Subject : gas Dynamic and Turbine Machine
Weekly Hours : Theoretical:2 UNITS:5
Tutorial : 1
Experimental : 1

موضوع : ديناميك غازات ومكائن تور بينية
الساعات الأسبوعية : نظري : 2 الوحدات :5
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عملي :1

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Chapter One

Fundamental of Fluid Dynamics

Introduction:

Gas dynamics is a branch of fluid mechanics which describe the flow of compressible fluid. Fluids which show appreciable variation in density as a results of the flow – such as gases- are called ***compressible fluids***. The variation in density is due mainly to variation in pressure and temperature.

The flow of a compressible fluid is governed by the first law of thermodynamics, which relates to energy balance, and by the second law of thermodynamics, which relates heat interaction and irreversibility to entropy. The flow is also affected by both kinetic and dynamic effects, which are described by Newton's laws of motion. An inertial frame of reference that is, a frame in which Newton's laws of motion are applicable- is generally used. In addition, the flow fulfils the requirement of conservation of mass.

These laws are not dependent on the properties of particular fluid, therefore in order to relate the motion to a particular fluid it is necessary to use subsidiary laws in addition to these fundamental principles , such as the equation of state for perfect gas.

$$p = \rho RT \quad \text{........(1)}$$

Although the most obvious application of compressible fluid flow theory are in the design of high speed aircraft, and this remains an important application to the subject, acknowledges of compressible fluid flow theory is required in the design and operation of many devices commonly encountered in engineering practice. Among these application are:

1- Gas Turbine: the flow in the balding and nozzle is compressible.
2- Steam turbine. Here, too, the flow in the nozzles and blades must be treated as compressible.
3- Reciprocating engines, flow of gases through the valves and intake and exhaust.
4- Natural gas transmission line.
5- Combustion chambers
6- Explosive.

1.1 *Conservation of Mass:*

The principle of conservation of mass, when referred to a system of fixed identity, simple states that the mass of the system is constant. Consider an arbitrary control volume through which fluid streams Fig.1. we wish to derive the form of the law of conservation of mass as it applied to this control volume. However, in order to apply the law, we must begin with a system of fixed identity, and so we defined our system as the fluid which some instant t occupies the control volume.

Next, we consider what happens during the succeeding time interval *dt*. By definition, the control volume remains fixed in space, but the system moves in the general direction of the streamline. The two position of the system are shown in fig.1 by dashed lines. For convenience in analysis, we consider three region of space denoted bt *I,II,III* in fig.1. At time *t* the system occupies spaces *I* and *III*, and at time *t+dt* it occupies space *I* and *II*. Thus, since the mass of the system is conserved, we write.

$$m_{I\,t} + m_{III\,t} = m_{I\,t+dt} + m_{II\,t+dt} \text{.....................}2$$

where m_{It} means the mass of the fluid in space I at time t, and so on. A simple rearrangement then gives.

$$\frac{m_{I\,t+dt} - m_{I\,t}}{dt} = \frac{m_{III\,t}}{dt} - \frac{m_{II\,t+dt}}{dt} \text{-----------------}3$$

The first term represent the time rate of change of mass within space I. But as dt goes to zero space I coincide with the control volume, and so in the limit.

$$\frac{m_{I\,t+dt} - m_{I\,t}}{dt} \rightarrow \frac{\partial}{\partial t}(m_{c.v.}) \text{----------------------}4$$

where m_{cv} denoted the instantaneous mass within the control volume.

The third term may be written.

$$\frac{m_{II\,t+dt}}{dt} = \frac{\sum \delta m_{II\,t+dt}}{dt} = \sum \frac{\delta m_{II\,t+dt}}{dt} = \int dm_{out} \text{-----------}5$$

where δm_{III+dt} represent the amount of mass crossing the elementary surface dA_{out} during the time dt. The ratio $\delta m_{III+dt}/dt$ is called the out going flux of mass cross the area dA_{out} Or the mass rate of flow and is denoted for convenience by dm_{out}.

similar reasoning yields for inlet.

$$\frac{m_{III\,t}}{dt} = \int dm_{in} \text{---------------}6$$

and so the conservation law may now be expressed as

$$\frac{\partial}{\partial t}(m_{c.v.}) = \int dm_{in} - \int dm_{out} \text{----------------}7$$

for detailed computation we note that at any instant

$$m_{c.v} = \int \delta m_{c.v} = \int_{c.v} \rho dv \text{-------------------}8$$

where dv is an element of control volume, ρ is the local mass density of that element and the integral is to be taken over the entire control volume.

$$\frac{\partial m_{c.v}}{dt} = \frac{\partial}{dt}\int_{c.v} \rho dv = \int_{c.v} \frac{\partial \rho}{\partial t} dv \text{----------------------}9$$

with the help of fig.1 we may express the mass rate of flow in the form.

$$dm_{out} = \frac{\delta m_{II\,t+dt}}{dt} = \frac{\rho(dA_{out})(V_n dt)}{dt} = \rho V_n dA_{out} \text{---------}10$$

where ρ is the local instantaneous mass density in the neighbourhood of dA_{out} and V_n is the corresponding local instantaneous component of velocity normal to dA_{out}, with the forgoing expression equation 7 may now written.

$$\int_{c.v} \frac{\partial \rho}{\partial t} dv = \int \rho V_n dA_{in} - \int \rho V_n dA_{out} \text{-----------------------}11$$

a form which is usually called the equation of continuity.

When the flow is steady, the identity of the fluid within the control; volume changes continuously, but the total mass remains constant or mathematically $\partial \rho/\partial t$ is zero for each element of control volume . Then equation 11 state that the incoming and outgoing mass rate of flow are identical.

$$\int \rho V_n dA_{in} = \int \rho V_n dA_{out} \quad \text{--------------------12}$$

For one dimensional steady state flow equation 12 for the inlet and outlet condition become.

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 \quad \text{--------------------13}$$

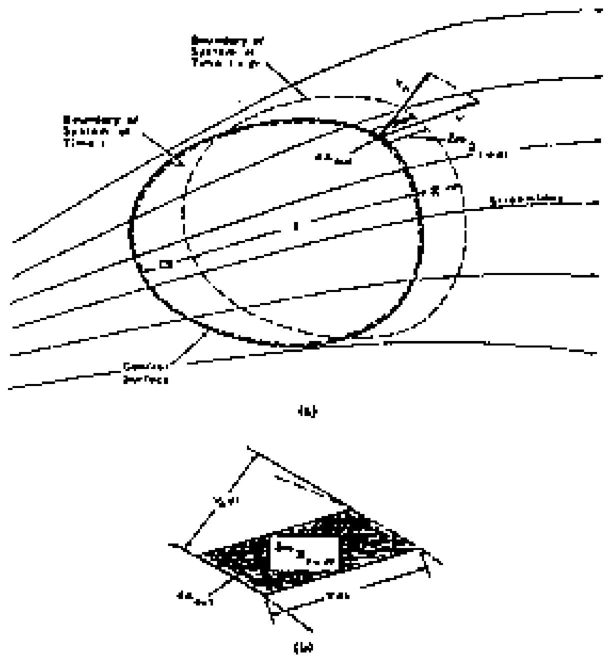


fig.(1) Flow through a control volume(continuity equation)

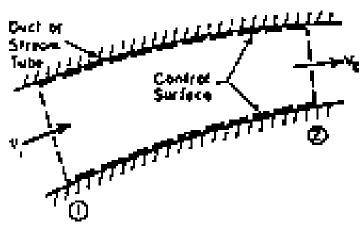


fig.2 One dimensional flow

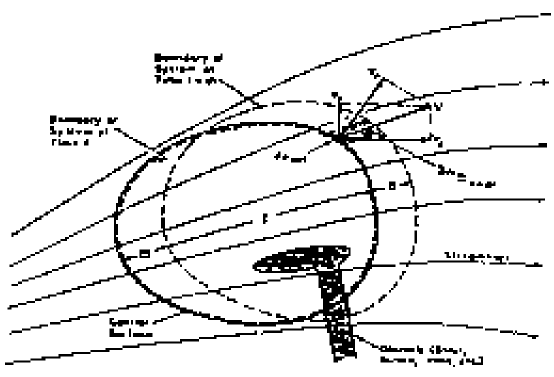


fig.3 flow through control volume with obstacle(momentum equation)

Example:1

Ten kg/sec of air enters a tank of $10 m^3$ in volume while 2 kg/sec is discharge from the tank as show in fig. If the temperature of the air inside the tank remains constant at 300K°. Find the rate of pressure rise inside the tank.

Solution:

Appling continuity equation

$$\int_v \frac{\partial \rho}{\partial t} dv = \int \rho V_n dA_{in} - \int \rho V_n dA_{out}$$

$10\frac{\partial \rho}{\partial t} = 10 - 2$, but $p = \rho RT$ so $\frac{\partial p}{\partial t} = RT\frac{\partial \rho}{\partial t}$

$$\frac{\partial p}{\partial t} = 287 * 300 * \frac{8}{10} = 68880 \ P_a/sec$$

Example:2

A tank 1 m^3 in volume contains air at an initial pressure of 6 atm (606.95 kPa) and an initial temperature of 25°C. Air is discharged isothermally from the tank at the rate of 0.1 m^3/s. Assuming that the discharged air has the same density as that of the air in the tank, find an expression for the time rate of change of density of the air in the tank. What would be the rate of pressure drop in the tank after 5 seconds?

solution:

Appling continuity equation $\int_v \frac{\partial \rho}{\partial t} dv = \int \rho V_n dA_{in} - \int \rho V_n dA_{out}$

$$1.0 \ \frac{\partial \rho}{\partial t} = -0.1\rho$$

or

$$\frac{\partial \rho}{\partial t} = -0.1\rho$$

Separating variables and integrating gives:

$$\rho = \rho_1 e^{-0.1t} = \left(\frac{p_1}{RT_1}\right) e^{-0.1t}$$

where subscript 1 refers to initial conditions in the tank. Pressure change may be expressed in terms of density change according to the relation:

$$p = \rho RT$$

so that:

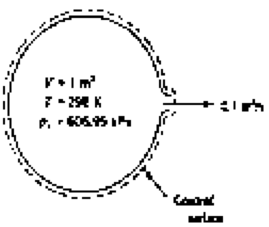
$$\frac{dp}{dt} = RT\frac{d\rho}{dt} = RT(-0.1\rho)$$

$$= -0.1RT\frac{p_1}{RT_1}e^{-0.1t}$$

$$= -0.1p_1e^{-0.1t}$$

Substituting numerical values gives:

$$\frac{dp}{dt} = -0.1 \times 606.95 \times e^{-0.1(5)} = -102.3 \text{ kPa/s}$$



1.2- *Momentum conservation theorem.*

The fundamental principle of dynamics is Newton's law of motion, and according to this law the resultant of force applied to a particle which may be at rest or in motion is equal to the rate of change of momentum of the particle in the direction of the resultant force. Newton's second law is vector relation. Consider the x-direction we write for the system.

$$\sum F_x = \frac{d}{dt}(mV_x) \qquad \text{----------14}$$

Where the left hand side represent the algebraic sum of the X-force acting on the system during the time interval dt, and the right hand side represent the time of change of the total momentum of the system see fig.3.

$$\frac{d}{dt}(mV_x) = \frac{(mV_x)_{I\,t+dt} + (mV_x)_{II\,t+dt} - (mV_x)_{I\,t} - (mV_x)_{III\,t}}{dt} \qquad \text{-----15}$$

$\frac{(mV_x)_{I\,t+dt} - (mV_x)_{I\,t}}{dt}$ -------- as dt goes to zero this term represent the time rate of change of the X-momentum within the control volume. $= \frac{\partial}{\partial t}(mV_x)_{c.v}$

so that :

$$\sum F_x = \frac{\partial}{\partial t}(mV_x)_{c.v} + \int V_x dm_{out} - \int V_x dm_{in} \qquad \text{-----------------16}$$

or

$$\sum F_x = \int_{cv} \frac{\partial \rho V_x dv}{\partial t} + \int \rho V_n V_x dA_{out} - \int \rho V_n V_x dA_{in} \qquad \text{---------17}$$

Example:3

Air flowing isentropically in a nozzle strikes a stationary blade when it leaves the nozzle as shown in fig. Determine ;

1- The magnitude of the reaction in the x-direction and in the y-direction needed to hold the blade in place.
2- The magnitude of the reaction in the x-direction and in the y-direction of the blade moves towed the nozzle at 80m/sec.

Solution:

$$T_2 = T_1\left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma} = 308\left(\frac{1}{1.5}\right)^{0.4/1.4} = 274.3\ \text{K}$$

The gas velocity at this section is obtained from the energy equation:

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$$

Therefore

$$\frac{V_2^2}{2} = c_p(T_1 - T_2) + \frac{V_1^2}{2}$$

$$= 1000(308 - 274.3) + \frac{(60)^2}{2}$$

from which $V_2 = 266.46$ m/s. The mass rate of flow is:

$$\dot{m} = \rho_1 A_1 V_1 = \left(\frac{p_1}{RT_1}\right)A_1 V_1$$

$$= \left(\frac{1.5\times 1.013\times 10^5}{287\times 308}\right)(25\times 10^{-4})(60)$$

$$= 0.258\ \text{kg/s}$$

Applying the momentum equation to the control volume shown gives:

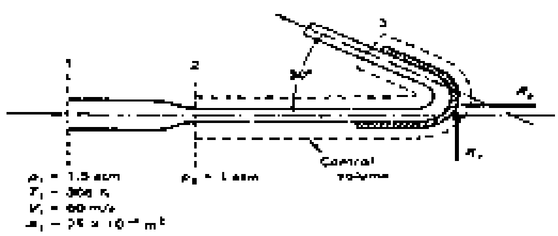
$$R_x = \dot{m}(V_{3x} - V_{2x}) = 0.258(V_3\cos 30 + V_2)$$

$$= 0.258(266.46\cos 30 + 266.46) = 128.28\ \text{N}$$

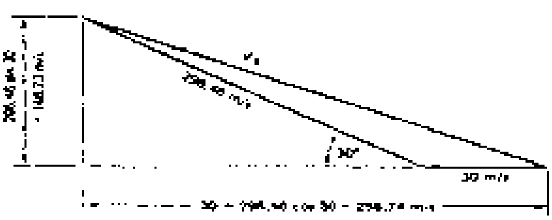
and

$$R_y = \dot{m}(V_{3y} - V_{2y}) = 0.258(V_3\sin 30 - 0)$$

$$= 0.258(266.46\sin 30) = 34.37\ \text{N}$$



(a)



(b) When the blade moves toward the nozzle, the relative velocity is 266.46 + 30 = 296.46 m/s. The mass striking the blade per unit time now becomes:

$$\dot{m} = 0.258\left(\frac{296.46}{266.46}\right) = 0.287 \text{ kg/s}$$

From the velocity diagram shown:

$$V_{3x} = 256.74 \text{ m/s} \quad \text{and} \quad V_{3y} = 148.23 \text{ m/s}$$

The momentum equation then gives:

$$R_x = \dot{m}(V_{3x} - V_{2x}) = 0.287(256.14 + 266.46) = 149.7 \text{ N}$$

and

$$R_y = \dot{m}(V_{3y} - V_{2y}) = 0.287(148.23 - 0) = 42.54 \text{ N}$$

Example:4

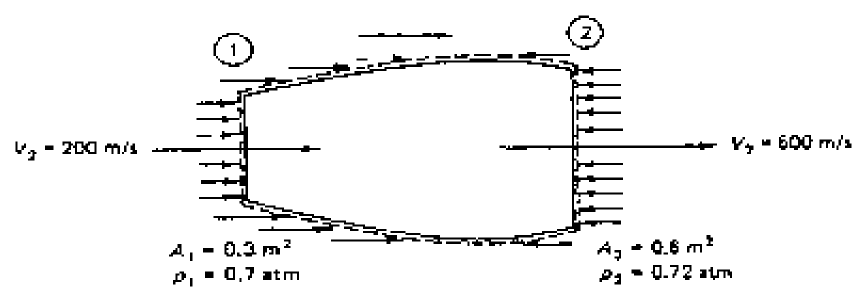
An airplane is traveling at a constant speed of 200 m/s. Air enters the jet engine's inlet at the rate of 40 kg/s while the combustion products are discharged at an exit velocity of 600 m/s relative to the airplane. The intake area is 0.3 m^2 and the exit area 0.6 m^2. The ambient pressure is 0.7 atm, and the pressure at the exit is 0.72 atm. Calculate the net thrust developed by the engine. Assume uniform steady conditions at the inlet and exit planes and the properties of the products of combustion to be the same as those of air.

Solution: consider the jet engine as a control volume as in fig. the air enters the engine with a speed of 200m/s. assuming horizontal flight and neglecting the momentum of the fuel, the net force opposite to thrust is:
Appling momentum equation:

$$\sum F_x = \int_v \frac{\partial \rho dv}{\partial t} + \int V_x dm_{out} - \int V_x dm_{in}$$

since the case is steady state thus mean that $\partial\rho/\partial t$=0 therefore the momentum equation become

$$\begin{aligned} F &= (p_2 A_2 + \dot{m} V_2) - (p_1 A_1 + \dot{m} V_1) \\ &= [(0.72 - 0.7)1.013 \times 10^5 \times 0.6 + 40 \times 600] - (0 + 40 \times 200) \\ &= 17{,}215.6 \text{ N} \end{aligned}$$



1.3The First Law of thermodynamic: (Energy Equation)

Energy is conveyed a cross the boundary of control volume in he form of heat and work. Consider the flow through the control volume with of fig., with the system defined as the material occcuping the control volume at time t. We consider what happens during the time interval dt. Passing through the control surface are a stationary strut and a rotating shaft attached to a turbo-machine, perhaps a compressor or turbine. The energy equation in a simple form can be written as following.

$$\frac{\delta Q}{dt}=\frac{dE}{dt}+\frac{\delta W}{dt} \quad \text{--------------------}$$

Rate of change of total energy E:

$$\frac{dE}{dt}=\frac{(E_{I\,t+dt}+E_{II\,t+dt})-(E_{I\,t}+E_{III\,t})}{dt} \quad \text{-----------------}$$

$$\frac{dE}{dt}=\frac{E_{I\,t+dt}-E_{I\,t}}{dt}+\int\frac{e\,\delta m_{out}}{dt}-\int\frac{e\,\delta m_{in}}{dt} \quad \text{------------------}$$

δ

$$\frac{dE}{dt}=(\frac{\partial E}{\partial t})_{cv}+\int e\,dm_{out}-\int e\,dm_{in} \quad \text{---------------------}$$

$$\frac{dE}{dt}=\int\frac{\partial e\rho dV}{\partial t}+\int e\,dm_{out}-\int e\,dm_{in} \quad \text{------------------}$$

Rate of work done.

Omitting from our consideration capillary, magnetic, and electrical force, the work done during the processes is the result of normal and shear stresses at the moving boundaries of the system.

A- Work Done by Normal Stresses.

Taking the normal stress at the boundary of the system as the hydrostatic pressure, the work done by the system owing to normal force at an element of area dA_{out} is $pdA_{out}dx$, where dx is the component of distance moved normal to dA_{out}. But $dA_{out}dx$ is the volume of the mass element $\delta m_{II\,t+dt}$ which volume may be writeen as $v\delta m_{II\,t+dt}$. The total rate of work done by normal stresses during the process may now be set down, with the aid of the foregoing, as

$$(\frac{\delta W}{dt})_{normal}=\frac{\int p\,v\delta m_{II\,t+dt}}{dt}-\frac{\int p\,v\delta m_{III}}{dt} \quad \text{--------------}$$

$$=\int p\,v dm_{out}-\int p\,v dm_{in} \quad \text{--------------}$$

B- Work Done by Shear Stresses: This work may be conveniently divided into two categories (i) the work done by the part of the shaft inside the system on the part outside the system, owing to the torque in the rotating shaft resulting from the shear stresses. (ii) the shear work done at the boundaries of the system on adjacent fluid which is in motion. Therefore the rate change of work can be written as follow.

$$\frac{\delta W}{dt}=W_{shaft}+W_{shear}+\int p\,v dm_{out}+\int p\,v dm_{in} \quad \text{--------------}$$

The total fluid energy per mass flow e is

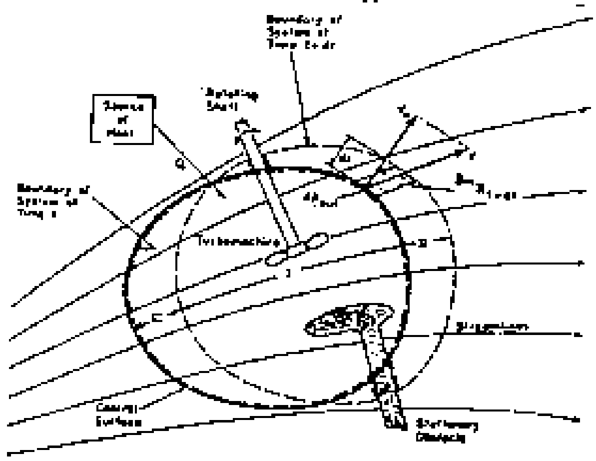
Total fluid energy= internal energy + kinetic energy + potential energy

$$e = u + \frac{V^2}{2} + gz \text{ ----------------}$$

$$u = h - pv = h - \frac{p}{\rho} \text{ --------------------}$$

Substitute these equations into the energy equation results

$$\frac{\delta Q}{dt} = W_{shaft} + W_{shear} + \int_{cv} \frac{\partial e \rho dv}{dt} + \int (h + \frac{V^2}{2} + gz) dm_{out} - \int (h + \frac{V^2}{2} + gz) dm_{in} \text{ ----------}$$



1.4 The second Law of Thermodynamics:

In a fixed-mass system entropy change occurs as a result of irreversible events or as a result of interaction with the environment in which there is heat transfer.

$$\oint_{cs} \frac{dQ}{T} \leq (\frac{\partial S}{\partial t})_{cv} + \int s dm_{out} + \int s dm_{in}$$

$$\oint_{cs} \frac{dQ}{T} \leq \int_{cv} \frac{\partial s \rho dv}{\partial t} + \oint_{cs} s \rho V dA$$

for steady –one dimension flow

$$m(s_2 - s_1) \geq \oint_{cs} \frac{dQ}{T}$$

for adiabatic flow $dQ=0$ therefore

$s_2 - s_1 \geq 0$ or $ds \geq 0$ for isentropic flow $ds = 0$ and flow adiabatic irreversible flow $ds>0$

1.5 The perfect Gas:

For most problem in gas dynamics, the assumption of perfect gas law is sufficiently in accord with the properties of real gases as to be a acceptable. We shall therefore set down here the special thermodynamics relations which apply to perfect gas.

