a Smooth Compression

It was shown in Section 15.3 that, at a smooth compressive turn in supersonic flow, Mach waves emanate from the wall, coalescing farther out in the stream to form an oblique shock wave. In the region from the wall out to the point of coalescence of the waves (see Figure 15.6), the flow is isentropic and possesses the same characteristics as Prandtl Meyer flow. Therefore, the equations derived for Prandtl Meyer flow can be applied to the isentropic flow region at a concave corner, even though a compression takes place at the corner. Naturally, the turning angle, \( \Delta v \) will here be negative, corresponding to a decrease in Mach number. The extent of the isentropic flow region at a concave corner depends on the curvature of the wall. For a sharp turn, the region that can be treated as Prandtl Meyer flow is negligible; for a gradual turn, with a large radius of curvature, a much greater region has the characteristics of Prandtl Meyer flow.

15.6 Maximum Turning Angle for Prandtl Meyer Flow

From Eq. (15.11), it can be seen that, as \( M \rightarrow \infty \), or as the static pressure \( p_2 \rightarrow 0 \) (see Figure 15.14), the turning angle approaches a finite value of 130.4°. This result has significance, for example, in a determination of the shape of the exhaust plume of an under-expanded nozzle discharging...
into the vacuum of Space. To prevent the impingement of rocket exhaust gases on a part of a Spacecraft, the designer must have knowledge of the shape of the rocket-nozzle exhaust plume; modification of a spacecraft geometrical design may be (required to prevent possible damage from the hot exhaust gases. Furthermore, the axial thrust of a rocket depends on the direction of the exhaust velocity vectors.

The actual magnitude of the maximum turning angle presented here has only academic interest, in that effects such as liquefaction of air gases and other departures from perfect gas flow would occur long before the ultimate pressure could be attained. However, the result does indicate the presence of a maximum turning angle for a supersonic expansion.

15.7 Reflections

When a Prandtl Meyer expansion flow impinges on a plane wall, as shown in Figure (15.15), sufficient waves must be generated to maintain the wall boundary condition; that is, at the wall surface, the flow must be parallel to the wall. Each Mach wave of the initial Prandtl Meyer fan, then, must reflect as an expansion Mach wave. The resultant wave interactions present complexities that render an exact analysis of the flow extremely difficult; however, the general nature of the flow can be recognized. An application is the expansion that takes place at the exit of an under-expanded, two-dimensional nozzle. Since, from symmetry, there can be no flow across the center streamline; this streamline can be replaced by a plane wall. The resultant flow situation is shown in Figure (15.16).