1- *Equation of state:*

$$pv = \frac{p}{\rho} = RT = \frac{\Re}{M} T \text{ --------------}$$

Where T is the absolute temperature (K^o), R is the gas constant($J/kg.mol.K^o$), $\Re$ is the universal gas constant and is equal to 8134.3 $J/kg.mol.K^o$, and M is the

molecular weight kg/kg.mol. For atmospheric air between 0 and 100 km, M=28.966, therefore the air gas constant is 287.04 J/kg.K°

When a perfect gas undergoes a thermodynamic process between to equilibrium state.

$$u_2 - u_1 = \int_1^2 cv.dT \quad \text{and} \quad h_2 - h_1 = \int_1^2 cp.dT$$

$$cv = (\frac{\partial u}{\partial T})_v = \frac{du}{dT} \quad \text{and} \quad Cp = (\frac{\partial h}{\partial T})_p = \frac{dh}{dT} \qquad \text{for perfect gas}$$

$$Cp - Cv = \frac{dh}{dT} - \frac{du}{dT} = \frac{d(u + pv)}{dt} - \frac{du}{dT} = \frac{d(RT)}{dT}$$

$$Cp - Cv = R \text{ --}$$

The specific heat ratio γ is $\gamma = \frac{Cp}{Cv}$ therefore $Cp = \frac{\gamma R}{\gamma - 1}$ and $Cv = \frac{R}{\gamma - 1}$

Changes of Entropy : Applying the special relation of a perfect gas to the general relation between s, u, v, we get

$$ds = \frac{du}{T} + \frac{pdv}{T} = Cv\frac{dT}{T} + R\frac{dv}{v}$$

and, upon integration

$$S_2 - S_1 = Cv \ln \frac{T_2}{T_1} + R \ln \frac{v_2}{v_1} = Cv \ln(\frac{T_2}{T_1}).(\frac{v_2}{v_1})^{\gamma - 1} \text{ ----------------------}$$

Alternatively, we may eliminate either T or v from this express the aid of $pv=RT$,and so obtain

$$S_2 - S_1 = Cv \ln \frac{p_2}{p_1} + Cp \ln \frac{v_2}{v_1} = Cv \ln(\frac{p_2}{p_1})(\frac{v_2}{v_1})^{\gamma} \text{ ------------------------}$$

$$S_2 - S_1 = Cp \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} = Cv \ln(\frac{T_2}{T_1})^{\gamma}.(\frac{p_2}{p_1})^{-\gamma - 1} \text{ ------------------------}$$

The isentropic. Often the isentropic process is taken as a model or as a limit for real adiabatic processes. If entropy is constant at each step of the processes, it follows from equation that T and v,p and v, and T and p are connected with each other during the processes by the following laws:

$$Tv^{\gamma - 1} = const. \qquad pv^{\gamma} = \frac{p}{\rho^{\gamma}} = const. \qquad \frac{T^{\frac{\gamma}{\gamma - 1}}}{p} = const. \text{ ------------------}$$

For isentropic flow process the enthalpy change is important. It is calculated in terms of the initial temperature and the pressure ratio as follows:

$$(\Delta h)_s = Cp(T_2 - T_1) = Cp.T_1\left[(\frac{T_2}{T_1}) - 1\right] = CpT_1\left[(\frac{p_2}{p_1})^{\frac{\gamma - 1}{\gamma}} - 1\right] \text{ --------------------}$$

Chapter Two

Wave Propagation in Compressible flow

2.1 Introduction:

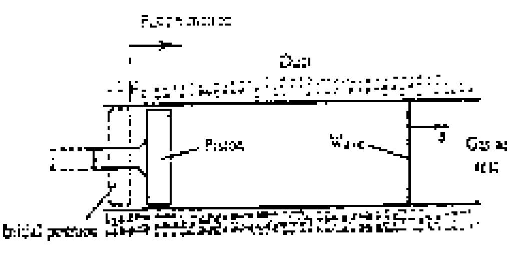
The term compressible flow implies variation in density through the field of flow. These variations are, in many cases, the result principally of pressure changes from one point to another. The rate of change of density with respect to pressure is, therefore, an important parameter in the analysis of compressible flow, and, as we shall see, it is closely connected with the velocity of propagation of small pressure disturbance, i.e. with the velocity of sound.

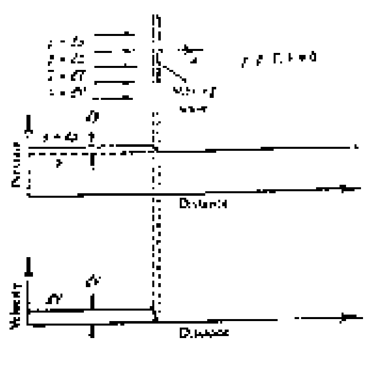
2.2 Wave Propagation in Elastic Media:

let us examine what happens when a solid elastic object such as steel bar is subjected to a sudden uniform distributed compressive stress applied at one end. in the first instant of time, a thin layer next to the point of application is compressed, while the remainder of the bar is unaffected. This compression is then transmitted to the next layer, and so on down the bar. Thus a disturbance created at the left side is eventually sensed at the opposite end. The compression wave initiated at the left side of the bar takes a finite time to travel to the right side, the wave velocity being dependent on the elasticity and density of the media.

Gases and liquid also are elastic substance and longitudinal wave can be propagated through these media in the same way that waves propagated through solid. Let a gas be confined in a long tube with a piston at the left hand. The piston is given a sudden push to the right. In the first instant a layer of gas piles up next to the piston and is compressed, the reminder of the gas is unaffected. The compression wave created by the piston then moves through the gas until eventually all the gas is able to sense the movement of the piston. If the impulse given to the gas is infinitesimally small, the wave is called a sound wave and the resultant compression wave move through the gas at velocity equal to the velocity of sound.

Let the pressure change across the wave be dp and let the corresponding density and temperature change be $d\rho$ and dT respectively. The gas into which the wave is propagated is assumed to be at rest. The wave will then induce a gas velocity dV behind it as it move through the gas. The changes across the wave are, therefore as shown in fig.2.2. In order to analyze the flow through the wave and thus to determine (a), it is convenient to use a coordinated system that is attached to the wave, i.e. is moving with the wave. In this coordinate system, the wave will of course be at rest and the gas will effectively flow through it with the velocity ,a, ahead of the wave and a velocity, a-dV, behind the wave. In this coordinate system, then, the changes through the wave are shown in fig.2.3. The pressure, temperature and density change, of course, independent of the coordinate system used.





The continuity and momentum equation are applied to a control volume of unit area across the wave as indicated in fig. For steady state the continuity equation for the control volume is:

$m^{\cdot} = \rho a = (\rho + d\rho)(a - dV)$ ------------------2.1

where m is the mass low rate per unit area through the wave. Since the case of a very weak is being consider, the second order term, $d\rho dV$ that arises in equation can be neglected and this equation then gives:

$$d\rho = \frac{\rho}{a} dV \text{ ------------------------2.2}$$

Conservation of momentum is next considered. The only force acting on the control volume are the pressure force. The momentum equation for steady state become:

$pA - (p + dp)A = m^{\cdot}[(a - dV) - a]$ ----------------------2.3

which lead to:

$$Adp = m^{\cdot} dV \quad \text{or} \quad dp = \rho a dV \text{ -------- 2.4}$$

Substitute equation 2.2 into equation 2.4 gives:

$$\frac{dp}{d\rho} = a^2 \text{ or } a = \sqrt{\frac{dp}{d\rho}} \text{ --------2.5}$$

In order to evaluate a using the above equation, it is necessary to know the process that the gas undergoes in passing through the wave. Because a very weak wave is being considered, the temperature and velocity changes through the wave are very small and the gradient of temperature and velocity within the wave remain small. For this reason, heat transfer and viscous effect for flow through the wave are assumed to be negligible. Hence, is passing through the wave, the gas is assumed to undergo an isentropic process. The flow through the wave is, therefore, assumed to satisfy:

$$\frac{p}{\rho^{\gamma}} = const. \text{ --------------------2.6}$$

putting this into logarithmic form, and differentiating the equation:

$\ln p - \gamma \ln \rho = const.$

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho} \text{ or } \frac{dp}{d\rho} = \frac{\gamma p}{\rho} \text{ -------------- 2.7}$$

noting that the fluid is compressible and is perfect gas, therefore $p = \rho RT$ substituting this into equation 2.7 and equation 2.5.

$$a = \frac{dp}{d\rho} = \sqrt{\gamma RT} \text{ ---------------2.8}$$

2.3 Pressure Field Created by a Moving Point Disturbance:

In order to illustrate the effect of the velocity of the body relative to the speed of sound on the flow field, consider the small body, i.e. essentially a point source of disturbance, to be moving at a uniform liner velocity, through the gas and let the speed of sound in the gas be a. Although the body is essentially emitted wave continuously, a series of wave emitted at time interval t will be consider. Since the body is moving through the gas, the origin of these waves will be continually changing. Wave generated at time 0,t,2t, and 3t will be considered. First, consider the case where the speed of the body is very small compared to the speed of sound. The pressure pattern which exists at any instant is then found by superposition of all the pressure pulses which were previously emitted. Fig. shows several pressure pulse pattern for different value of the speed of the source compared with the speed of sound in the fluid.

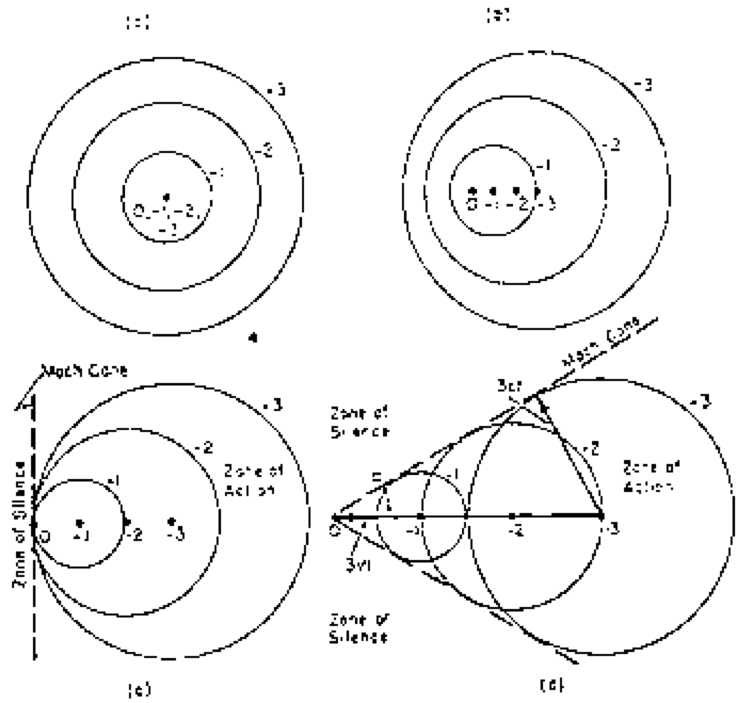


Fig (2.3). Pressure field produced by a point source of disturbance moving at uniform speed leftwards.

(a) Incompressible fluid ($V/c = 0$).
(b) Subsonic motion ($V/c = \frac{1}{2}$).
(c) Transonic motion ($V/c = 1$).
(d) Supersonic motion, illustrating Karman's three rules of supersonic flow ($V/c = 2$).

*- **Incompressible Flow:** When the medium is incompressible (fig.2.3a). or when the speed of the moving point disturbance is small compared with the speed of sound. the pressure pulse spread uniformly in all direction.

٣- **Subsonic Flow**: When the source move at subsonic speeds, Fig.2.3b, the pressure disturbance is felt in all direction and at all points in space, but the pressure pattern is no longer symmetrical.
٤- **Supersonic Flow**: For supersonic speed Fig.2.3d indicates that the phenomena are entirely different from those at subsonic speed. All the pressure disturbance are included in a cone which has the point source of disturbance. The cone within which the disturbances are confined is called the Mach cone. Fig.2.3c shows the pressure pattern at the boundary between subsonic and supersonic, that is, for the case where the stream velocity is identical with the sonic velocity; here the wave front is a plane.

Karman's Rules of Supersonic Flow : Fig 2.3 illustrates the three rules of supersonic flow proposed by Van Karman's .

1- **The Rules of Forbidden Signals.** The effect of pressure change produced by a body moving at a speed faster than sound cannot reach point ahead of the body.
2- **The Zone of Action and the Zone of Silences.** A stationary point source in a supersonic stream produces effect only on point that lie on or inside the Mach cone extending downstream from the point source. Conversely, the pressure and velocity at an arbitrary point of the stream can be influenced only by disturbance acting at point that lies on or inside a cone extending upstream from the point considered and having the same vertex angle as the Mach cone.
3- **The Rule of Concentrated Action.** The pressure disturbance is largely concentrated in the neighbourhood of the Mach cone that forms the outer limit of the zone of action.

2.4 The Mach Number and the Mach Angle:

It was shown that the nature of the flow pattern depends on the comparative magnitudes of the stream velocity and the sonic velocity. The ratio of these two velocity is called the Mach Number. Thus,

$$M = \frac{V}{a} \quad \text{.......................2.9}$$

The semi-angle of the Mach cone is related to the Mach number by

$$\sin\alpha = \frac{1}{M} \quad \text{.......................2-10}$$

Note that the mach angle is imaginary for subsonic flow.

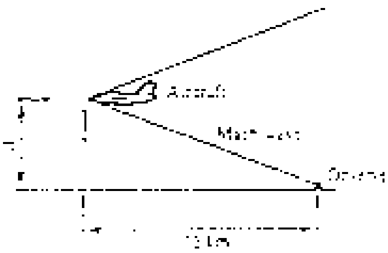
Example:

An observer on the ground finds that an airplane flying horizontally at an altitude of 5000 m has traveled **12 km** from the overhead position before the sound of the airplane is first heard. Estimate the speed at which the airplane is flying.

Solution

It is assumed that the net disturbance produced by the aircraft is weak, i.e., that, as indicated by the wording of the question, basically what is being investigated is how far the aircraft will have traveled from the overhead position when the sound waves emitted by the aircraft are first heard by the observer. If the discussion of Mach waves given above is considered, it will be seen that, as indicated in Fig. E3.9, the aircraft will first be heard by the observer when the Mach wave emanating from the nose of the aircraft reaches the observer.

Now, since the temperature varies through the atmosphere, the speed of sound varies as the sound waves pass down through the atmosphere which means that the Mach waves from the aircraft are actually curved. This effect is, however, small and will be neglected here, the sound speed at the average temperature between the ground and the aircraft being used to describe the Mach wave.



Now as discussed in Example 3.3, for altitudes, H, of from 0 m (sea-level) to 11019 m, the temperature in the atmosphere is given by $T = 288.16 - 0.0065H$ so, at the mean altitude of 2500 m, the temperature is $288.16 - 0.0065 \times 2500 = 271.9$ K. Hence, the mean speed of sound is given by:

$$a = \sqrt{\gamma RT} = \sqrt{1.4 \times 287.04 \times 271.9} = 330.6 \text{ m/s}$$

From the above figure it will be seen that if α is the Mach angle based on the mean speed of sound then

$$\tan\alpha = 5000/12\,000 = 0.417$$

But since $\sin\alpha = 1/M$, it follows that $\tan\alpha = 1/\sqrt{M^2 - 1}$ so

$$M = \sqrt{(1/0.417)^2 - 1} = 2.6$$

Hence, it follows that:

$$\text{Velocity of aircraft} = 2.6 \times 330.6 = 859.6 \text{ m/s}$$

Problem:

2.1 Air at a temperature of 25°C is flowing with a velocity of 180 m/s. A projectile is fired into the air stream with a velocity of 800 m/s in the opposite direction to that of the air flow. Calculate the angle that the Mach waves from the projectile make to the direction of motion.

2.2 An observer at sea level does not hear an aircraft that is flying at an altitude of 7000 m until it is a distance of 13 km from the observer. Estimate the Mach number at which the aircraft is flying. In arriving at the answer, assume that the average temperature of the air between sea level and 7000 m is −10°C.

2.3 An observer on the ground finds that an airplane flying horizontally at an altitude of 2500 m has traveled 6 km from the overhead position before the sound of the airplane is first heard. Assuming that, overall, the aircraft creates a small disturbance, estimate the speed at which the airplane is flying. The average air temperature between the ground and the altitude at which the airplane is flying is 10°C. Explain the assumptions you have made in arriving at the answer.

In the absent of electromagnetic force and with friction negligible, the only force acting on the control surface are pressure force. Assume that a pressure $p-dp/2$ acts on the side surface of the control volume.

$$pA + (p + \frac{dp}{2})dA - (p + dp)(A + dA) = (\rho AV)(V + dV - V),$$

Simplifying yields.

$$dp + \rho VdV = 0 \quad \text{------------------------3.2}$$

The energy equation with no external heat transfer and no work, for stady one-dimensional flow become.

$$\oint_S (h + \frac{V^2}{2})(\rho VdA) = 0 \quad \text{------------------------}$$

or $\quad dh + d\frac{V^2}{2} = 0$

An expression for the second law of thermodynamic is given :

$Tds = dh - \frac{dp}{\rho}$ and for isentropic flow $ds=0$,therefore $dh = \frac{dp}{\rho}$

Combining these equation we obtain:

$\frac{dp}{\rho} = -d\frac{V^2}{2}$ or $dp + \rho VdV = 0$ which is the same as the momentum equation

3.3 Isentropic flow Through a Varying Area Channel.

Combining the continuity and momentum equation for isentropic flow result in.

$$dp + \rho V^2\left[-\frac{d\rho}{\rho} - \frac{dA}{A}\right] = 0$$

But

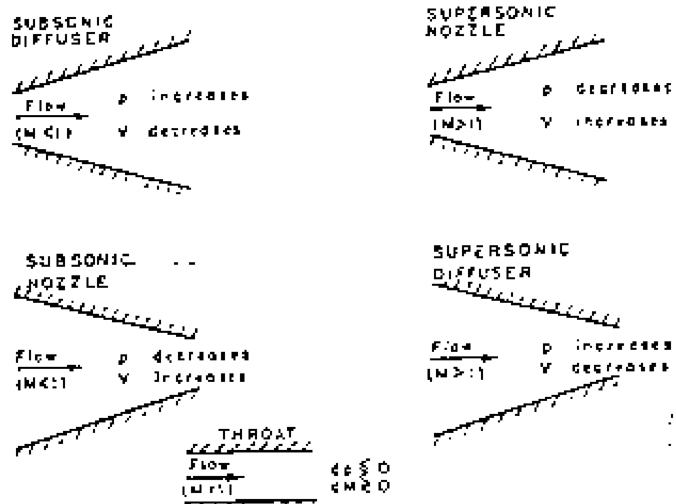
$\frac{\partial p}{\partial \rho} = a^2$ Therefore, for isentropic flow

$$dp - \rho V^2(-\frac{dp}{\rho a^2} - \frac{dA}{A}) = 0 \text{ and } M = \frac{V}{a}$$

$$dp(1 - M^2) = \rho V^2\frac{dA}{A} \quad \text{--------------------3.3}$$

Equation 3.3 demonstrates the influence of Mach number on that flow. For M<1, subsonic flow, the term $1-M^2$ is positive. Therefore, an increase in area result in an increase in pressure and from equation 3.2 a decrease in velocity. Likewise, a decrease in area results in decrease in pressure and an increase in velocity. For supersonic flow, the term $1-M^2$ in equation 3.3 is negative, and opposite variation occur. The result illustrate in fig have ramifications. Subsonic flow cannot be accelerated to a velocity greater than the velocity of sound in a converging nozzle. This is true irrespective of the pressure difference imposed on the flow through the nozzle. If it is desired to accelerate a stream from negligible velocity to supersonic velocity. A convergent-divergent channel must be used as show in fig.

Fig 3.2 Show the variation of the pressures and velocity in different shape of area change for subsonic and supersonic flow.



M = 1

3.4 Stagnation ~~and~~ Properties:

Stagnation properties are useful in that they define a reference state for compressible flow. Stagnation enthalpy or total enthalpy, at appoint in flow is defined as the enthalpy attained by bringing the flow adiabatically to rest at that point. For adiabatic process energy equation become

$$h_o = h + \frac{V^2}{2}$$

Where h_o is the stagnation or total enthalpy per unit mass. Likewise, stagnation temperature T_t or T_o can be defined as the temperature measured by bringing a flow adiabatically to rest at a point. For a perfect gas with constant specific heats the energy equation becomes:

$$c_p T_o + \frac{V_o^2}{2} = c_p T + \frac{V^2}{2}$$ since $V_o=0$, therefore

$$c_p T_o = c_p T + \frac{V^2}{2} \quad \text{or} \quad T_o = T + \frac{V^2}{2c_p} \quad \text{or} \quad \frac{T_o}{T} = (1 + \frac{V^2}{2c_pT}) \quad \text{since } c_p = \frac{\gamma R}{\gamma - 1}$$

Therefore. $\frac{T_o}{T} = (1 + \frac{(\gamma-1)V^2}{2\gamma RT})$ whereas $a = \sqrt{\gamma RT}$, and $M = \frac{V}{a}$ Therefore.

$$\frac{T_o}{T} = (1 + \frac{\gamma - 1}{2} M^2) \quad \text{-----------------------3.4}$$

For isentropic flow the relation between pressure, temperature and density of perfect gas are: $\frac{p}{p_o} = (\frac{\rho}{\rho_o})^\gamma$ and $\frac{T}{T_o} = (\frac{p}{p_o})^{\frac{\gamma-1}{\gamma}}$ Therefore the pressure and density relation become.

$$\frac{p_o}{p} = (1 + \frac{\gamma - 1}{2} M^2)^{\frac{\gamma}{\gamma-1}} \quad \text{-----------------------3.5}$$ Apply Isentropic only

$$\frac{\rho_o}{\rho} = (1 + \frac{\gamma - 1}{2} M^2)^{\frac{1}{\gamma-1}} \quad \text{-----------------------3.6}$$

3.5 Flow per Unit Area.

Next we will derive a useful relation between the flow per unit area, stagnation temperature, pressure and Mach number for perfect gas. Starting with the equation of continuity we make the following arrangements:

$$\frac{m^*}{A} = \rho V = \frac{p}{RT}V = \frac{pV}{\sqrt{\gamma RT}}\sqrt{\frac{\gamma}{R}}\sqrt{\frac{T_o}{T}}\frac{1}{\sqrt{T_o}}$$

Substitute equation 3.4 for adiabatic flow

$$\frac{m^*}{A} = \sqrt{\frac{\gamma}{R}}\frac{p}{\sqrt{T_o}}M\sqrt{1-\frac{\gamma-1}{2}M^2} \quad \text{--------------3.7}$$

To find a conventional formula for the mass flow per unit area in terms of M, we eliminate p in the equation above by means of the isentropic law relation , or substitute equation 3.5.

$$\frac{m^*}{A} = \sqrt{\frac{\gamma}{R}}\frac{p_o}{\sqrt{T_o}}\frac{M}{(1-\frac{\gamma-1}{2}M^2)^{\frac{\gamma+1}{2(\gamma-1)}}} \quad \text{--------------3.8}$$

3.6 Maximum Flow per Unit Area: To find the condition of maximum flow per unit area we could differentiate equation 3.8 with respect to M and set this derivative equal to zero. At this condition, we would find that M=1. Therefore to find m/A_{max} we need only set M=1 in equation 3.8 , thus we find.

$$\left(\frac{m^*}{A}\right)_{max} = \frac{m^*}{A^*} = \sqrt{\frac{\gamma}{R}}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\frac{p_o}{\sqrt{T_o}} \quad \text{----------------3.9}$$

For a given gas, therefore, the maximum flow per unit area depends only on the ratio $p_o/\sqrt{T_o}$. For a given value of the stagnation pressure and stagnation temperature and for a passage with minimum area, Equation 3.9 shows that maximum flow which can be passed is relatively large for gases of high molecular weight and relatively small for gases of low molecular weight. Doubling the pressure level doubles the maximum flow, whereas doubling the absolute temperature level reduce the maximum flow by a bout 29 per cent. For air with γ=1.4 and R=287 J/kg.K° the maximum mass flow per unit area is:

$$\frac{m^*\sqrt{T_o}}{A^*p_o} = 0.04042$$

The particular value of the temperature, pressure and density ratios at the critical state (i.e at the minimum area) are found by setting M=1 in equations 3.4, 3.5, 3.6. We will refer to the critical properties by superscript asterisk (*).

$$\frac{T^*}{T_o} = \left(\frac{2}{\gamma+1}\right) \quad \text{for air} = 0.833$$

$$\frac{p^*}{p_o} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \quad \text{for air} = 0.5283 \qquad \frac{\rho^*}{\rho_o} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \quad \text{for air} = 0.6339$$

3.7 The area Ratio.

Just as we have found it convenient to work with the dimensionless ratio p/p_o etc. it is convenient to introduce a dimensionless area ratio. Obviously the appropriate reference area is A^*, and so we compute from equation 3.8 and 3.9 the formula.

$$\frac{A}{A^*} = \frac{m}{m}\frac{A^*}{A} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma+1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} \quad \text{............... 3.10}$$

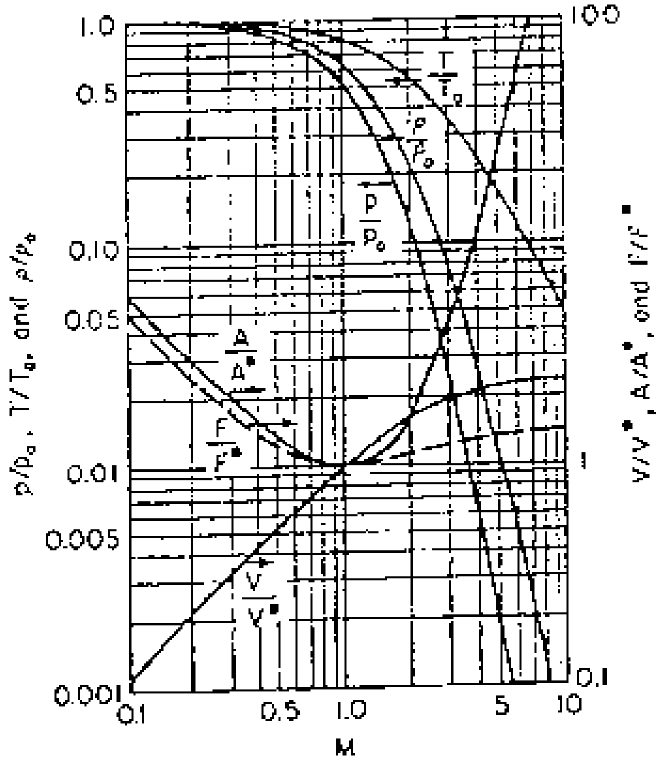
The area ration is always grater than unity, and for any given value of A/A^* there always correspond two value of M, one for subsonic flow and the other for supersonic flow

3.8 Working Charts and Tables for isentropic Flow

Since the formulas thus fae derived lead to tedious numerical calculation , of the of a trial-error natural, practical computation are greatly facilitated by working chart and tables.

Chart for Isentropic Flow.

Fig. represent in graphical form the various dimensionless ratio foe isentropic flow with M as independent variable. Since changes of fluid properties in isentropic flow are brought a bout through change in cross-sectional area, the key curve on this chart is that of A/A^*. The effects of change in area on other properties may easily be found by tracing the curve of A/A^*, keeping in mind that A^*, p_o ,etc. are all constant reference value for a given problem. For example, an increase in area at subsonic speed produces a decrease in velocity, an increase in p, T, ρ.



Working Tables.

For accurate or extensive calculation tables is available. lists the various isentropic function for γ=1.4 with Mach number as independent argument.

M	p/p_o	T/T_o	ρ/ρ_o	A/A^*	$\frac{A}{A^*}\cdot\frac{p}{P_o}$
0.5	0.843	0.95238	0.8893	1.3398	1.1295
2.0	0.1278	0.55556	0.2300	1.6875	0.21567

3.9 Isentropic Flow in Convergent Nozzle:

Consider a fluid stored in a large reservoir is to be discharge through a converging nozzle to region where the back pressure P_b is controllable by means of a valve. For a constant reservoir pressure P_o it is desired to study the effects of the variations in back pressure on the rat of mass flow through the nozzle, the pressure distribution along the passage and on the exit-plane pressure P_e. These effect are portrayed graphically in Fig am b. and c, respectively.

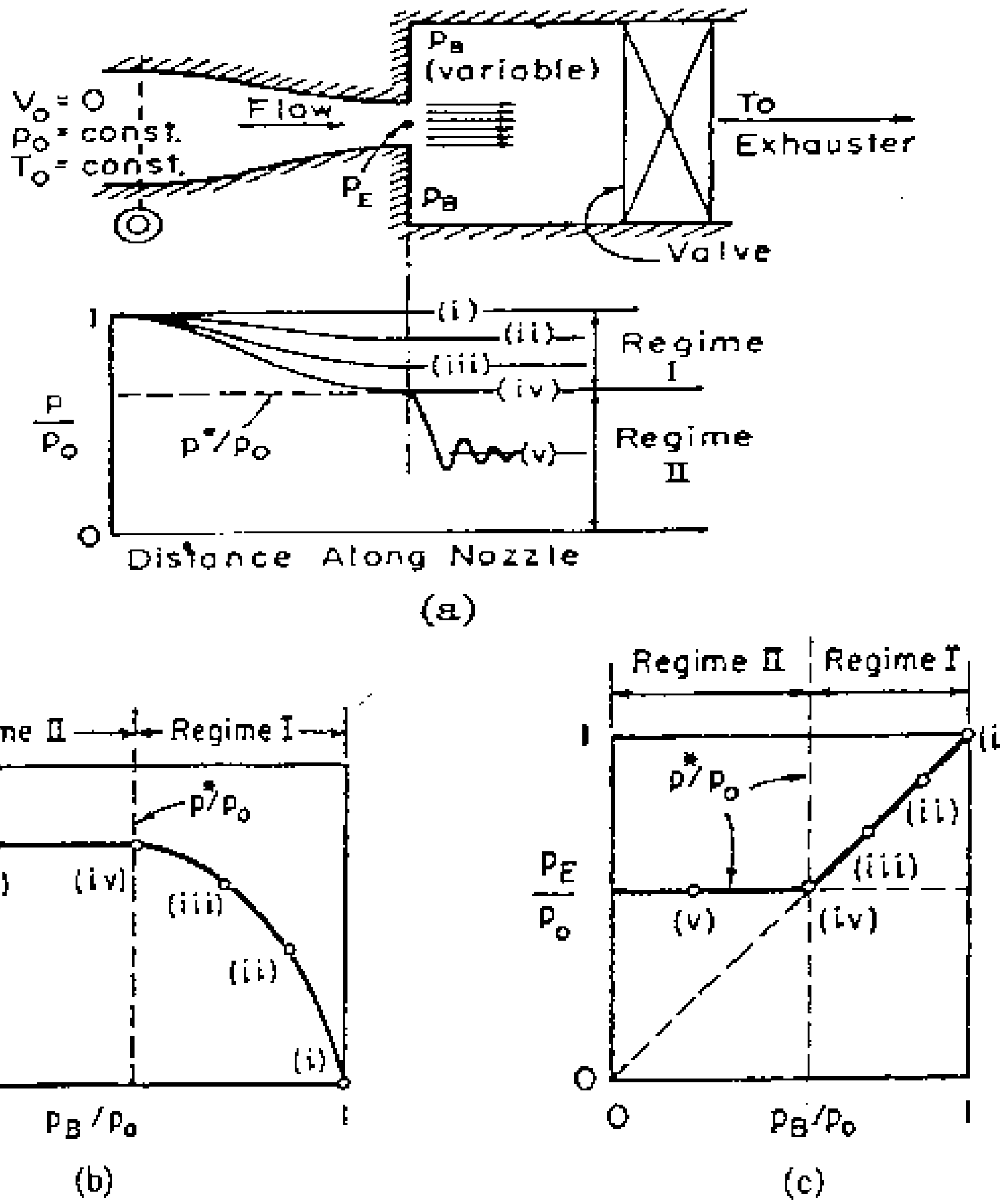


Fig. operation of converging nozzle at various back pressure.

To begin with, suppose that $P_b/P_o=1$, shown as condition (i) in fig.. The pressure is then constant through the nozzle, and there is no flow. If P_b is now reduced to a value slightly less than P_o as shown by condition (ii), there will be flow with a constantly decreasing pressure through the nozzle. Because the exit flow is subsonic, the exit-plan pressure Pe must be the same as the back pressure P_b. A further reduction in P_b to condition(iii) acts to increase the flow rate and to change the pressure distribution , but there is no qualitative change in performance. Similar consideration apply until condition (v) is reach at which point P_b/P_o equal the critical pressure ratio and the value of Me equal unity. Further reduction in Pb/Po, say to condition (v), cannot produce further change in condition within the nozzle, for the value of Pe/Po cannot be made less than the critical pressure ratio unless there is a throat upstream of the exit section (it is assumed here that the stream fills the passage). Consequently at condition(v), the pressure distribution within the nozzle, the value of Pe/Po , and the flow rate are all identical with the corresponding quantities for condition (iv). When the flow reach the condition the flow is called to be chocked.

To summarize the proceeding discussion, the two different type of flow will be denoted as regime I and regime II. These regimes may be compared as follows.

Regime I	Regime II
$Pb/Po > P^*/Po$	$Pb/Po < P^*/Po$
$Pe/Po = Pb/Po$	$Pb/Po = P^*/Po$
M<1	M=1
$\dfrac{m\sqrt{To}}{Ae.Po}$ dependent on Pb/Po	$\dfrac{m\sqrt{To}}{Ae.Po}$ independent on Pb/Po

3-10 Convergent-Divergent Nozzles:

Consider an experiment similar to the one describe, except that a converging –diverging nozzle is to be used. Fig. With Pb less than Po by a small amount , the flow is similar to that through a venture passage, and it may be treated approximately as incompressible. The corresponding pressure distribution is shown by curve(i) and (ii) in fig. When Pb/Po is reduced to the value corresponding to curve (iii). The Mach Number at the throat is unity, and no further reduction in Pt/Po are possible if the stream fills the passage. We consider next the operation when the flow is entirely supersonic, corresponding to curve(iv). The value of Pb/Po for curve(iv) corresponds exactly to the area ratio of the nozzle. Ae/At, as given by isentropic table(in this case At=A*, since Mt=1). This is often called the ***design pressure ratio of the nozzle.***

- No flow pattern fulfilling the condition of isentropic and one-dimensional flow can be found which will correspond to values of *Pb/Po* between those of curves (iii) and (iv) in fig. One method of finding solutions foe these boundary condition is to suppose that irreversible discontinuity involving entropy increase occur somewhere within the passage.

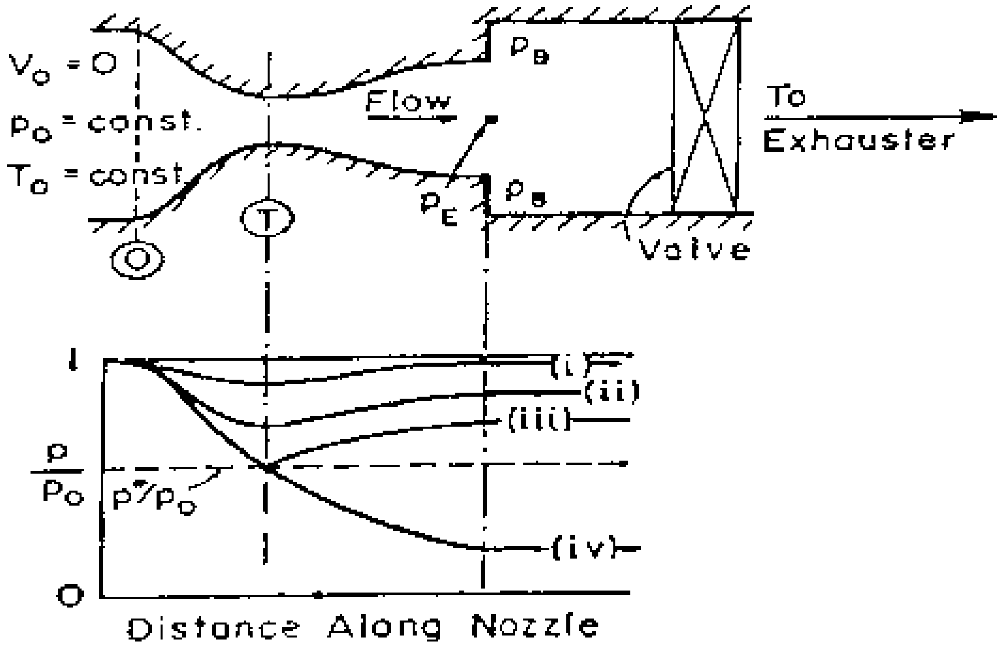


Fig. Operation of converging-diverging nozzle at various back pressure.

3-11 Some Application of Isentropic Flow.

Thrust of Rocket Motor. Rocket motor is generally consist of two parts, the combustion chamber which is a container where the fuel is burn and the thrust unit where the thrust is develop. The thrust unit is almost a convergent-divergent nozzle. The combustion chamber is generate gasses steadily at a stagnation pressure of Po and stagnation temperature of To and then the gas is expanded isentopically in the thrust unit as show in fig.

The converging-diverging nozzle has a throat area of At and exit area of Ae. The generated gases discharge to the atmosphere at pressure of Pa. Most rocket engine gases at about 3600kPa and operate in atmospheres with pressure of 101,3kPa or less, therefore, such a reduction in pressure is only possible by converging-diverging nozzle. The net thrust acting on the rocket

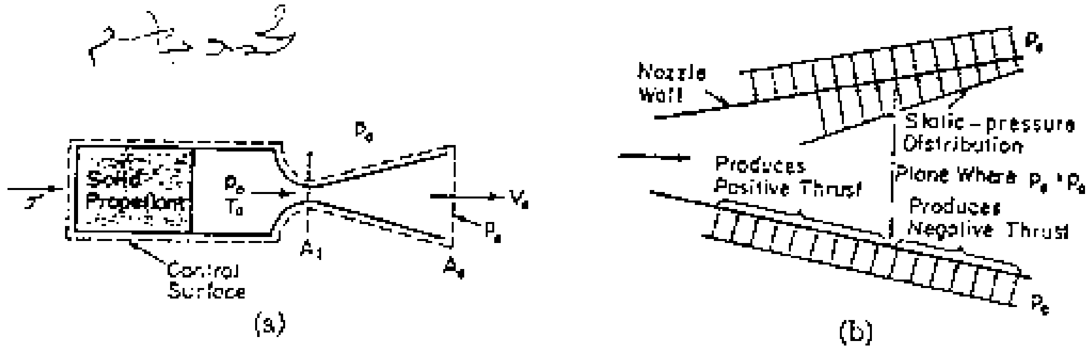


Fig Isentropic flow in rocket motor.

engine may now be obtained by applying the momentum equation on the free body diagrams of the control volume.

$$\Im = \dot m V_e + A_e(P_e - P_a) \text{ ---------------------- } 3.11$$

which is then put into dimensionless for through division by $P_o A_t$.

$$\frac{\Im}{P_o A_t} = \frac{\dot m}{P_o A_t} V_e + \frac{A_e}{A_t}\left(\frac{P_e}{P_o} - \frac{P_a}{P_o}\right) \text{ ---------------- } 3.12$$

From choked flow equation

$$\frac{\dot m}{P_o A_t} = \sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}\frac{1}{\sqrt{T_o}} \text{ ----------------------- }$$

and from the energy equation :

$$V_e = \sqrt{2C_p(T_o - T_e)} = \sqrt{2C_pT_o}\sqrt{1-\frac{T_e}{T_o}} = \sqrt{2C_pT_o}\sqrt{1-\left(\frac{P_e}{P_o}\right)^{\frac{\gamma-1}{\gamma}}} \text{ --------------- }$$

Subsisting these into the thrust equation and rearranging, there results.

$$\frac{\Im}{P_o A_t} = \gamma\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}\sqrt{1-\left(\frac{P_e}{P_o}\right)^{\frac{\gamma-1}{\gamma}}} + \frac{A_e}{A_t}\left(\frac{P_e}{P_o} - \frac{P_a}{P_o}\right) \text{ ------------------- } 3.13$$

Since the pressure ratio P_e/P_o depends only on the area ratio equation 3.13 , indicates that the trust for a nozzle of given size and geometry depends only on Po and the ratio P_e/P_o and is independent of the temperature T_o.

Effect of Area Ration

We now ask, for given value of At, Po and Pa what exit area should be used in order to obtain maximum thrust?. By appling the calculus to equation 3.13 it may be shown after a laborious calculation that $\Im$ is a maximum when the area ratio is chosen in such a way to make the pressure in the exit plane exactly equal to Pa. Therfore equation 3.13 become.

$$\frac{\dot{m}_{max}}{Po.At} = \gamma \sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}\sqrt{1-\left(\frac{Pe}{Po}\right)^{\frac{\gamma-1}{\gamma}}} \quad \text{-------------------- 3.14}$$

Performance of Real Nozzle:

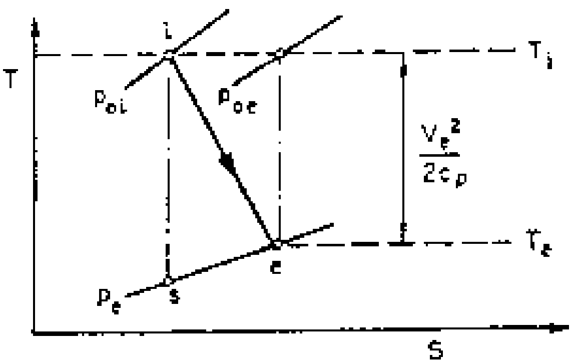
The performance of real nozzle differs slightly from that computed by isentropic flow owing to the friction effect. Since departure from isentropic flow are usually small, tha usual design procedure is based on the use of isentropic flow function which then modified by empirically determined coefficient. These coefficient are the nozzle efficiency and the nozzle discharge coefficient.

The nozzle efficiency η_N may defined as the rate of the exit kinetic energy to the kinetic energy which may be obtained by expanding the gas isentropically to the same final pressure.

$$\eta_N = \frac{V_e^2/2}{V_{se}^2/2} \quad \text{---------------------- 3.15}$$

The nozzle discharge coefficient Cd is defined as the rate of the actual mass flow rate $\dot{m}$ to the isentropic mass flow rate $\dot{m}_{se}$ which would be obtained by expanding the gas isentropically to the same final pressure.

$$Cd = \frac{\dot{m}}{\dot{m}_{se}} \quad \text{----------------3.16}$$



The figure at the right hand side shows the isentropic and the real expansion process through the nozzle. When the first law of thermodynamic applying at the expansion process for both isentropic and the real process.

$h_{se} = h_{oi} - \frac{V_{se}^2}{2}$ and $h = c_p.T$, therefore

$V_{se}^2 = 2.c_p.T_{oi}(1-\frac{T_s}{T_{oi}})$ and for isentropic process $\frac{T_s}{T_{oi}} = (\frac{P_e}{P_{oi}})^{\frac{\gamma-1}{\gamma}}$ therefore ,

$$V_{se}^2 = 2c_p.T_{oi}\left[1-\left(\frac{P_e}{P_{oi}}\right)^{\frac{\gamma-1}{\gamma}}\right] \quad \text{-----------------3.17}$$

similarly one might consider the imaginary isentropic process between the actual exit state and its stagnation state oe.

$h_{oe} = h_e + \frac{V_e^2}{2}$ and $h = c_p.T$, therefore

$V_e^2 = 2.c_p.T_{oe}(1-\frac{T_e}{T_{oe}})$ and for isentropic process $\frac{T_e}{T_{oe}} = (\frac{P_e}{P_{oe}})^{\frac{\gamma-1}{\gamma}}$ therefore ,

$$V_e^2 = 2c_p T_{oe}\left[1-\left(\frac{P_e}{P_{oe}}\right)^{\frac{\gamma-1}{\gamma}}\right] \text{.................} 3.18$$

The process within the nozzle is adiabatic this mean that $T_{oi}=T_{oe}$, substitute equation 3.18 and 3.17 into equation 3.16 and simplifying.

$$\frac{P_e}{P_{oi}} = \left[1-\eta_N\left(1-\left(\frac{P_s}{P_{oi}}\right)^{\frac{\gamma-1}{\gamma}}\right)\right]^{\frac{\gamma}{\gamma-1}} \text{..........} 3.19$$

The mass per unit area for isentropic flow can be evaluated as a function of pressure ratio instated of Mach Number, if one can substitute equation 3.5 into equation 3.7.

$$\frac{m_{ise}^{\cdot}}{A} = P_{oi}\sqrt{\frac{\gamma}{RT_{oi}}}\sqrt{\frac{2}{\gamma-1}\left[\left(\frac{P_s}{P_{oi}}\right)^{\frac{2}{\gamma}}-\left(\frac{P_s}{P_{oi}}\right)^{\frac{\gamma+1}{\gamma}}\right]} \text{.............} 3.20$$

Similarly the actual mass flux may be obtain.

$$\frac{m^{\cdot}}{A} = P_{oi}\sqrt{\frac{\gamma}{RT_{oi}}}\sqrt{\frac{2}{\gamma-1}\left[\left(\frac{P_e}{P_{oi}}\right)^{\frac{2}{\gamma}}-\left(\frac{P_e}{P_{oi}}\right)^{\frac{\gamma+1}{\gamma}}\right]} \text{................} 3.21$$

Substituting equation 3.21, 3.20 into equation 3.16 to find the discharge coefficient in term of pressure ratio.

$$Cd = \frac{\left(\frac{P_e}{P_{oi}}\right)^{\frac{\gamma-1}{\gamma}}}{\left(\frac{P_s}{P_{oi}}\right)^{\frac{\gamma-1}{\gamma}}}\sqrt{\frac{1-\left(\frac{P_e}{P_{oi}}\right)^{\frac{\gamma-1}{\gamma}}}{1-\left(\frac{P_s}{P_{oi}}\right)^{\frac{\gamma-1}{\gamma}}}} \text{................} \quad 3.22$$

Substitute equation 3.19 into the above equation to find the discharge coefficient in term of isotropic pressure ratio and nozzle efficiency.

$$Cd = \frac{\sqrt{\eta_N\left(\frac{P_s}{P_{oi}}\right)^{\frac{\gamma-1}{\gamma}}}}{1-\eta_N\left[1-\left(\frac{P_s}{P_{oi}}\right)^{\frac{\gamma-1}{\gamma}}\right]} \text{......................} 3.23$$

PROBLEMS

3.1. Air flows at the rate of 1 kg/s through a convergent-divergent nozzle. The entrance area is 2×10^{-3} m^2 and the inlet temperature and pressure are 438 K and 580 kPa. If the exit pressure is 140 kPa and the expansion is isentropic, find:

(a) The velocity at entrance.

(b) The stagnation temperature and stagnation pressure.
(c) The throat and exit areas.
(d) The exit velocity.

3.2. A convergent nozzle has an exit area 6.5×10^{-4} m^2. Air enters the nozzle at $p_0 = 680$ kPa, $T_0 = 370$ K. If the flow is isentropic, determine the mass rate of flow for back pressure of:
(a) 359 kPa.
(b) 540 kPa.
(c) 200 kPa.

3.3. A convergent-divergent steam nozzle has an exit area of 3.2×10^{-4} m^2 and an exit pressure of 270 kPa. The inlet conditions are 1 MPa and 590 K with negligible velocity. Assume ideal flow, i.e., no losses, and

$$\frac{p^*}{p_0} = 0.545$$

Find:
(a) The mass rate of flow for this nozzle.
(b) The throat area.
(c) The sonic velocity at the throat.

3.4. Air flows isentropically through a convergent-divergent passage with inlet area 5.2 cm^2, minimum area 3.2 cm^2 and exit area 3.87 cm^2. At the inlet the air velocity is 100 m/s, pressure is 680 kPa, and temperature 345 K. Determine:
(a) The mass rate of flow through the nozzle.
(b) The Mach number at the minimum-area section.
(c) The velocity and the pressure at the exit section.

3.5. Air is flowing in a convergent nozzle. At a particular location within the nozzle the pressure is 280 kPa, the stream temperature is 345 K, and the velocity is 150 m/s. If the cross-sectional area at this location is 9.29×10^{-3} m^2, find:
(a) The Mach number at this location.
(b) The stagnation temperature and pressure.
(c) The area, pressure, and temperature at the exit where $M = 1.0$.
(d) The mass rate of flow for the nozzle.
Indicate any assumptions you may make and the source of data used in the solution.

3.6. Air flows isentropically at the rate of 0.5 kg/s through a supersonic convergent-divergent nozzle. At the inlet, the pressure is 680 kPa, the temperature 295 K, and the area is 6.5 cm^2. If the exit area is 13 cm^2, calculate:
(a) The stagnation pressure and temperature.
(b) The exit Mach number.
(c) The exit pressure and temperature.
(d) The area and the velocity at the throat.
(e) What will be the maximum rate of flow and the corresponding exit Mach number if the flow is completely subsonic in the nozzle?

3.7. A stream of carbon dioxide is flowing in a 7.5 cm I.D. pipe at a stream pressure of 680 kPa and a stream temperature of $\boxed{365}$ K. A 7.5 cm $\times$ 5 cm venturimeter installed in this pipe shows a pressure differential reading of 1.68 mm Hg. Assuming ideal flow, determine:
(a) The mass rate of flow of CO_2. Compare your answer with that obtained if the gas is considered incompressible.

(4)

(b) If the mass rate of flow of CO_2 were to be doubled, what would be the new pressure differential reading for the venturimeter?

(c) If the fluid were hydrogen instead of CO_2, other conditions being the same as given in the problem statement, what would be the mass rate of flow?

(d) If the temperature of the CO_2 were 440 K instead of 365 K, other conditions being the same as given in the problem statement, what would be the mass rate of flow for the CO_2?

3.8. A 0.14 m^3 tank of compressed air discharges through a 2.2 cm diameter converging nozzle located in the side of the tank. If the mass flow coefficient of the nozzle based on isentropic flow through it is 0.95 and the gas within the tank expands isothermally from 1 MPa to 350 kPa, plot the pressure in the tank versus elapsed time as the pressure decreases. Assume the temperature of the tank is 295 K and the surrounding pressure is 101.3 kPa.

3.9. Air at stagnation conditions of 2 MPa and 750 K flows isentropically through a converging-diverging nozzle. If the maximum flow rate is 5.4 kg/s, determine:

(a) The throat area in m^2.

(b) The velocity, pressure, and temperature at the nozzle exit if the exit area is three times as large as the throat area.

3.10. Find the throat and exit areas in m^2 for a critical-flow nozzle handling air at the rate of 6.7 kg/s when the desired exit velocity is 1100 m/s with the stream at $p = 170$ kPa and $T = 310$ K. Assume isentropic flow and $\gamma = 1.4$.

3.11. Air flows reversibly and adiabatically in a nozzle. At section 1 of the nozzle the velocity, pressure, temperature, and area are 165 m/s, 350 kPa, 480 K, and 13×10^{-4} m^2. At section 2 in nozzle the area is 26×10^{-4} m^2. Find:

(a) The mass flow rate in the nozzle.

(b) V_2, M_2, p_2, t_2 and v_2.

(*Note:* There are two independent answers for this condition. Calculate both cases. If there is a throat, determine its area.)

3.12. Air at a pressure of 680 kPa and a temperature of 833 K enters a converging-diverging nozzle through a line of 4.6×10^{-3} m^2 area and expands to a delivery-region pressure of 33 kPa. Assuming isentropic expansion and a mass rate of flow of 1 kg/s, find:

(a) The stagnation enthalpy.

(b) The temperature and enthalpy at discharge.

(c) The Mach number and velocity of the air stream at discharge.

(d) The maximum mass flow rate per unit area.

3.13. Air flows isentropically at the rate of 1 kg/s through a duct. At one section of the duct the cross-sectional area is 9.3×10^{-3} m^2, static pressure is 200 kPa, and stagnation temperature is 550 K. Determine the velocity of the stream and the minimum area at the exit of the duct that causes no reduction in the mass rate of flow.

3.14. Air flows isentropically through a converging nozzle. At the inlet of the nozzle the pressure $p_1 = 340$ kPa, the temperature T_1 is 550 K, the velocity V_1 is 200 m/s, and the cross-sectional area A_1 is 9.3×10^{-3} m^2. Consider air to be an ideal gas with $\gamma = 1.4$ and find:

(a) The stagnation temperature and pressure.

(b) The sonic velocity and the Mach number at the inlet.

(c) The area, pressure, temperature, and velocity at the exit if $M = 1$ at exit.

0.18

(d) Draw graphs of G, M, V, and v versus pressure, indicating the values at the inlet and exit of the nozzle.

3.15. Superheated steam expands isentropically in a convergent-divergent nozzle from an initial state in which the pressure is 2.0 MPa and the superheat is 378 K to a pressure of 680 kPa. The rate of flow is 0.5 kg/s.

(a) Find the velocity of the steam and the cross-sectional area of the nozzle at the sections where the pressures are 1.0 MPa and 1.2 MPa.

(b) Determine the pressure, velocity, and cross-sectional area at the throat.

(c) Determine the velocity and cross-sectional area at discharge.

Assume $\dfrac{p^*}{p_0} = 0.55$.

3.16. A convergent nozzle receives steam at a pressure of 3.4 MPa and a temperature of 640 K with negligible velocity. The nozzle discharges into a chamber at which the pressure is maintained at 1.36 MPa. If the throat area of the nozzle is 2.3×10^{-4} m^2 and the discharge chamber area is 0.056 m^2, find

(a) The velocity at the throat.

(b) The mass rate of flow.

Assume $\dfrac{p^*}{p_0} = 0.55$ and the flow is isentropic.

3.17. Air flows isentropically through the convergent-divergent nozzle shown in Fig. 3.24. The inlet pressure is 80 kPa, the inlet temperature 295 K, and the back

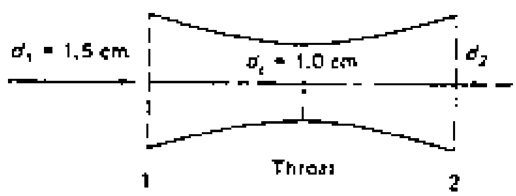


FIGURE 3.24

pressure 1.013 kPa. What should be the exit diameter of the nozzle which corresponds to the maximum obtainable value of Mach number at the exit? What are the mass rate of flow, the exit Mach number, and the exit temperature?

3.18. A rocket motor is fitted with a convergent-divergent nozzle having a throat diameter 2.5 cm. If the chamber pressure is 1 MPa and the chamber temperature is 2200 K, determine:

(a) The mass flow rate through the nozzle.

(b) The Mach number at the exit ($p_{back} = 101.3$ kPa).

(c) The thrust developed at sea level.

Assume that the products of combustion behave like a perfect gas ($\gamma = 1.4$, $R = 540$ J/kg K) and the expansion through the nozzle is isentropic.

3.19. Air is flowing through a section of a straight convergent nozzle. At the entrance to the nozzle section the area is 4×10^{-3} m^2, the velocity is 100 m/s, the air pressure is 680 kPa, and the air temperature is 365 K. At the exit of the section the area is 2×10^{-3} m^2. Assume reversible adiabatic flow. Calculate the magnitude and direction of the force exerted by the fluid upon the given nozzle section.

Chapter Four

Normal Shock Waves

Normal

Normal Shock Waves

Introduction:

The shock process represent an abrupt change in fluid properties, in which finite variation in pressure temperature and density occur over a shock thickness comparable to the mean free path of the gas molecules. It has been established that supersonic flow adjust to the pressure of a body by mean of such shock wave, whereas subsonic flow can adjust by gradual change in flow properties. Shock may also occur in the flow through nozzle or duct and have a decisive effect on these flow.

How Shock Wave Take Place:

Consider a piston in a tube and its given a steady velocity to the right of magnitude dv. A sound wave travels a head of the piston through the medium in the tube. Suppose the piston is now given a second increment of velocity dv, casing a second wave to move into the compressed gas behind the first wave. The location of the wave and the pressure distribution in the tube after a time t are shown in figure. Each wave travel at the velocity of sound with respect to the gas into which its moving. since the second wave is moving into a gas that is already moving to the right with velocity dv. The second wave is moving into a compressed gas having a slightly elevated temperature, therefore the second wave travel with a greater absolute velocity than the first wave and gradually over take it. A series of this induced wave after its over take each other will produce a shock wave or a sudden change in pressure and other properties.

$a \propto \sqrt{T}$ $\qquad$ $T3>T2>T1$ $\qquad$ therefore $\qquad$ $a3>a2>a1$

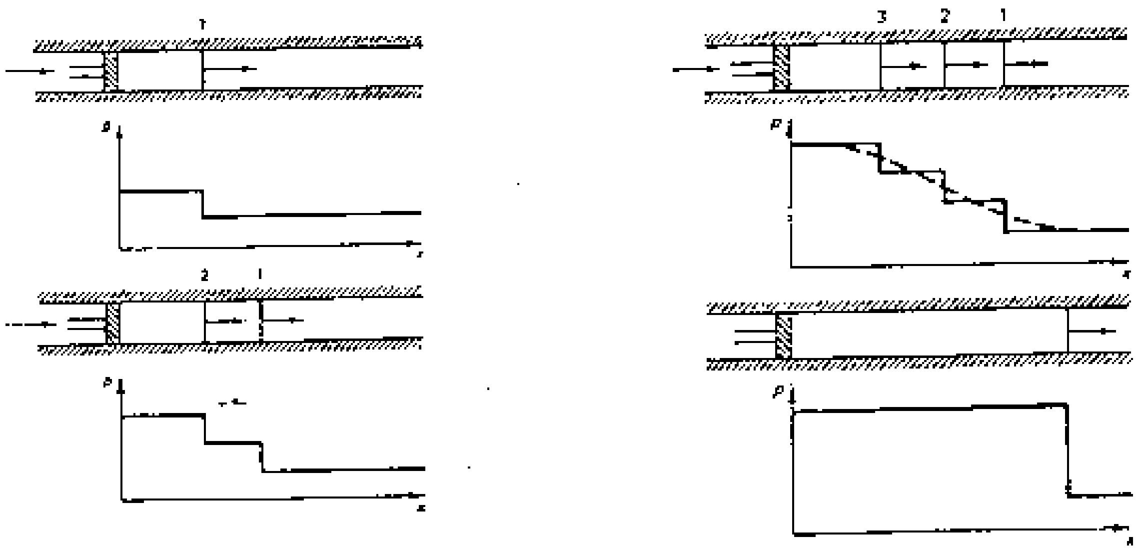


Fig shows one and two, three and the over take of the sound wave propagate a head of the piston

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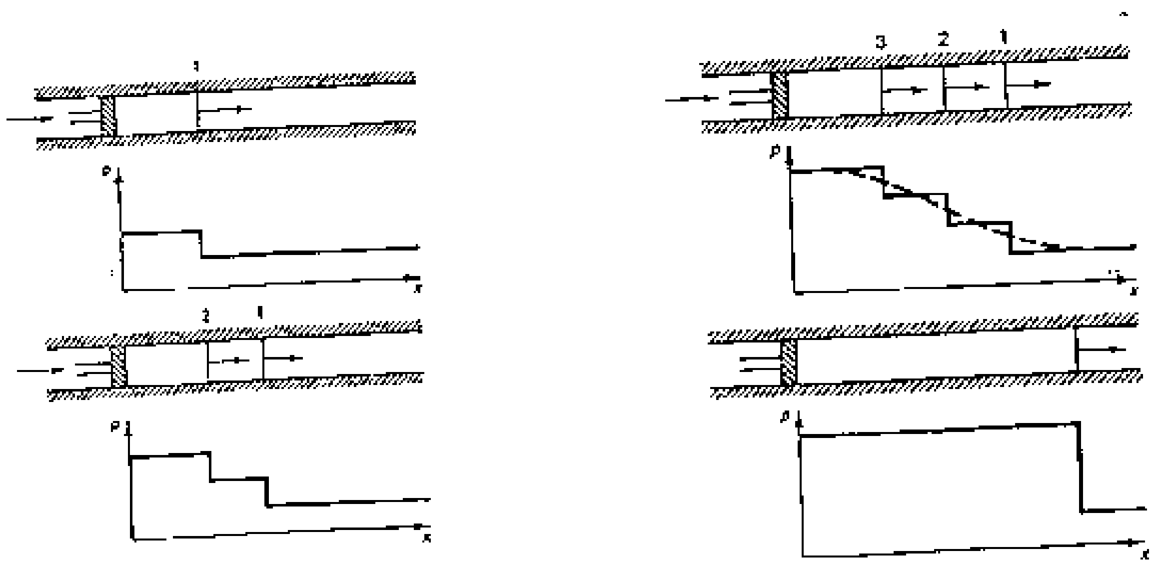
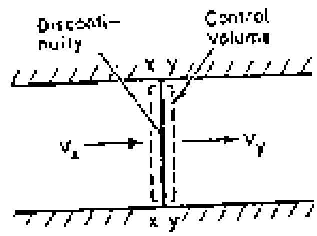


Fig shows one and two, three and the over take of the sound wave propagate a head of the piston

Stationary Normal Shock Waves:

In order to analyze the flow through a stationary normal shock wave, consider a control volume of the form shown. This control volume has cross sectional area S normal to the flow direction. The shock wave relations are obtained by applying the laws of conservation of mass, momentum and energy to the control volume for steady state flow. We will refer to the properties of the flow upstream of the shock by subscript "x" and downstream by "y".



$m^\cdot = \rho_x V_x A_x = \rho_y V_y A_y$

The shock wave thickness is very small therefore $A_x = A_y$.

$\rho_x V_x = \rho_y V_y$ --------------- 4.1

For perfect gas

$$\frac{P_x}{RT_x} M_x \sqrt{\gamma R T_x} = \frac{P_y}{RT_y} M_y \sqrt{\gamma R T_y} \quad \text{-------4.2}$$

Since the only force acting on the control volume in the flow direction are the pressure force, conservation of momentum is.

$P_x A_x - P_y A_y = m^\cdot (V_y - V_x)$

Combine of equation 4.1 into the above equation, where $m^\cdot = \rho_x V_x A_x = \rho_y V_y A_y$

$$P_x + \rho_x V_x^2 = P_y + \rho_y V_y^2 \quad \text{-------4.3}$$

For perfect gas $P = \rho . R . T$

$P_x + \rho_x V_x^2 = P_x (1 + \gamma M_x^2)$

$P_y + \rho_y V_y^2 = P_y (1 + \gamma M_y^2)$

$$P_x (1 + \gamma M_x^2) = P_y (1 + \gamma M_y^2) \quad \text{-------------4.4}$$

The flow through the control volume is adiabatic and the energy equation become.

$c_p T_x + \frac{V_x^2}{2} = c_p T_y + \frac{V_y^2}{2} = c_p + T_o$ For adiabatic flow the stagnation temperature does nit change across the shock wave this mean that $T_{ox} = T_{oy}$.

$$T_x \left(1 + \frac{\gamma + 1}{2} M_x^2\right) = T_y \left(1 + \frac{\gamma + 1}{2} M_y^2\right) \quad \text{---------------4.5}$$

Substitute energy equation 4.5 and momentum equation 4.4 into the continuity equation 4.2.

$$\frac{M_x}{1 + \gamma M_x^2} \sqrt{1 + \frac{\gamma + 1}{2} M_x^2} = \frac{M_y}{1 + \gamma M_y^2} \sqrt{1 + \frac{\gamma + 1}{2} M_y^2} \quad \text{----------------4.6}$$

By inspection its evident that one solution 4.6 is the trivial one, $M_x = M_y$. This solution involving no change in properties in constant area flow corresponding to isentropic flow and that is not of interest for the irreversible of normal shock. Equation 4.6 can be solve to yield M_y in term of M_x.

$$M_y^2 = \frac{M_x^2 + \frac{2}{\gamma - 1}}{\frac{2\gamma}{\gamma - 1} M_x^2 - 1} \quad \text{------------------4.7}$$

Now to find the pressure ratio after and before the shock, substitute equation 4.7 into equation 4.4 .

$$\frac{P_y}{P_x} = \frac{2\gamma M_x^2 - (\gamma - 1)}{\gamma + 1} \quad \text{----------------4.8}$$

also to find the temperature ratio after and before the shock, one may substitute equation 4.7 into equation 4.5

$$\frac{T_y}{T_x} = \frac{\left[2\gamma M_x^2 - (\gamma - 1)\right]\left[2 + (\gamma - 1)M_x^2\right]}{(\gamma + 1)M_x^2} \quad \text{---------------4,9}$$

and if we substitute equation 4.7 into equation 4.1 we can find the density and the velocity ratio.

$$\frac{\rho_y}{\rho_x} = \frac{V_x}{V_y} = \frac{(\gamma + 1)M_x^2}{2 + (\gamma - 1)M_x^2} \quad \text{------------------4.10}$$

The ratio of stagnation pressure is a measure of the irreversibility in the shock process. It may be found by observing that:

$$\frac{P_{oy}}{P_{ox}} = \frac{P_{oy}}{P_y}\frac{P_y}{P_x}\frac{P_x}{P_{ox}}$$

Now P_y/P_x is given by Eq. 4.8, and P_{oy}/P_y and P_x/P_{ox} may be found from Eq.3.5. Using Eq. 4.7 for the value of M_y we get after algebraic simplification,

$$\frac{P_{oy}}{P_{ox}} = \left[\frac{2\gamma}{\gamma + 1} M_x^2 - \frac{\gamma - 1}{\gamma + 1}\right]^{\frac{1}{\gamma - 1}} \left[\frac{(\gamma + 1)M_x^2}{2 + (\gamma + 1)M_x^2}\right]^{\frac{\gamma}{\gamma - 1}} \quad \text{--------------------4.11}$$

To evaluate the entropy change across the shock, we employ the perfect gas formula,

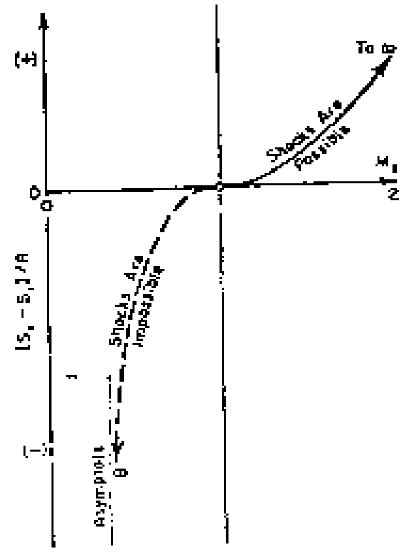
$$S_y - S_x = c_p \ln\frac{T_y}{T_x} - R\ln\frac{P_y}{P_x} \quad \text{---------------- 4.12}$$

substitute Eq. 4.8 and 4.9 into Eq. 4.12 then.

$$\frac{S_y - S_x}{R} = \frac{1}{\gamma - 1}\ln\left[\frac{2\gamma}{\gamma + 1} M_x^2 - \frac{\gamma - 1}{\gamma + 1}\right] + \frac{\gamma}{\gamma - 1}\ln\left[\frac{(\gamma + 1)M_x^2}{2 + (\gamma + 1)M_x^2}\right] \quad \text{----4.12}$$

Impossibility of a Rerefaction Shock.

Carful study of Eq.4.12 indicate that for gases with $1<\gamma<1.67$ the entropy change is always positive when Mx is greater than unity, and is always negative when Mx is less than unity. The general form of Eq.4.12 is shown in Fig. It is proven rigorously that for perfect gas only the shock from supersonic to subsonic is possible. Since the shock process is adiabatic and according to second law of thermodynamic the entropy change must be positive.



Comparing Eq. 4.12 for entropy change and Eq.4.11 for stagnation pressure ration, one can conclude the following correlation:

$$\frac{S_y - S_x}{R} = -\ln\frac{P_{oy}}{P_{ox}} \quad \text{------------4.13}$$

According to the second law of thermodynamic the rate of change of entropy is positive $ds>0$, and referring to Eq.4.13 this mean that P_{oy} is less than P_{ox}.

The shock wave take place in-order to keep the flow continuation this mean that the flow is steady and the mass flow does not change across the shock.

$$m^*_x = m^*_y$$

we have seen from the previous chapter that the maximum mass flow rate can be achieved at the choked condition and the mass flow rate in term of stagnation properties and the critical area is.

$$\frac{P_{ox}A_x^* \, constant}{\sqrt{T_{ox}}} = \frac{P_{oy}A_y^* \, constant}{\sqrt{T_{oy}}}$$

the flow through the shock is adiabatic therefore $T_{ox}=T_{oy}$

$P_{ox}A^*_x=P_{oy}A^*_y$ or $\quad \dfrac{P_{oy}}{P_{ox}} = \dfrac{A_x^*}{A_y^*}$ since $\quad P_{oy}< P_{ox}$ this mean that $\quad A^*_y>A^*_x$

Normal Shock Table:

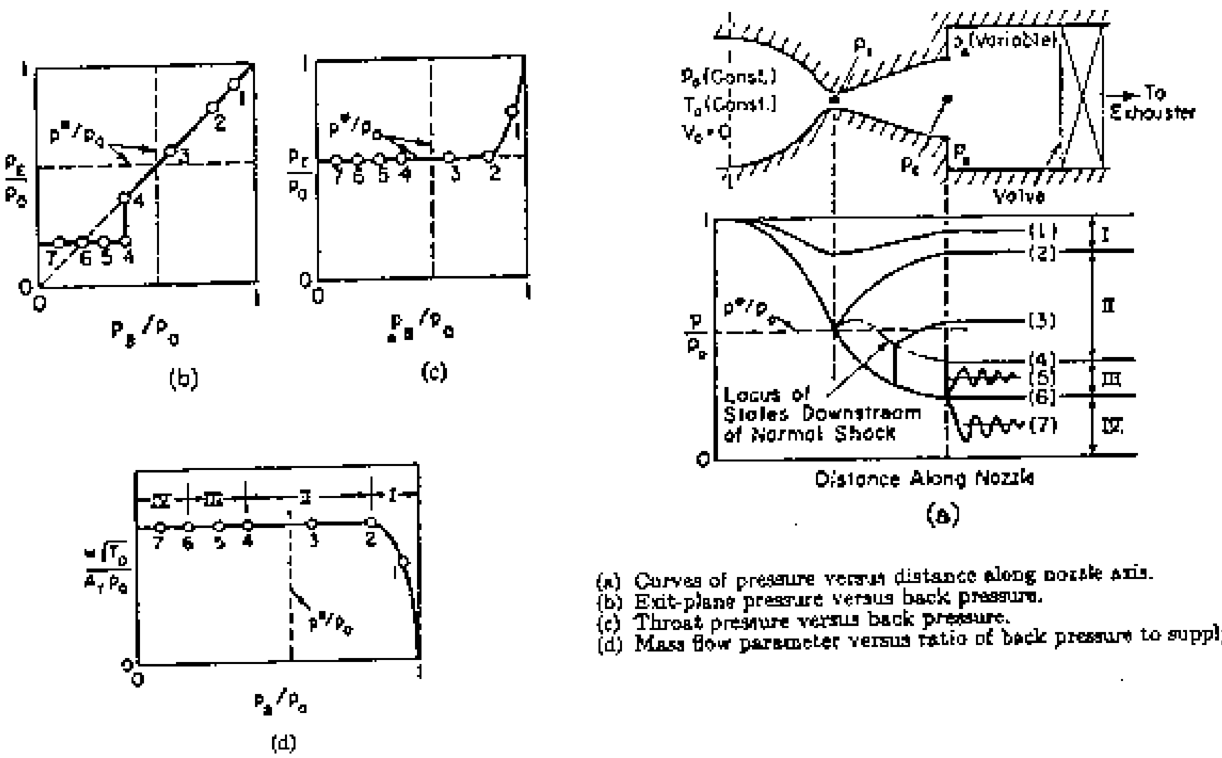
Table is available which list the ratio of the various flow variable such as pressure, temperature , and density across the normal shock wave and the downstream Mach Number as a function of the upstream Mach Number.

$M_x \quad M_y \quad P_y/P_x \quad T_y/T_x \quad \rho_y/\rho_x \quad P_{oy}/P_{ox}$ or A^*_x/A^*_y

Convergent-Divergent Nozzle:

We return now to the problem of the operating characteristics of converging-diverging nozzle under pressure ratio, discussed previously in chapter two. *Fig.* show the characteristic performance of convergent divergent nozzle with various back pressure to the supply pressure.

Four different regimes are possible . In regime I the flow is entirely subsonic, and the passage behave like a conventional venture tube. The flow rate is sensitive to change in back pressure. At condition *2*, which forms the dividing line between *I* and *II*, the Mach Number at the throat is unity. As regime *II* is entered, a normal shock appears down stream of the throat, and the process aft of the shock comprises subsonic deceleration. As the back pressure is lowered, the shock move down the nozzle until, at condition *4* it appears in the exit plane of the nozzle. In regime *II*, as in regime *I*, the exit plane pressure P_E is virtually identical with the back pressure P_B. On the other hand, the flow rate in regime *II* is constant and is unaffected by the back pressure. This is in accord



(a) Curves of pressure versus distance along nozzle axis.
(b) Exit-plane pressure versus back pressure.
(c) Throat pressure versus back pressure.
(d) Mass flow parameter versus ratio of back pressure to supply

with the fact that throughout regime *II* all stream properties at the throat section are constant.

In regime *III*. As for condition *5*, the flow within the entire nozzle is supersonic, and the pressure in the exit plane is lower than the back pressure. The compression which subsequently occurs outside the nozzle involve oblique shock wave which cannot be treated on one-dimensional grounds. Condition *6* is termed the design condition for the nozzle under supersonic condition, since the exit-plane pressure is then identical with the back pressure. A reduction in the back pressure below that corresponding to condition 6 has no effect whatsoever on the flow pattern within the nozzle. In regime *IV* the expansion from the exit-plane pressure to the back pressure occurs outside the nozzle in

the form of oblique expansion waves which also cannot be studied by one-dimensional analysis.

In both regimes *III* and *IV* the flow pattern within the nozzle is independent of back pressure, and corresponds to the flow pattern for the design condition. Adjustment to the back pressure are made outside the nozzle.

For subsonic flow, there are an infinite number of possible pressure distance curves. For the supersonic region of flow, however, the pressure-distance curve is unique. To put it differently, in subsonic flow the pressure ratio does not depend solely on the area ratio; in supersonic flow the pressure ratio does depend solely on the area ratio.

Only over a narrow range of back pressure ratio, namely, the range covered by regime *I*, does the flow rate depend on the back pressure. For regime *II, III, IV*, the flow rate is independent of the back pressure, since $M=1$ at the throat, may be computed form choked flow equation.

Converging- Diverging Supersonic Diffuser.

A diffuser is a device that cause the static pressure of a gas to rise while the gas is decelerating. When deceleration is isentropic, the maximum pressure that can be attained is the isentropic stagnation pressure. Diffusers are either subsonic or supersonic, depending on the Mach Number of the approaching stream. In a subsonic diffuser the cross-sectional area increases in the direction of flow, while in a supersonic diffuser the cross sectional area first decrease and then increases.

A supersonic diffuser is located at the inlet to such air-breathing engines as the supersonic turbojet and the ramjet,. The high velocity air is decelerated by the diffuser before it is compressed in the axial flow compressor of the turbojet or before it undergoes combustion in the ramjet. An idéal supersonic diffuser consists of a convergent-divergent passageway in which the flow is shock-free and isentropic. Deceleration of the flow to M=1 at the throat is followed by a further deceleration to subsonic speed downstream of the throat. In real application, however, starting transients and off-design interfere in establishing the desired flow pattern. The maximum pressure that can be achieved in the diffuser is the isentropic stagnation pressure. Any loss in available energy (or stagnation pressure) in the diffuser will have a harmful effect on the operation of the engine as a whole. For a supersonic diffuser it would be highly desirable to provide shock free isentropic flow.

For any configuration of the converging-diverging diffuser, there are two values of Mach number in which the flow is isentropically compressed, this will called subsonic design Mach number(M_{Dsub})and supersonic design Mach number(M_{Dsup}).The following cases will show how the flow is established from the starting-up to the design flying Mach number.

1- When the flying Mach Number is below M_{Dsub} value , this mean that the actual throat area is grater than the critical area, therefore the flow at the throat is subsonic and the flow is continue to compressed at the divergent part as show in fig.a .
2- When the flying Mach number reach the M_{Dsub} value, this mean that the actual throat area is equal to the critical area of the flying Mach number, therefore the flow at the throat is sonic M=1 and the flow is continue to compressed at the divergent part and the exit Mach number will be subsonic fig.b.

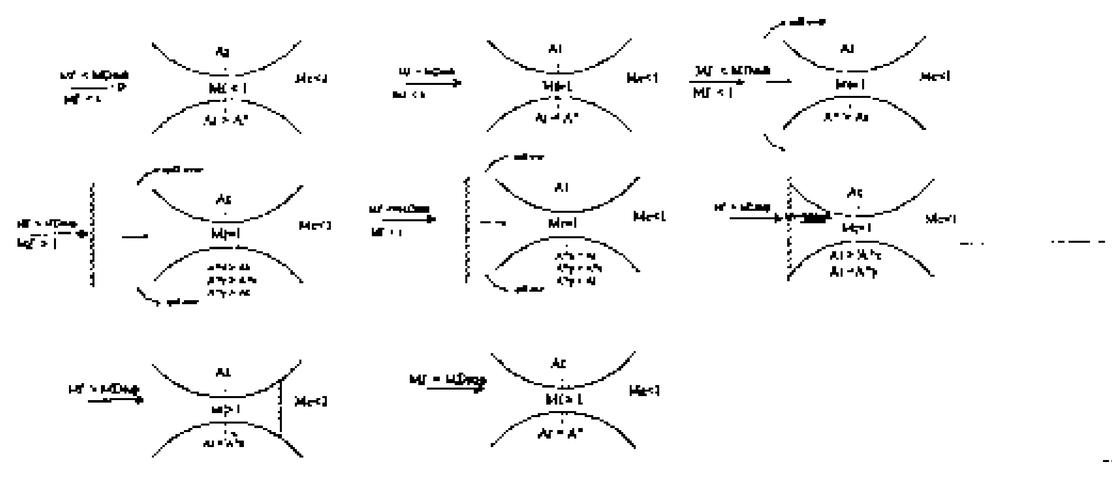
3- When the flying Mach number is grater than M_{Dsub} value, this mean that the actual throat area is less than the critical area this mean that the throat area is too small to accommodate the flow. The pressure is instantaneously increased at the throat area and part of the incoming flow is divert or spill over the inlet cowl of the diffuser as show in fig.c This mean that as the flying Mach number increase the different between the throat area and the required area increase and hence mass spill over is increase.

4- When the flying Mach number is grater than one but is less than the M_{Dsup} , in this case the throat area is less than the critical area or the required area to accommodate the flow. Therefore the instantaneously pressure built up at the throat area. A curved or normal shock is appears in the front of the diffuser inlet. The subsonic flow downstream of the shock is partially spilled over the diffuser inlet, reducing the mass flow through the inlet, this will lower the combustion pressure and a loss in thrust.

5- When the flying Mach number is equal to the M_{Dsup} value, in this case the existing of the shock wave will caused of stagnation pressure loss. The critical area behind the existing shock is increased and this mean that the critical area upstream of the shock is equal to the throat area but the area down stream of the shock is still grater than the throat area. Therefore the normal shock is still existing and the flow spill over is continue as show in fig. d.

6- To over come the existing shock the engine have to speed over the design supersonic Mach number until the shock located at the diffuser inlet. At this case the Mach number down stream of the shock wave is equal to the M_{Dsub} so that the mach number at the throat is equal to sonic. A little increase in speed will make the shock wave to swallowed and stand at the divergent part of the diffuser as show in fig. e.

7- To return back to the design condition the engine have to slow down to the design supersonic flying Mach number, in this case the shock wave is drawn back toward the throat and it strength will reduce gradually until it vanished at the throat when the flying Mach number is equal to the M_{Dsup} as show in fig.f



PROBLEMS

4.1. Air with initial stagnation conditions of 700 kPa and 330 K passes through a convergent-divergent nozzle at the rate of 1 kg/s. At the exit area of the nozzle the stagnation pressure is 550 kPa and the stream pressure is 500 kPa. The nozzle is insulated and there is no irreversibility except for the occurrence of a shock.

(a) What is the nozzle throat area?
(b) What is the Mach number before and after the shock?
(c) What is the nozzle area at the point of shock and at the exit?
(d) What is the stream density at the exit?

4.2. A perfect gas ($\gamma = 1.4$) enters a converging-diverging nozzle with a Mach number of 0.50 and local pressure and temperature values of 280 kPa and 280 K, respectively. The nozzle throat area is 6.5×10^{-4} m^2 and the nozzle exit area is 26×10^{-4} m^2. The nozzle exit pressure is 170 kPa.

(a) What are the values of the Mach number and the stream temperature at the exit?
(b) At what area does the shock occur?

Show your method of solution on a skeleton flow chart.

4.3. An air nozzle has an exit area 1.6 times the throat area. If a normal shock occurs at a plane where the area is 1.2 times the throat area, find the pressure, temperature, and Mach number at the exit. The stagnation temperature and pressure before the shock are 310 K and 700 kPa.

4.4. Air enters a supersonic nozzle with inlet conditions $A_1 = 6.5 \times 10^{-4}$ m^2, $M_1 = 1.8$, $p_1 = 35$ kPa, and $T_1 = 260$ K. A normal shock occurs in the nozzle resulting in an increase in entropy of $\Delta s = 113$ J/kg K. If the Mach number at the exit $M_2 = 0.3$, find:

(a) The area of the normal shock A_{xy}.
(b) The Mach numbers before and after the shock M_x, M_y.
(c) The pressure at the exit p_2.
(d) The mass rate of flow per unit area at exit.
(e) Show the process on a schematic flow chart and a Fanno-Rayleigh plot.

Assume isentropic flow except for the normal shock.

4.5. An impact (stagnation) tube in an air stream reads 186 kPa. If the local temperature is 293 K and the local Mach number is 0.8, determine:

(a) The local pressure.
(b) The mass rate of flow per unit area.

4.6. A Pitot tube and a thermocouple give the following measurements pertaining to air flow in a duct:

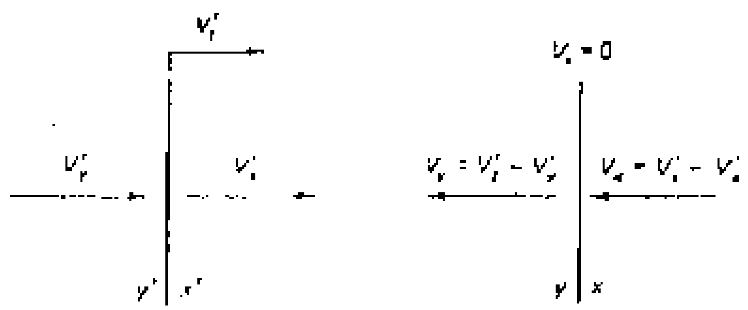
$$p_0 = 180 \text{ kPa}, \qquad p = 157 \text{ kPa}, \qquad T_0 = 1250 \text{ K}$$

Moving Shock Wave:

Previous section have dealt with the fixed normal shock wave. However, many physical situation arise in which a normal shock is moving. When an explosive occurs, a shock propagates through the atmosphere from the point of the explosion. As a blunt body re-enters the atmosphere from space, a shock travels a short distance a head of the body. When a valve in a gas line is suddenly closed, a shock propagates back through the gas. To treat these cases, it is necessary to extend the procedures already develop for the ficed normal shock wave.

Consider a normal shock moving at constant velocity into still air as show in fig. Let Vs= absolute shock velocity and Vg= velocity of the gases behind the wave, both velocities are measured with respect to a fixed observer. For a fixed observer, the flow is not steady, since condition at a point are dependent on whether or not the shock has passed over that point.

Now consider the same physical situation with an observer moving at the shock-wave velocity, a situation, for instant, with the observer "sitting on the shock wave". The shock is now fixed with respect to the observer as shown in fig. But this the same case already covered in the normal shock section. Relation have been derived and result tabulated for the fixed normal shock. To apply these result to the moving shock, consideration must be given to the effect of observer velocity on static and stagnation properties.



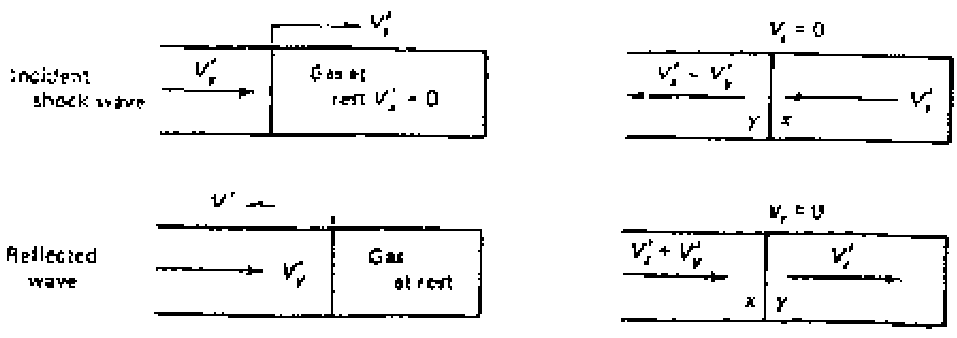
(a) Moving wave (b) Stationary wave

Since static properties are independent of the observer velocity, the transformation of the coordinate system has no effect on static properties. Stagnation properties on the other hand depend on the observer velocity and consequently are affected by the choice of the coordinate system. Table 4.1 show properties in a fixed coordinate system and I a moving coordinate system.

TABLE 4.1

Static properties:	
$p_x = p'_x$	$p_y = p'_y$
$T_x = T'_x$	$T_y = T'_y$
$c_x = c'_x$	$c_y = c'_y$
Mach numbers:	
$M_x = \dfrac{V_x}{c_x} = \dfrac{V'_s - V'_x}{c_x}$	$M_y = \dfrac{V_y}{c_y} = \dfrac{V'_s - V'_y}{c_y}$
$M'_x = \dfrac{V'_x}{c'_x} = \dfrac{V'_s - V_x}{c_x}$	$M'_y = \dfrac{V'_y}{c'_y} = \dfrac{V'_s - V_y}{c_y}$
Stagnation properties:	
$T_{0x} = T_x\left(1 + \dfrac{\gamma - 1}{2} M_x^2\right)$	$T'_{0x} = T_x\left(1 + \dfrac{\gamma - 1}{2} M'^2_x\right)$
$T_{0y} = T_y\left(1 + \dfrac{\gamma - 1}{2} M_y^2\right)$	$T'_{0y} = T_y\left(1 + \dfrac{\gamma - 1}{2} M'^2_y\right)$
$p_{0x} = p_x\left(1 + \dfrac{\gamma - 1}{2} M_x^2\right)^{\gamma/(\gamma-1)}$	$p'_{0x} = p_x\left(1 + \dfrac{\gamma - 1}{2} M'^2_x\right)^{\gamma/(\gamma-1)}$
$p_{0y} = p_y\left(1 + \dfrac{\gamma - 1}{2} M_y^2\right)^{\gamma/(\gamma-1)}$	$p'_{0y} = p_y\left(1 + \dfrac{\gamma - 1}{2} M'^2_y\right)^{\gamma/(\gamma-1)}$

When a normal shock wave travels in a closed-end, the gas between the shock wave and the closed end remains at rest. The gas behined the shock, however, moves at a velocity V_y' as shown in fig. The incident shock is reflected at the closed end of the tube and propagates back through the incoming gas. For an observer moving with the wave the velocity appear as shown in fig. Since the gas velocity decres across the reflected wave, the incident shock wave is reflected at the end of the tube as a shock wave.



(a) Velocities relative to a fixed coordinate system

(b) Velocities relative to a system moving with the wave

Chapter 9

Fanno Flow

9.1 INTRODUCTION

At the start of Chapter 1 we mentioned that area changes, friction, and heat transfer are the most important factors affecting the properties in a flow system. Up to this point we have considered only one of these factors, that of variations in area. However, we have also discussed the various mechanisms by which a flow adjusts to meet imposed boundary conditions of either flow direction or pressure equalization. We now wish to take a look at the subject of friction losses.

To study only the effects of friction, we analyze flow in a constant-area duct without heat transfer. This corresponds to many practical flow situations that involve reasonably short ducts. We consider first the flow of an arbitrary fluid and discover that its behavior follows a definite pattern which is dependent on whether the flow is in the subsonic or supersonic regime. Working equations are developed for the case of a perfect gas, and the introduction of a reference point allows a table to be constructed. As before, the table permits rapid solutions to many problems of this type, which are called *Fanno flow*.

9.2 OBJECTIVES

After completing this chapter successfully, you should be able to:

1. List the assumptions made in the analysis of Fanno flow.
2. *(Optional)* Simplify the general equations of continuity, energy, and momentum to obtain basic relations valid for any fluid in Fanno flow.
3. Sketch a Fanno line in the h–v and the h–s planes. Identify the sonic point and regions of subsonic and supersonic flow.
4. Describe the variation of static and stagnation pressure, static and stagnation temperature, static density, and velocity as flow progresses along a Fanno line. Do for both subsonic and supersonic flow.

5. (*Optional*) Starting with basic principles of continuity, energy, and momentum, derive expressions for property ratios such as T_2/T_1, p_2/p_1, and so on, in terms of Mach number (M) and specific heat ratio (γ) for Fanno flow with a perfect gas.
6. Describe (include T–s diagram) how the Fanno table is developed with the use of a * reference location.
7. Define *friction factor*, *equivalent diameter*, *absolute* and *relative roughness*, *absolute* and *kinematic viscosity*, and *Reynolds number*, and know how to determine each.
8. Compare similarities and differences between Fanno flow and normal shocks. Sketch an h–s diagram showing a typical Fanno line together with a normal shock for the same mass velocity.
9. Explain what is meant by *friction choking*.
10. (*Optional*) Describe some possible consequences of adding duct in a choked Fanno flow situation (for both subsonic and supersonic flow).
11. Demonstrate the ability to solve typical Fanno flow problems by use of the appropriate tables and equations.

9.3 ANALYSIS FOR A GENERAL FLUID

We first consider the general behavior of an arbitrary fluid. To isolate the effects of friction, we make the following assumptions:

Steady one-dimensional flow	
Adiabatic	$\delta q = 0, ds_e = 0$
No shaft work	$\delta w_s = 0$
Neglect potential	$dz = 0$
Constant area	$dA = 0$

We proceed by applying the basic concepts of continuity, energy, and momentum.

Continuity

$$\dot{m} = \rho A V = \text{const}$$

but since the flow area is constant, this reduces to

$$\rho V = \text{const} \tag{9.1}$$

We assign a new symbol G to this constant (the quantity ρV), which is referred to as the *mass velocity*, and thus

$$\boxed{\rho V = G = \text{const}} \tag{9.2}$$

What are the typical units of G?

Energy

We start with:

$$h_{t1} + q = h_{t2} + w$$

For adiabatic and no work, this becomes

$$h_{t1} = h_{t2} = h_t = \text{const} \tag{9.3}$$

If we neglect the potential term, this means that

$$h_t = h + \frac{V^2}{2g_c} = \text{const} \tag{9.4}$$

Substitute for the velocity from equation (9.2) and show that

$$\boxed{h_t = h + \frac{G^2}{\rho^2 2g_c} = \text{const}} \tag{9.5}$$

Now for any given flow, the constant h_t and G are known. Thus equation (9.5) establishes a unique relationship between h and ρ. Figure 9.1 is a plot of this equation in the h–v plane for various values of G (but all for the same h_t). Each curve is called a *Fanno line* and represents flow at a particular *mass velocity*. Note carefully that this is constant G and not constant $\dot{m}$. Ducts of various sizes could pass the same mass flow rate but would have different mass velocities.

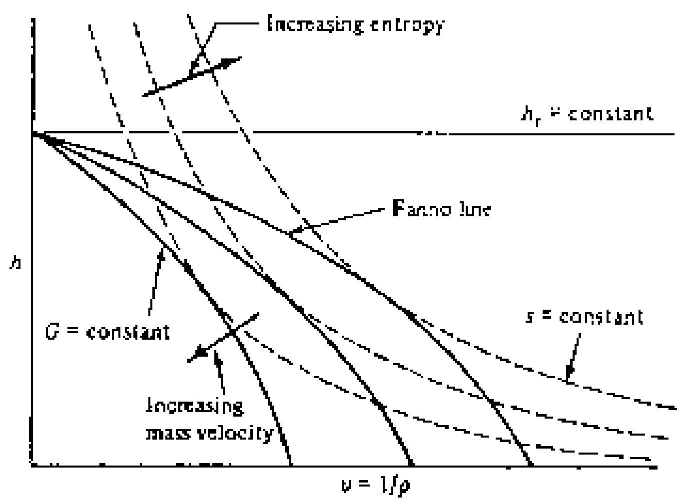


Figure 9.1 Fanno lines in h–v plane.

Once the fluid is known, one can also plot lines of constant entropy on the h–v diagram. Typical curves of s = constant are shown as dashed lines in the figure. It is much more instructive to plot these Fanno lines in the familiar h–s plane. Such a diagram is shown in Figure 9.2. At this point, a significant fact becomes quite clear. Since we have assumed that there is no heat transfer ($ds_e = 0$), the *only* way that entropy can be generated is through irreversibilities (ds_i). Thus *the flow can only progress toward increasing values of entropy*! Why? Can you locate the points of maximum entropy for each Fanno line in Figure 9.1?

Let us examine one Fanno line in greater detail. Figure 9.3 shows a given Fanno line together with typical pressure lines. All points on this line represent states with the same mass flow rate per unit area (mass velocity) and the same stagnation enthalpy. Due to the irreversible nature of the frictional effects, the flow can only proceed to the right. Thus the Fanno line is divided into two distinct parts, an upper and a lower branch, which are separated by a limiting point of maximum entropy.

What does intuition tell us about adiabatic flow in a constant-area duct? We normally feel that frictional effects will show up as an internal generation of "heat" with a corresponding reduction in density of the fluid. To pass the same flow rate (with constant area), continuity then forces the velocity to increase. This increase in kinetic energy must cause a decrease in enthalpy, since the stagnation enthalpy remains constant. As can be seen in Figure 9.3, this agrees with flow along the *upper branch* of the Fanno line. It is also clear that in this case both the static and stagnation pressure are decreasing.

But what about flow along the *lower branch*? Mark two points on the lower branch and draw an arrow to indicate proper movement along the Fanno line. What is happening to the enthalpy? To the density [see equation (9.5)]? To the velocity [see equation (9.2)]? From the figure, what is happening to the static pressure? The stagnation pressure? Fill in Table 9.1 with *increase*, *decrease*, or *remains constant*.

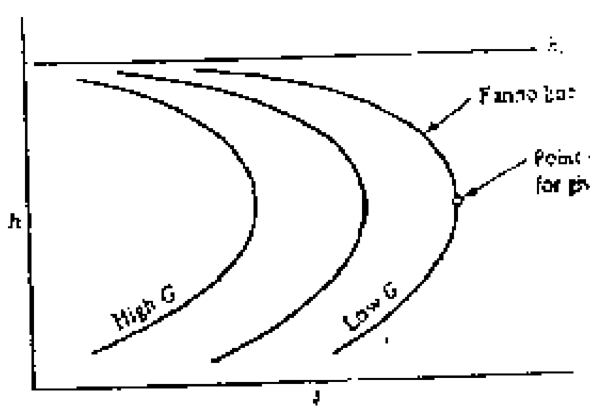


Figure 9.2 Fanno lines in h–s plane.

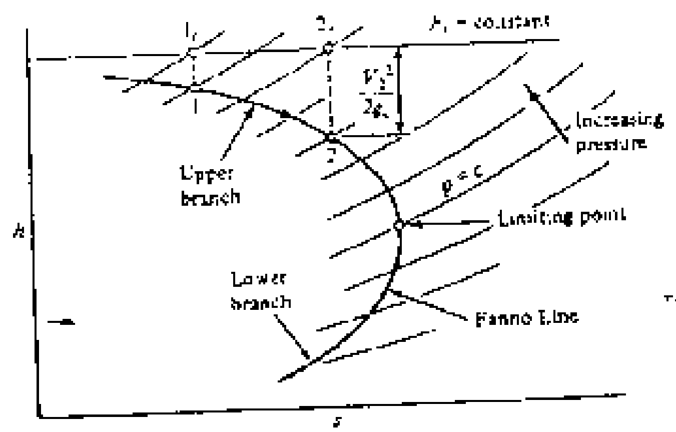


Figure 9.3 Two branches of a Fanno line.

Table 9.1 Analysis of Fanno Flow for Figure 9.3

Property	Upper Branch	Lower Branch
Enthalpy		
Density		
Velocity		
Pressure (static)		
Pressure (stagnation)		

Notice that on the lower branch, properties do not vary in the manner predicted by *intuition*. Thus this must be a flow regime with which we are not very familiar. Before we investigate the limiting point that separates these two flow regimes, let us note that these flows do have one thing in common. Recall the stagnation pressure energy equation

STAGNATION PRESSURE–ENERGY EQUATION

Consider the two section locations on the physical system shown in Figure. If we let the distance between these locations approach zero, we are dealing with an infinitesimal control volume with the thermodynamic states differentially separated, as shown in Figure below. Also shown are the corresponding stagnation states for these two locations.

We may write the following property relation between points 1 and 2:

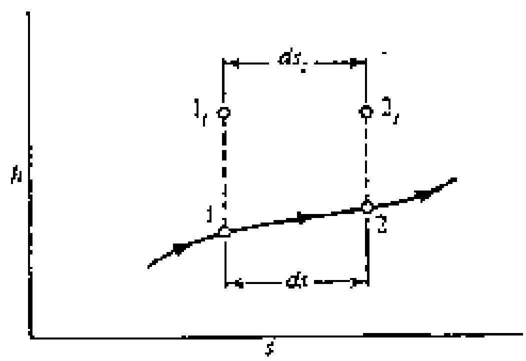


Figure Infinitesimally separated static states with associated stagnation states.

$$T\,ds = dh - v\,dp \tag{A.1}$$

Note that even though the stagnation states do not actually exist, they represent legitimate thermodynamic states, and thus any valid property relation or equation may be applied to these points. Thus we may also apply equation (A.1) between states 1_t and 2_t:

$$T_t\,ds_t = dh_t - v_t\,dp_t \tag{A.2}$$

However,

$$ds_t = ds \tag{A.3}$$

and

$$ds = ds_e + ds_i \tag{A.4}$$

Thus we may write

$$T_t(ds_e + ds_i) = dh_t - v_t\,dp_t \tag{A.5}$$

Recall the energy equation written in the form

$$\delta q = \delta w_s + dh_t \tag{A.6}$$

By substituting dh_t from equation (A.5) into (A.6), we obtain

$$\delta q = \delta w_s + T_t(ds_e + ds_i) + v_t\,dp_t \tag{A.7}$$

Now also recall that

$$\delta q = T\,ds_e \tag{A.8}$$

Substitute equation (A.8) into (A.7) and note that $v_t = 1/\rho_t$ and you should obtain the following equation, called the *stagnation pressure–energy equation*:

$$\boxed{\frac{dp_t}{\rho_t} + ds_e(T_t - T) + T_t\,ds_i + \delta w_s = 0} \tag{A.9}$$

For Fanno flow, $ds_e = \delta w_s = 0$.

Thus any frictional effect must cause a decrease in the total or stagnation pressure! Figure 9.3 verifies this for flow along both the upper and lower branches of the Fanno line.

Limiting Point

From the energy equation we had developed,

$$h_t = h + \frac{V^2}{2g_c} = \text{constant} \tag{9.4}$$

Differentiating, we obtain

$$dh_t = dh + \frac{V\,dV}{g_c} = 0 \tag{9.6}$$

From continuity we had found that

$$\rho V = G = \text{constant} \tag{9.2}$$

Differentiating this, we obtain

$$\rho\,dV + V\,d\rho = 0 \tag{9.7}$$

which can be solved for

$$dV = -V\frac{d\rho}{\rho} \tag{9.8}$$

Introduce equation (9.8) into (9.6) and show that

$$dh = \frac{V^2\,d\rho}{g_c\rho} \tag{9.9}$$

Now recall the property relation

$$T\,ds = dh - v\,dp$$

which can be written as

$$T\,ds = dh - \frac{dp}{\rho} \tag{9.10}$$

Substituting for dh from equation (9.9) yields

$$\boxed{T\,ds = \frac{V^2\,d\rho}{g_c\rho} - \frac{dp}{\rho}} \tag{9.11}$$

We hasten to point out that this expression is valid for *any* fluid and between two differentially separated points *anyplace* along the Fanno line. Now let's apply equation (9.11) to two adjacent points that surround the limiting point of maximum entropy. At this location s = const; thus $ds = 0$, and (9.11) becomes

$$\frac{V^2\,d\rho}{g_c} = dp \qquad \text{at limit point} \tag{9.12}$$

or

$$V^2 = g\left(\frac{dp}{d\rho}\right)_{\text{at limit point}} = g_c\left(\frac{\partial p}{\partial \rho}\right)_{s = \text{const}} \tag{9.13}$$

This should be a familiar expression (dp/dρ= $\sqrt{\gamma RT}$) and we recognize that *the velocity is sonic at the limiting point.* The upper branch can now be more significantly called the *subsonic branch*, and the lower branch is seen to be the *supersonic branch*

Now we begin to see a reason for the failure of our intuition to predict behavior on the lower branch of the Fanno line. From our privous studies, it shows that fluid behavior in supersonic flow is frequently contrary to our expectations. This points out the fact that we live most of our lives "subsonically," and, in fact, our knowledge of fluid phenomena comes mainly from experiences with incompressible fluids. It should be apparent that we cannot use our intuition to guess at what might be happening, particularly in the supersonic flow regime. We must learn to get religious and put faith in our carefully derived relations.

Momentum

The foregoing analysis was made using only the continuity and energy relations. We now proceed to apply momentum concepts to the control volume shown in Figure 9.4 The x-component of the momentum equation for steady, one-dimensional flow is

$$\sum F_x = \frac{\dot{m}}{g_c}\left(V_{\text{out}_x} - V_{\text{in}_x}\right)$$

From Figure 9.4 we see that the force summation is

$$\sum F_x = p_1 A - p_2 A - F_f \tag{9.14}$$

where F_f represents the total wall frictional force on the fluid between sections 1 and 2. Thus the momentum equation in the direction of flow becomes

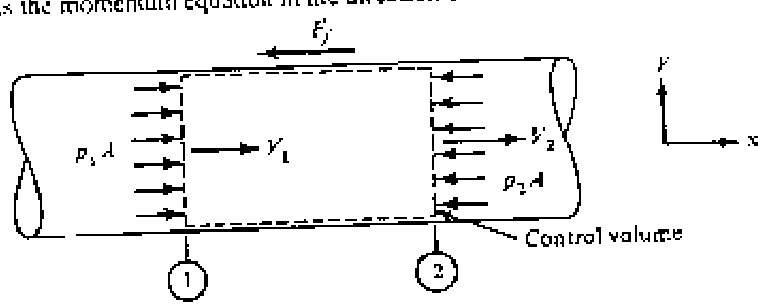


Figure 9.4 Momentum analysis for Fanno flow.

$$(p_1 - p_2)A - F_f = \frac{\dot{m}}{g_c}(V_2 - V_1) = \frac{\rho A V}{g_c}(V_2 - V_1) \tag{9.15}$$

Show that equation (9.15) can be written as

$$p_1 - p_2 - \frac{F_f}{A} = \frac{\rho_2 V_2^2}{g_c} - \frac{\rho_1 V_1^2}{g_c} \tag{9.16}$$

or

$$\boxed{\left(p_1 + \frac{\rho_1 V_1^2}{g_c}\right) - \frac{F_f}{A} = p_2 + \frac{\rho_2 V_2^2}{g_c}} \tag{9.17}$$

In this form the equation is not particularly useful except to bring out one significant fact. *For the steady, one-dimensional, constant-area flow of any fluid,* the value of $p + \rho V^2/g_c$ *cannot* be constant if frictional forces are present. This fact will be recalled later in the chapter when Fanno flow is compared with normal shocks.

Before leaving this section on fluids in general, we might say a few words about Fanno flow at low Mach numbers. A glance at Figure 9.3 shows that the upper branch is asymptotically approaching the horizontal line of constant total enthalpy. Thus the extreme left end of the Fanno line will be nearly horizontal. This indicates that flow at very low Mach numbers will have almost constant velocity. This checks our previous work, which indicated that we could treat gases as incompressible fluids if the Mach numbers were very small.

9.4 WORKING EQUATIONS FOR PERFECT GASES

We have discovered the general trend of property variations that occur in Fanno flow, both in the subsonic and supersonic flow regime. Now we wish to develop some specific working equations for the case of a perfect gas. Recall that these are relations between properties at arbitrary sections of a flow system written in terms of Mach numbers and the specific heat ratio.

Energy

We start with the energy equation developed in Section 9.3 since this leads immediately to a temperature ratio:

$$h_{t1} = h_{t2} \tag{9.3}$$

But for a perfect gas, enthalpy is a function of temperature only. Therefore,

$$T_{t1} = T_{t2} \tag{9.18}$$

Now for a perfect gas with constant specific heats,

$$T_t = T\left(1 + \frac{\gamma - 1}{2} M^2\right)$$

Hence the energy equation for Fanno flow can be written as

$$T_1\left(1 + \frac{\gamma - 1}{2} M_1^2\right) = T_2\left(1 + \frac{\gamma - 1}{2} M_2^2\right) \tag{9.19}$$

or

$$\boxed{\frac{T_2}{T_1} = \frac{1 + [(\gamma - 1)/2]M_1^2}{1 + [(\gamma - 1)/2]M_2^2}} \tag{9.20}$$

Continuity

From Section 9.3 we have

$$\rho V = G = \text{const} \tag{9.2}$$

or

$$\rho_1 V_1 = \rho_2 V_2 \tag{9.21}$$

If we introduce the perfect gas equation of state

$$p = \rho R T$$

the definition of Mach number

$$V = Ma$$

and sonic velocity for a perfect gas

$$a = \sqrt{\gamma g_c R T}$$

equation (9.21) can be solved for

$$\frac{p_2}{p_1} = \frac{M_1}{M_2}\left(\frac{T_2}{T_1}\right)^{1/2} \tag{9.22}$$

Can you obtain this expression? Now introduce the temperature ratio from (9.20) and you will have the following working relation for static pressure:

$$\boxed{\frac{p_2}{p_1} = \frac{M_1}{M_2}\left(\frac{1+[(\gamma-1)/2]M_1^2}{1+[(\gamma-1)/2]M_2^2}\right)^{1/2}} \qquad (9.23)$$

The density relation can easily be obtained from equations (9.20), (9.23), and the perfect gas law.

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{M_1}{M_2}\left(\frac{1+[(\gamma-1)/2]M_2^2}{1+[(\gamma-1)/2]M_1^2}\right)^{1/2}} \qquad (9.24)$$

Entropy Change

We start with an expression for entropy change that is valid between any two points:

$$\Delta s_{1-2} = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1} \qquad (1.53)$$

Equation (4.15) can be used to substitute for c_p and we nondimensionalize the equation to

$$\frac{s_2 - s_1}{R} = \frac{\gamma}{\gamma - 1}\ln\frac{T_2}{T_1} - \ln\frac{p_2}{p_1} \qquad (9.25)$$

If we now utilize the expressions just developed for the temperature ratio (9.20) and the pressure ratio (9.23), the entropy change becomes

$$\frac{s_2 - s_1}{R} = \frac{\gamma}{\gamma - 1}\ln\left(\frac{1+[(\gamma-1)/2]M_1^2}{1+[(\gamma-1)/2]M_2^2}\right)$$
$$- \ln\frac{M_1}{M_2}\left(\frac{1+[(\gamma-1)/2]M_1^2}{1+[(\gamma-1)/2]M_2^2}\right)^{1/2} \qquad (9.26)$$

Show that this entropy change between two points in Fanno flow can be written as

$$\frac{s_2 - s_1}{R} = \ln\frac{M_2}{M_1}\left(\frac{1+[(\gamma-1)/2]M_1^2}{1+[(\gamma-1)/2]M_2^2}\right)^{(\gamma+1)/2(\gamma-1)} \qquad (9.27)$$

Now recall that in Section 4.5 we integrated the stagnation pressure–energy equation for adiabatic no-work flow of a perfect gas, with the result

$$\frac{p_{t2}}{p_{t1}} = e^{-\Delta s/R} \qquad (4.28)$$

and equation (9.26) becomes

$$\frac{T}{T^*} = \frac{(\gamma + 1)/2}{1 + [(\gamma - 1)/2]M^2} = f(M,\gamma) \tag{9.41}$$

We see that $T/T^* = f(M,\gamma)$ and we can easily construct a table giving values of T/T^* versus M for a particular γ. Equation (9.25) can be treated in a similar fashion. In this case

$$p_2 \Rightarrow p \qquad M_2 \Rightarrow M \quad \text{(any value)}$$

$$p_1 \Rightarrow p^* \qquad M_1 \Rightarrow 1$$

and equation (9.25) becomes

$$\frac{p}{p^*} = \frac{1}{M}\left(\frac{(\gamma + 1)/2}{1 + [(\gamma - 1)/2]M^2}\right)^{1/2} = f(M,\gamma) \tag{9.42}$$

The density ratio can be obtained as a function of Mach number and γ from equation (9.24). This is particularly useful since it also represents a velocity ratio. Why?

$$\frac{\rho}{\rho^*} = \frac{V^*}{V} = \frac{1}{M}\left(\frac{1 + [(\gamma - 1)/2]M^2}{(\gamma + 1)/2}\right)^{1/2} = f(M,\gamma) \tag{9.43}$$

Apply the same techniques to equation (9.28) and *show* that

$$\frac{p_t}{p_t^*} = \frac{1}{M}\left(\frac{1 + [(\gamma - 1)/2]M^2}{(\gamma + 1)/2}\right)^{(\gamma+1)/2(\gamma-1)} = f(M,\gamma) \tag{9.44}$$

We now perform the same type of transformation on equation (9.40); that is, let

$$x_2 \Rightarrow x \qquad M_2 \Rightarrow M \quad \text{(any value)}$$

$$x_1 \Rightarrow x^* \qquad M_1 \Rightarrow 1$$

with the following result:

$$\frac{f(x - x^*)}{D_e} = \left(\frac{\gamma + 1}{2\gamma}\right) \ln\left(\frac{1 + [(\gamma - 1)/2]M^2}{(\gamma + 1)/2}\right) - \frac{1}{\gamma}\left(\frac{1}{M^2} - 1\right) - \frac{\gamma + 1}{2\gamma} \ln M^2 \tag{9.45}$$

But a glance at the physical diagram in Figure 9.5 shows that $(x^* - x)$ will always be a negative quantity; thus it is more convenient to change all signs in equation (9.45) and simplify it to

$$\ln T - \ln\left(1 + \frac{\gamma - 1}{2}M^2\right) = \ln \text{ const} \tag{9.32}$$

and then differentiating, we obtain

$$\frac{dT}{T} - \frac{d\{1 + [(\gamma - 1)/2]M^2\}}{1 + [(\gamma - 1)/2]M^2} = 0 \tag{9.33}$$

which can be used to substitute for dT/T in (9.30).

The continuity relation [equation (9.21)] put in terms of a perfect gas becomes

$$\frac{pM}{\sqrt{T}} = \text{ const} \tag{9.34}$$

By logarithmic differentiation (take the natural logarithm and then differentiate) *show* that

$$\frac{dp}{p} + \frac{dM}{M} - \frac{1}{2}\frac{dT}{T} = 0 \tag{9.35}$$

We can introduce equation (9.33) to eliminate dT/T, with the result that

$$\frac{dp}{p} = -\frac{dM}{M} + \frac{1}{2}\frac{d\{1 + [(\gamma - 1)/2]M^2\}}{1 + [(\gamma - 1)/2]M^2} \tag{9.36}$$

which can be used to substitute for dp/p in (9.30).

Make the indicated substitutions for dp/p and dT/T in the momentum equation, neglect the potential term, and *show* that equation (9.30) can be put into the following form:

$$f\frac{dx}{D_e} = \frac{d\{1 + [(\gamma - 1)/2]M^2\}}{1 + [(\gamma - 1)/2]M^2} - \frac{dM^2}{M^2} + \frac{2}{\gamma}\frac{dM}{M^3}$$
$$+ \frac{1}{\gamma M^2}\frac{d\{1 + [(\gamma - 1)/2]M^2\}}{1 + [(\gamma - 1)/2]M^2} \tag{9.37}$$

The last term can be simplified for integration by noting that

$$\frac{1}{\gamma M^2}\frac{d\{1 + [(\gamma - 1)/2]M^2\}}{1 + [(\gamma - 1)/2]M^2} = \frac{(\gamma - 1)}{2\gamma}\frac{dM^2}{M^2}$$
$$- \frac{(\gamma - 1)}{2\gamma}\frac{d\{1 + [(\gamma - 1)/2]M^2\}}{1 + [(\gamma - 1)/2]M^2} \tag{9.38}$$

The momentum equation can now be written as

$$f\frac{dx}{D_e} = \frac{\gamma+1}{2\gamma}\frac{d[1+[(\gamma-1)/2]M^2]}{1+[(\gamma-1)/2]M^2} + \frac{2}{\gamma}\frac{dM}{M^3} - \frac{\gamma+1}{2\gamma}\frac{dM^2}{M^2} \tag{9.39}$$

Equation (9.39) is restricted to steady, one-dimensional flow of a perfect gas, with no heat or work transfer, constant area, and negligible potential changes. We can now integrate this equation between two points in the flow and obtain

$$\boxed{\begin{aligned}\frac{f(x_2-x_1)}{D_e} &= \frac{\gamma+1}{2\gamma}\ln\frac{1+[(\gamma-1)/2]M_2^2}{1+[(\gamma-1)/2]M_1^2} \\ &\quad - \frac{1}{\gamma}\left(\frac{1}{M_2^2}-\frac{1}{M_1^2}\right) - \frac{\gamma+1}{2\gamma}\ln\frac{M_2^2}{M_1^2}\end{aligned}} \tag{9.40}$$

Note that in performing the integration we have held the friction factor constant. Some comments will be made on this in a later section. If you have forgotten the concept of equivalent diameter, you may want to review the last part of Section 3.8 and equation (3.61).

9.5 REFERENCE STATE AND FANNO TABLE

The equations developed in Section 9.4 provide the means of computing the properties at one location in terms of those given at some other location. The key to problem solution is predicting the Mach number at the new location through the use of equation (9.40). The solution of this equation for the unknown M_2 presents a messy task, as no explicit relation is possible. Thus we turn to a technique similar to that used with isentropic flow in Chapter.

We introduce *another* * reference state, which is defined in the same manner as before (i.e., "that thermodynamic state which would exist if the fluid reached a Mach number of unity *by a particular process*"). In this case we imagine that we continue *by Fanno flow* (i.e., more duct is added) until the velocity reaches Mach 1. Figure 9.5 shows a physical system together with its T–s diagram for a subsonic Fanno flow. We know that if we continue along the Fanno line (remember that we always move to the right), we will eventually reach the limiting point where sonic velocity exists. The dashed lines show a hypothetical duct of sufficient length to enable the flow to traverse the remaining portion of the upper branch and reach the limit point. This is the * reference point *for Fanno flow*.

The *isentropic* * reference points have also been included on the T–s diagram to emphasize the fact that the Fanno * reference is a totally different thermodynamic state. One other fact should be mentioned. If there is any entropy difference between two points (such as points 1 and 2), their isentropic * reference conditions are not the same and we have always taken great care to label them separately as 1^* and 2^*.

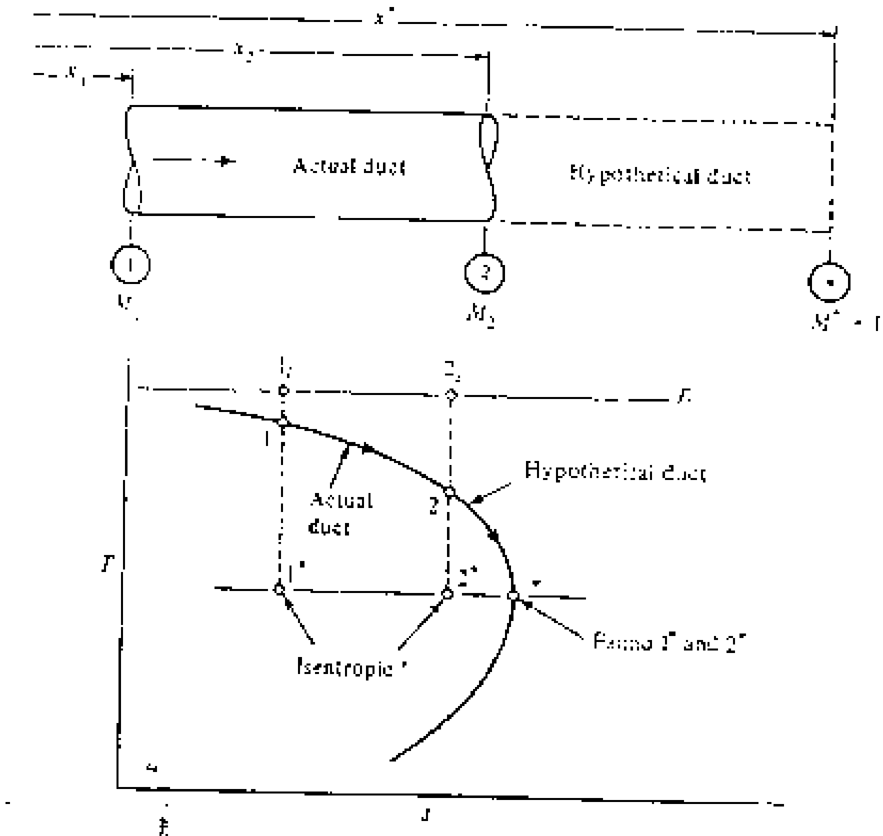


Figure 9.5 The * reference for Fanno flow.

However, proceeding from either point 1 or point 2 *by Fanno flow* will ultimately lead to the same place when Mach 1 is reached. Thus we do not have to talk of 1* or 2* but merely * in the case of Fanno flow. Incidentally, why are all three * reference points shown on the same horizontal line in Figure 9.5? (You may need to review Section 4.6.)

We now rewrite the working equations in terms of the Fanno flow * reference condition. Consider first

$$\frac{T_2}{T_1} = \frac{1 + [(\gamma - 1)/2]M_1^2}{1 + [(\gamma - 1)/2]M_2^2} \tag{9.20}$$

Let point 2 be an arbitrary point in the flow system and let its Fanno * condition be point 1. Then

$$T_2 \Rightarrow T \qquad M_2 \Rightarrow M \quad \text{(any value)}$$

$$T_1 \Rightarrow T^* \qquad M_1 \Rightarrow 1$$

and equation (9.20) becomes

$$\frac{T}{T^*} = \frac{(\gamma + 1)/2}{1 + [(\gamma - 1)/2]M^2} = f(M, \gamma) \tag{9.41}$$

We see that $T/T^* = f(M, \gamma)$ and we can easily construct a table giving values of T/T^* versus M for a particular γ. Equation (9.23) can be treated in a similar fashion. In this case

$$p_2 \Rightarrow p \qquad M_2 \Rightarrow M \quad \text{(any value)}$$

$$p_1 \Rightarrow p^* \qquad M_1 \Rightarrow 1$$

and equation (9.23) becomes

$$\frac{p}{p^*} = \frac{1}{M}\left(\frac{(\gamma + 1)/2}{1 + [(\gamma - 1)/2]M^2}\right)^{1/2} = f(M, \gamma) \tag{9.42}$$

The density ratio can be obtained as a function of Mach number and γ from equation (9.24). This is particularly useful since it also represents a velocity ratio. Why?

$$\frac{\rho}{\rho^*} = \frac{V^*}{V} = \frac{1}{M}\left(\frac{1 + [(\gamma - 1)/2]M^2}{(\gamma + 1)/2}\right)^{1/2} = f(M, \gamma) \tag{9.43}$$

Apply the same techniques to equation (9.28) and *show* that

$$\frac{p_t}{p_t^*} = \frac{1}{M}\left(\frac{1 + [(\gamma - 1)/2]M^2}{(\gamma + 1)/2}\right)^{(\gamma+1)/2(\gamma-1)} = f(M, \gamma) \tag{9.44}$$

We now perform the same type of transformation on equation (9.40); that is, let

$$x_2 \Rightarrow x \qquad M_2 \Rightarrow M \quad \text{(any value)}$$

$$x_1 \Rightarrow x^* \qquad M_1 \Rightarrow 1$$

with the following result:

$$\frac{f(x - x^*)}{D_e} = \left(\frac{\gamma + 1}{2\gamma}\right) \ln\left(\frac{1 + [(\gamma - 1)/2]M^2}{(\gamma + 1)/2}\right) - \frac{1}{\gamma}\left(\frac{1}{M^2} - 1\right) - \frac{\gamma + 1}{2\gamma} \ln M^2 \tag{9.45}$$

But a glance at the physical diagram in Figure 9.5 shows that $(x^* - x)$ will always be a negative quantity; thus it is more convenient to change all signs in equation (9.45) and simplify it to

$$\frac{f(x^* - x)}{D_e} = \left(\frac{\gamma + 1}{2\gamma}\right) \ln\left(\frac{[(\gamma + 1)/2]M^2}{1 + [(\gamma - 1)/2]M^2}\right)$$

$$+ \frac{1}{\gamma}\left(\frac{1}{M^2} - 1\right) = f(M, \gamma) \qquad (9.46)$$

The quantity $(x^* - x)$ represents the amount of duct that would have to be added to cause the flow to reach the Fanno * reference condition. It can alternatively be viewed as the maximum duct length that may be added without changing some flow condition. Thus the expression

$$\frac{f(x^* - x)}{D_e} \quad \text{is called} \quad \frac{fL_{max}}{D_e}$$

and is listed in table along with the other Fanno flow parameters: T/T^*, p/p^*, V/V^*, and p_t/p_t^*. In the next section we shall see how this table greatly simplifies the solution of Fanno flow problems. But first, some words about the determination of friction factors.

Dimensional analysis of the fluid flow problem shows that the friction factor can be expressed as

$$f = f(\mathrm{Re}, \varepsilon/D) \qquad (9.47)$$

where Re is the *Reynolds number*,

$$\mathrm{Re} \equiv \frac{\rho V D}{\mu g_c} \qquad (9.48)$$

and

$$\varepsilon/D \equiv \textit{relative roughness}$$

Typical values of ε, the *absolute roughness* or average height of wall irregularities, are shown in Table 9.2.

The relationship among f, Re, and ε/D is determined experimentally and plotted on a chart similar to Figure 9.6, which is called a *Moody diagram*.
If the flow rate is known together with the duct size and

Table 9.2 Absolute Roughness of Common Materials

Material	ε (ft)
Glass, brass, copper, lead	smooth < 0.00001
Steel, wrought iron	0.00015
Galvanized iron	0.0005
Cast iron	0.00085
Riveted steel	0.03

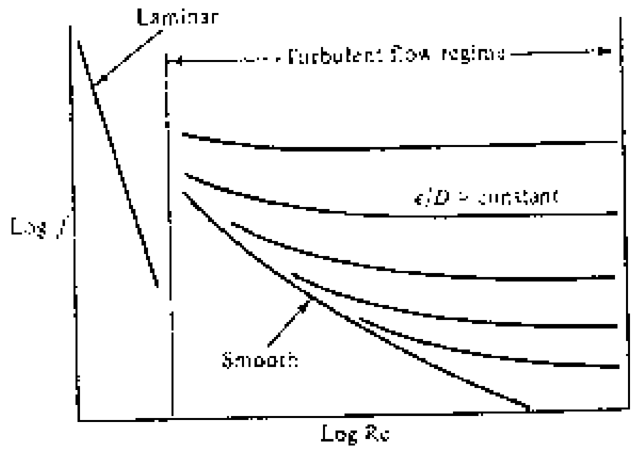


Figure 9.6 Moody diagram for friction factor in circular ducts.

material, the Reynolds number and relative roughness can easily be calculated and the value of the friction factor is taken from the diagram. The curve in the laminar flow region can be represented by

$$f = \frac{64}{\text{Re}} \tag{9.49}$$

For noncircular cross sections the *equivalent diameter* as described in Section 3.8 can be used.

$$D_e = \frac{4A}{P} \tag{3.61}$$

This equivalent diameter may be used in the determination of relative roughness and Reynolds number, and hence the friction factor. However, care must be taken to work with the *actual* average velocity in all computations. Experience has shown that the use of an equivalent diameter works quite well in the turbulent zone. In the laminar flow region this concept is not sufficient and consideration must also be given to the aspect ratio of the duct.

In some problems the flow rate is not known and thus a trial-and-error solution results. As long as the duct size is given, the problem is not too difficult; an excellent approximation to the friction factor can be made by taking the value corresponding to where the ϵ/D curve begins to *level off*. This converges rapidly to the final answer, as most engineering problems are well into the turbulent range.

9.6 APPLICATIONS

The following steps are recommended to develop good problem-solving technique:

1. Sketch the physical situation (including the hypothetical * reference point).
2. Label sections where conditions are known or desired.
3. List all given information with units.
4. Compute the equivalent diameter, relative roughness, and Reynolds number.
5. Find the friction factor from the Moody diagram.
6. Determine the unknown Mach number.
7. Calculate the additional properties desired.

The procedure above may have to be altered depending on what type of information is given, and occasionally, trial-and-error solutions are required. You should have no difficulty incorporating these features once the basic straightforward solution has been mastered. In complicated flow systems that involve more than just Fanno flow, a T–s diagram is frequently helpful in solving problems.

For the following examples we are dealing with the steady one-dimensional flow of air ($\gamma = 1.4$), which can be treated as a perfect gas. Assume that $Q = W_s = 0$ and negligible potential changes. The cross-sectional area of the duct remains constant. Figure E9.1 is common to Examples 9.1 through 9.3.

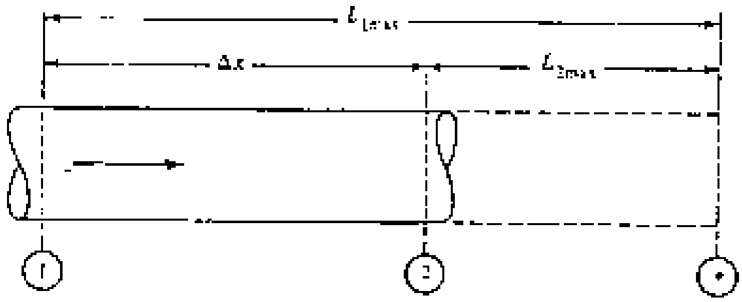


Figure E9.1

Example 9.1 Given $M_1 = 1.80$, $p_1 = 40$ psia, and $M_2 = 1.20$, find p_2 and $f\,\Delta x/D$.
Since both Mach numbers are known, we can solve immediately for

$$p_2 = \frac{p_2}{p^*}\frac{p^*}{p_1}p_1 = (0.8044)\left(\frac{1}{0.4741}\right)(40) = 67.9 \text{ psia}$$

Check Figure E9.1 to see that

$$\frac{f\,\Delta x}{D} = \frac{fL_{1max}}{D} - \frac{fL_{2max}}{D} = 0.2419 - 0.0336 = 0.208$$

Example 9.2 Given $M_2 = 0.94$, $T_1 = 400$ K, and $T_2 = 350$ K, find M_1 and p_2/p_1.
To determine conditions at section 1 in Figure E9.1, we must establish the ratio

$$\frac{T_2}{T^*} = \frac{T_2}{T_1}\frac{T_1}{T^*} = \left(\frac{400}{350}\right)(1.0198) = 1.1655$$

From Fanno table at $M = 0.94$

Given

Look up $T/T^* = 1.1655$ in the Fanno table and determine that $M_2 = 0.385$.
Thus

$$\frac{p_2}{p_1} = \frac{p_2}{p^*}\frac{p^*}{p_1} = (1.0743)\left(\frac{1}{2.8046}\right) = 0.383$$

Notice that these examples confirm previous statements concerning static pressure changes. In subsonic flow the static pressure decreases, whereas in supersonic flow the static pressure increases. *Compute* the stagnation pressure ratio and show that the friction losses cause p_{t2}/p_{t1} to decrease in each case.

For Example 9.1:

$$\frac{p_{t2}}{p_{t1}} = \qquad\qquad (p_{t2}/p_{t1} = 0.7161)$$

For Example 9.2:

$$\frac{p_{t2}}{p_{t1}} = \qquad\qquad (p_{t2}/p_{t1} = 0.611)$$

Example 9.3 Air flows in a 6-in.-diameter, insulated, galvanized iron duct. Initial conditions are $p_1 = 20$ psia, $T_1 = 70°F$, and $V_1 = 406$ ft/sec. After 70 ft, determine the final Mach number, temperature, and pressure.

Since the duct is circular we do not have to compute an equivalent diameter. From Table 9.2 the absolute roughness ε is 0.0005. Thus the relative roughness

$$\frac{\varepsilon}{D} = \frac{0.0005}{0.5} = 0.001$$

We compute the Reynolds number at section 1 (Figure E9.1) since this is the only location where information is known.

$$\rho_1 = \frac{p_1}{RT_1} = \frac{(20)(144)}{(53.3)(530)} = 0.102 \text{ lbm/ft}^3$$

$$\mu_1 = 3.8 \times 10^{-7} \text{ lbf-sec/ft}^2 \text{ (Air properties table)}$$

Thus

$$\mathrm{Re}_1 = \frac{\rho_1 V_1 D_1}{\mu_1 g_c} = \frac{(0.102)(406)(0.5)}{(3.8 \times 10^{-7})(32.2)} = 1.69 \times 10^6$$

From the Moody diagram at $\mathrm{Re} = 1.69 \times 10^6$ and $\varepsilon/D = 0.001$, we determine that the friction factor is $f = 0.0198$. To use the Fanno table (or equations), we need information on Mach numbers.

$R = 53.3$

$\mu = 3.8 \times 10^{-7}$

$g_c = 32.2$

$$a_1 = (\gamma g_c R T_1)^{1/2} = [(1.4)(32.2)(53.3)(530)]^{1/2} = 1128 \text{ ft/sec}$$

$$M_1 = \frac{V_1}{a_1} = \frac{406}{1128} = 0.36$$

From the Fanno table at $M_1 = 0.36$, we find that

$$\frac{p_1}{p^*} = 3.0042 \qquad \frac{T_1}{T^*} = 1.1697 \qquad \frac{fL_{1\,max}}{D} = 3.1801$$

The key to completing the problem is in establishing the Mach number at the outlet, and this is done through the *friction length*.

$$\frac{f\,\Delta x}{D} = \frac{(0.0198)(70)}{0.5} = 2.772$$

Looking at the physical sketch it is apparent (since f and D are constants) that

$$\frac{fL_{2\,max}}{D} = \frac{fL_{1\,max}}{D} - \frac{f\,\Delta x}{D} = 3.1801 - 2.772 = 0.408$$

We enter the Fanno table with this friction length and find that

$$M_2 = 0.623 \qquad \frac{p_2}{p^*} = 1.6939 \qquad \frac{T_2}{T^*} = 1.1136$$

Thus

$$p_2 = \frac{p_2}{p^*}\frac{p^*}{p_1}p_1 = (1.6939)\left(\frac{1}{3.0042}\right)(20) = 11.28 \text{ psia}$$

and

$$T_2 = \frac{T_2}{T^*}\frac{T^*}{T_1}T_1 = (1.1136)\left(\frac{1}{1.1697}\right)(530) = 505°\text{R}$$

In the example above, the friction factor was assumed constant. In fact, this assumption was made when equation (9.39) was integrated to obtain (9.40), and with the introduction of the * reference state, this became equation (9.46), which is listed in the Fanno table. Is this a reasonable assumption? Friction factors are functions of Reynolds numbers, which in turn depend on velocity and density—both of which can change quite rapidly in Fanno flow. Calculate the velocity at the outlet in Example 9.3 and compare it with that at the inlet. ($V_2 = 686$ ft/sec and $V_1 = 406$ ft/sec.)

But don't despair. From continuity we know that the product of ρV is always a constant, and thus the only variable in Reynolds number is the viscosity. Extremely large temperature variations are required to change the viscosity of a gas significantly, and thus variations in the Reynolds number are small for any given problem. We are also fortunate in that most engineering problems are well into the turbulent range where the friction factor is relatively insensitive to Reynolds number. A greater potential error is involved in the estimation of the duct roughness, which has a more significant effect on the friction factor.

Example 9.4 A converging-diverging nozzle with an area ratio of 5.42 connects to an 8-ft-long constant-area rectangular duct (see Figure E9.4). The duct is 8 × 4 in. in cross section and has a friction factor of $f = 0.02$. What is the minimum stagnation pressure feeding the nozzle if the flow is supersonic throughout the entire duct and it exhausts to 14.7 psia?

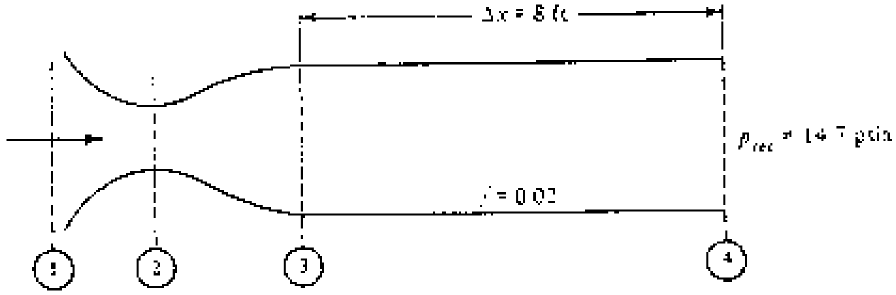


Figure E9.4

$$D_e = \frac{4A}{P} = \frac{(4)(32)}{24} = 5.334 \text{ in.}$$

$$\frac{f\,\Delta x}{D} = \frac{(0.02)(8)(12)}{5.334} = 0.36$$

To be supersonic with $A_3/A_2 = 5.42$, $M_3 = 3.26$, $p_3/p_{t3} = 0.0185$, $p_3/p^* = 0.1901$, and $fL_{3\max}/D = 0.5582$.

$$\frac{fL_{4\max}}{D} = \frac{fL_{3\max}}{D} - \frac{f\,\Delta x}{D} = 0.5582 - 0.36 = 0.1982$$

Thus

$$M_4 = 1.673 \quad \text{and} \quad \frac{p_4}{p^*} = 0.5243$$

and

$$p_{t1} = \frac{p_{t1}}{p_{t3}}\frac{p_{t3}}{p_3}\frac{p_3}{p^*}\frac{p^*}{p_4}p_4 = (1)\left(\frac{1}{0.0185}\right)(0.1901)\left(\frac{1}{0.5243}\right)(14.7) = 288 \text{ psia}$$

Any pressure above 288 psia will maintain the flow system as specified but with expansion waves outside the duct. (Recall an underexpanded nozzle.) Can you envision what would happen if the inlet stagnation pressure fell below 288 psia? (Recall the operation of an overexpanded nozzle.)

9.7 CORRELATION WITH SHOCKS

As you have progressed through this chapter you may have noticed some similarities between Fanno flow and normal shocks. Let us summarize some pertinent information.

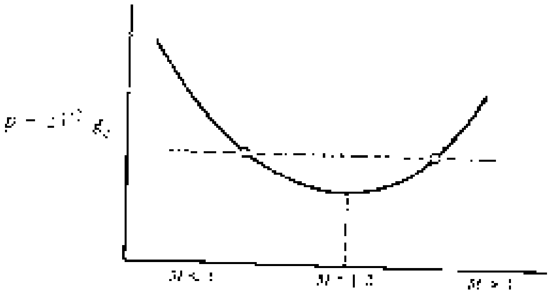


Figure 9.7 Variation of $p + \rho V^2/g_c$ in Fanno flow

The points just before and after a normal shock represent states with the same mass flow per unit area, the same value of $p + \rho V^2/g_c$, and the same stagnation enthalpy. These facts are the result of applying the basic concepts of continuity, momentum, and energy to any arbitrary fluid. This analysis resulted in equations (6.2), (6.3), and (6.9).

A Fanno line represents states with the same mass flow per unit area and the same stagnation enthalpy. This is confirmed by equations (9.2) and (9.5). To move *along* a Fanno line requires friction. At the end of Section 9.3 [see equation (9.17)] it was pointed out that it is this very friction which causes the value of $p + \rho V^2/g_c$ to change.

The variation of the quantity $p + \rho V^2/g_c$ along a Fanno line is quite interesting. Such a plot is shown in Figure 9.7. You will notice that for every point on the supersonic branch of the Fanno line there is a corresponding point on the subsonic branch with the same value of $p + \rho V^2/g_c$. Thus these two points satisfy all three conditions for the end points of a normal shock and could be connected by such a shock.

Now we can imagine a supersonic Fanno flow leading into a normal shock. If this is followed by additional duct, subsonic Fanno flow would occur. Such a situation is shown in Figure 9.8a. Note that the shock merely causes the flow to jump from the supersonic branch to the subsonic branch of the *same* Fanno line. [See Figure 9.8b.]

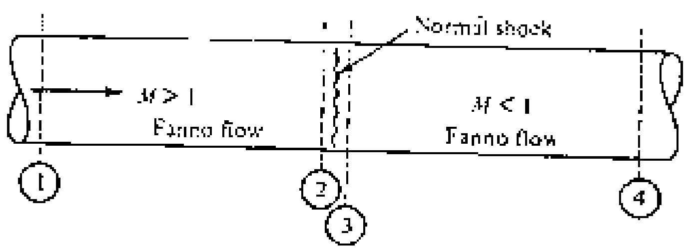


Figure 9.8a Combination of Fanno flow and normal shock (physical system).

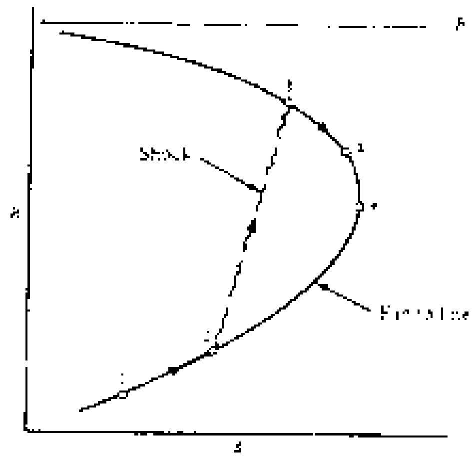


Figure 9.8b Combination of Fanno flow and normal shock

Example 9.5 A large chamber contains air at a temperature of 300 K and a pressure of 5 bar abs (Figure E9.5). The air enters a converging–diverging nozzle with an area ratio of 2.4. A constant-area duct is attached to the nozzle and a normal shock stands at the exit plane. Receiver pressure is 3 bar abs. Assume the entire system to be adiabatic and neglect friction in the nozzle. Compute the $f\,\Delta x/D$ for the duct.

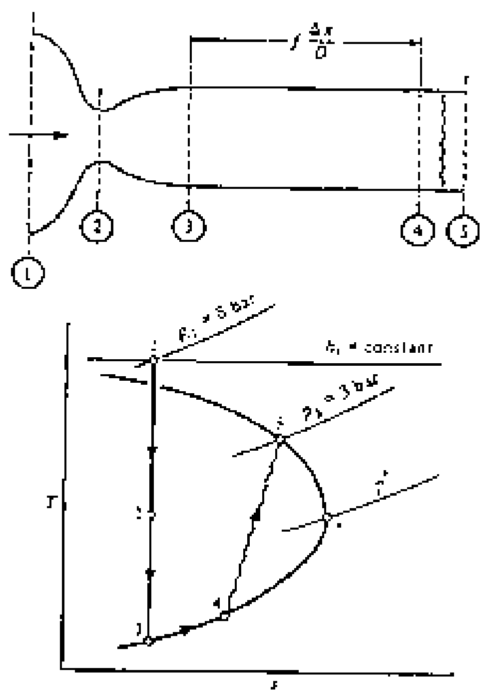


Figure E9.5

For a shock to occur as specified, the duct flow must be supersonic, which means that the nozzle is operating at its third critical point. The inlet conditions and nozzle area ratio fix conditions at location 3. We can then find p^* at the tip of the Fanno line. Then the ratio p_5/p^* can be computed and the Mach number after the shock is found from the Fanno table. This solution probably would not have occurred to us had we not drawn the T–s diagram and recognized that point 5 is on the same Fanno line as 3, 4, and *.

For $A_3/A_t = 2.4$, $M_3 = 2.4$ and $p_3/p_{t3} = 0.06840$. We proceed immediately to compute p_5/p^*:

$$\frac{p_5}{p^*} = \frac{p_5}{p_{t5}}\frac{p_{t5}}{p_{t3}}\frac{p_{t3}}{p_3}\frac{p_3}{p^*} = \left(\frac{3}{8}\right)(1)\left(\frac{1}{0.06840}\right)(0.3111) = 1.7056$$

From the Fanno table we find that $M_5 = 0.619$, and then from the shock table $M_4 = 1.764$. Returning to the Fanno table, $fL_{3\max}/D = 0.4099$ and $fL_{4\max}/D = 0.2382$. Thus

$$\frac{f\,\Delta x}{D} = \frac{fL_{3\max}}{D} - \frac{fL_{4\max}}{D} = 0.4099 - 0.2382 = 0.172$$

9.8 FRICTION CHOKING

In Chapter 5 we discussed the operation of nozzles that were fed by constant stagnation inlet conditions (see Figures 5.6 and 5.8). We found that as the receiver pressure was lowered, the flow through the nozzle increased. When the *operating pressure ratio* reached a certain value, the section of minimum area developed a Mach number of unity. The nozzle was then said to be choked. Further reduction in the pressure ratio did not increase the flow rate. This was an example of *area choking*.

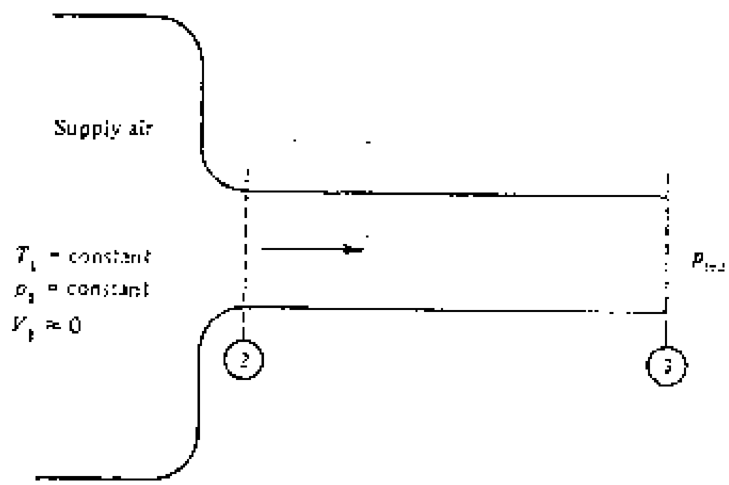


Figure 9.9 Converging nozzle and constant-area duct combination.

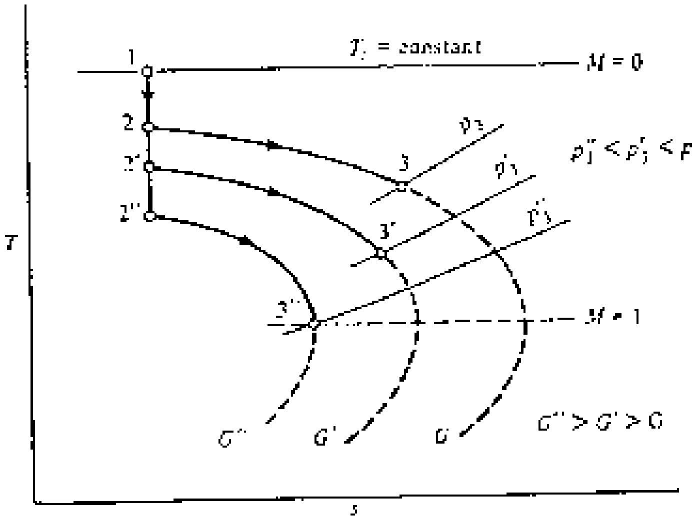


Figure 9.10 T–s diagram for nozzle–duct combination.

The subsonic Fanno flow situation is quite similar. Figure 9.9 shows a given length of duct fed by a large tank and converging nozzle. If the receiver pressure is below the tank pressure, flow will occur, producing a T–s diagram shown as path 1–2–3 in Figure 9.10. Note that we have isentropic flow at the entrance to the duct and then we move along a Fanno line. As the receiver pressure is lowered still more, the flow rate and exit Mach number continue to increase while the system moves to Fanno lines of higher mass velocities (shown as path 1–2′–3′). It is important to recognize that the receiver pressure (or more properly, the operating pressure ratio) is controlling the flow. This is because in subsonic flow the pressure at the duct exit must equal that of the receiver.

Eventually, when a certain pressure ratio is reached, the Mach number at the duct exit will be unity (shown as path 1–2″–3″). This is called *friction choking* and any further reduction in receiver pressure would not affect the flow conditions *inside* the system. What would occur as the flow leaves the duct and enters a region of reduced pressure?

Let us consider this last case of choked flow with the exit pressure equal to the receiver pressure. Now *suppose that the receiver pressure is maintained at this value* but more duct is added to the system. (Nothing can physically prevent us from doing this.) What happens? We know that we cannot move *around the Fanno line*, yet somehow we must reflect the added friction losses. This is done by moving to a new Fanno line at a *decreased* flow rate. The T–s diagram for this is shown as path 1–2‴–3‴–4 in Figure 9.11. Note that pressure equilibrium is still maintained at the exit but

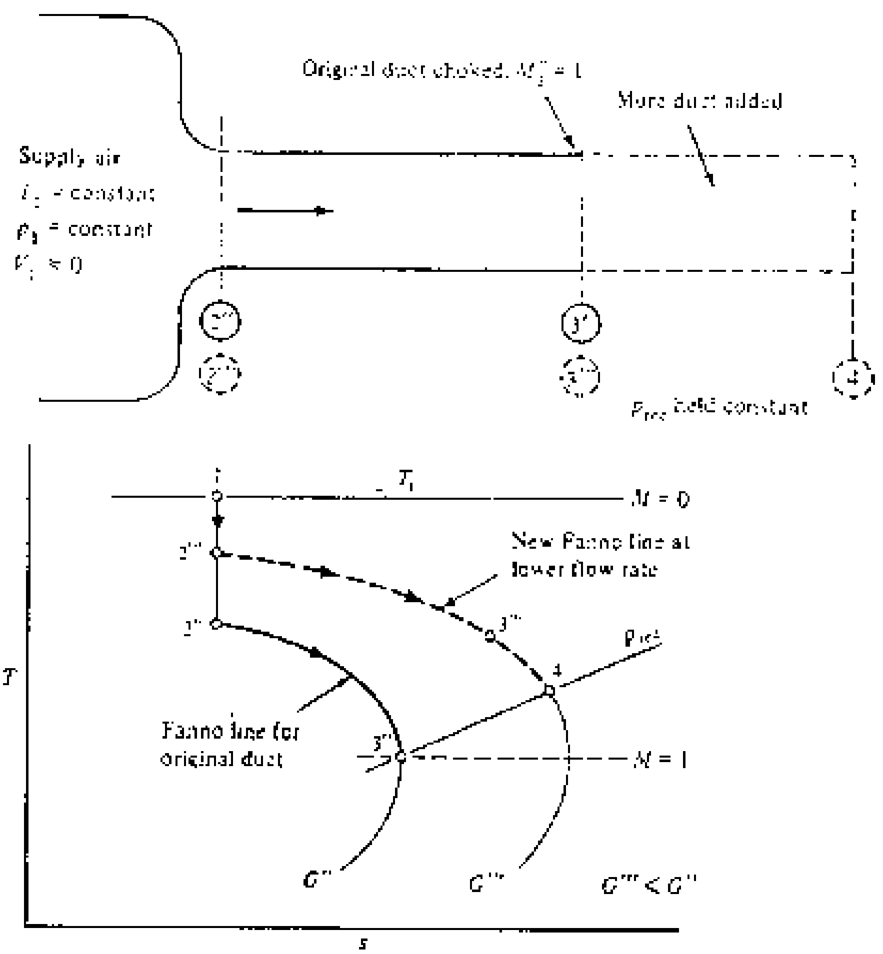


Figure 9.11 Addition of more duct when choked.

the system is no longer choked, although the flow rate has decreased. What would occur if the receiver pressure were now lowered?

In summary, when a *subsonic* Fanno flow has become *friction choked* and more duct is added to the system, the flow rate must decrease. Just how much it decreases and whether or not the exit velocity remains sonic depends on how much duct is added and the receiver pressure imposed on the system.

Now suppose that we are dealing with *supersonic* Fanno flow that is *friction choked*. In this case the addition of more duct causes a normal shock to form inside the duct. The resulting subsonic flow can accommodate the increased duct length at the same flow rate. For example, Figure 9.12 shows a Mach 2.18 flow that has an fL_{max}/D value of 0.356. If a normal shock were to occur at this point, the Mach number after the shock would be about 0.550, which corresponds to an fL_{max}/D

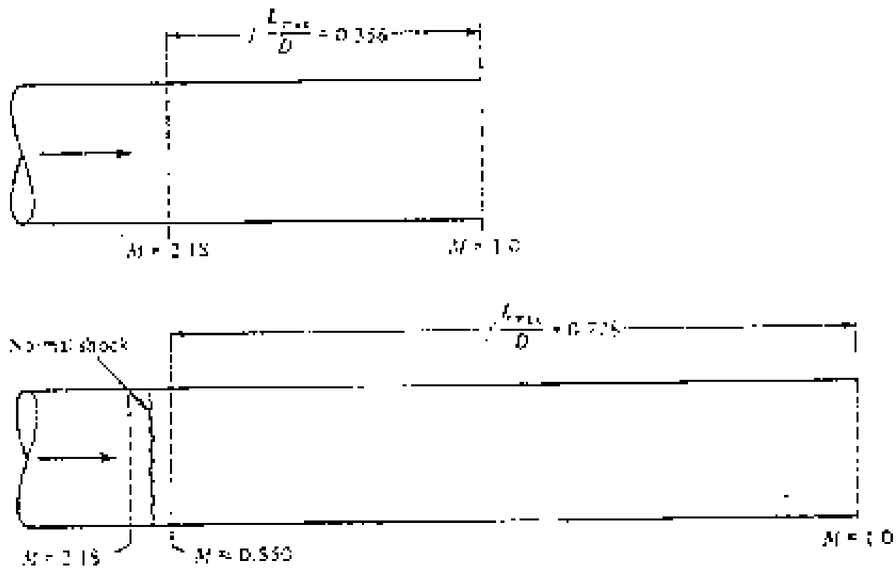


Figure 9.12 Influence of shock on maximum duct length.

value of 0.728. Thus, in this case, the appearance of the shock permits over twice the duct length to the choke point. This difference becomes even greater as higher Mach numbers are reached.

The shock location is determined by the amount of duct added. As more duct is added, the shock moves upstream and occurs at a higher Mach number. Eventually, the shock will move into that portion of the system that precedes the constant-area duct. (Most likely, a converging–diverging nozzle was used to produce the supersonic flow.) If sufficient friction length is added, the entire system will become subsonic and then the flow rate will decrease. Whether or not the exit velocity remains sonic will again depend on the receiver pressure.

9.9. WHEN γ IS NOT EQUAL TO 1.4

As indicated earlier, the Fanno flow table is for $\gamma = 1.4$. The behavior of fL_{max}/D, the friction function, is given in Figure 9.13 for $\gamma = 1.13$, 1.4, and 1.67 for Mach numbers up to $M = 5$. Here we can see that the dependence on γ is rather noticeable for $M \geq 1.4$. Thus, below this Mach number the tabulation in fanno table may be used with little error for any γ. This means that for subsonic flows, where most Fanno flow problems occur, there is little difference between the various gases. The desired accuracy of results will govern how far you want to carry this approximation into the supersonic region.

Strictly speaking, these curves are only representative for cases where γ variations are *negligible within the flow*. However, they offer hints as to what magnitude of

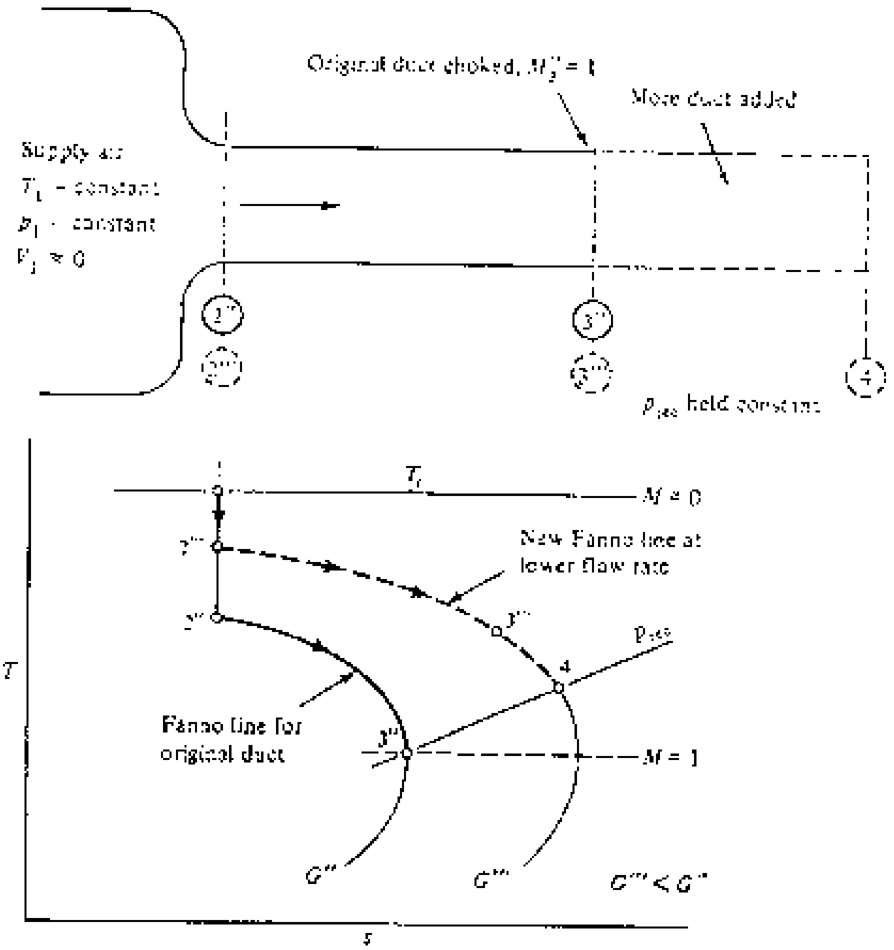


Figure 9.11 Addition of more duct when choked

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Listed below are the precise inputs and program that you use in the computer.

```
[ > g := 1.4;   X := [illegible];
[ > Y := ((g + 1)/(2*g))*ln(((g + 1)*(X^2)/2)/(1 +
    (g - 1)*(X^2)/2)) + (1/g)*((1/X^2) - 1);
                              Y := 3.180117523
```

We can proceed to find the Mach number at station 2. The new value of Y is $3.1801 - 2.772 = 0.408$. Now we use the same equation (9.46) but solve for M_2 as shown below. Note that since M is implicit in the equation, we are going to utilize "fsolve." Let

$g \equiv \gamma$, a parameter (the ratio of specific heats)

$X \equiv$ the dependent variable (which in this case is M_2)

$Y \equiv$ the independent variable (which in this case is fL_{max}/D)

Listed below are the precise inputs and program that you use in the computer.

```
[ > g2 := 1.4;   Y2 := 0.408;
[ > fsolve(Y2 = ((g2 + 1)/(2*g2))*log(((g2 + 1)*(X2^2)/2)/(1 +
    (g2 - 1)*(X2^2)/2)) + (1/g2)*((1/X2^2) - 1), X2, 0..1);
                              .6227097475
```

The answer of $M_2 = 0.6227$ is consistent with that obtained in Example 9.3. We can now proceed to calculate the required static properties, but this will be left as an exercise for the reader. ♦

9.11 SUMMARY

We have analyzed flow in a constant-area duct with friction but without heat transfer. The fluid properties change in a predictable manner dependent on the flow regime as shown in Table 9.3. The property variations in subsonic Fanno flow follow an intuitive pattern but we note that the supersonic flow behavior is completely different. The

Table 9.3 Fluid Property Variation for Fanno Flow

Property	Subsonic	Supersonic
Velocity	Increases	Decreases
Mach number	Increases	Decreases
Enthalpy[a]	Decreases	Increases
Stagnation enthalpy[a]	Constant	Constant
Pressure	Decreases	Increases
Density	Decreases	Increases
Stagnation pressure	Decreases	Decreases

[a] Also temperature if the fluid is a perfect gas.

only common occurrence is the decrease in stagnation pressure, which is indicative of the loss.

Perhaps the most significant equations are those that apply to all fluids:

$$\rho V = G = \text{constant} \tag{9.2}$$

$$h_t = h + \frac{G^2}{\rho^2 2g_c} = \text{constant} \tag{9.5}$$

Along with these equations you should keep in mind the appearance of Fanno lines in the h–s and T–s diagrams (see Figures 9.1 and 9.2). Remember that each Fanno line represents points with the same mass velocity (G) and stagnation enthalpy (h_t), and a normal shock can connect two points on opposite branches of a Fanno line which have the same value of $p + \rho V^2/g_c$. *Families* of Fanno lines could represent

1. Different values of G for the same h_t (such as those in Figure 9.10), or
2. The same G for different values of h_t (see Problem 10.17).

Detailed working equations were developed for perfect gases, and the introduction of a * reference point enabled the construction of a Fanno table which simplifies problem solution. The * condition for Fanno flow has no relation to the one used previously in isentropic flow (except in general definition). All Fanno flows proceed toward a limiting point of Mach 1. Friction choking of a flow passage is possible in Fanno flow just as area choking occurs in varying-area isentropic flow. An h–s (or T–s) diagram is of great help in the analysis of a complicated flow system. *Get into the habit of drawing these diagrams.*

PROBLEMS

In the problems that follow you may assume that all systems are completely adiabatic. Also, all ducts are of constant area unless otherwise indicated. You may neglect friction in the varying-area sections. You may also assume that the friction factor shown in charts applies to noncircular cross sections when the equivalent diameter concept is used and the flow is turbulent.

9.1. Conditions at the entrance to a duct are $M_1 = 3.0$ and $p_1 = 8 \times 10^4$ N/m². After a certain length the flow has reached $M_2 = 1.5$. Determine p_2 and $f\,\Delta x/D$ if $\gamma = 1.4$.

9.2. A flow of nitrogen is discharged from a duct with $M_2 = 0.85$, $T_2 = 500$°R, and $p_2 = 28$ psia. The temperature at the inlet is 560°R. Compute the pressure at the inlet and the mass velocity (G).

9.3. Air enters a circular duct with a Mach number of 3.0. The friction factor is 0.01.

(a) How long a duct (measured in diameters) is required to reduce the Mach number to 2.0?

(b) What is the percentage change in temperature, pressure, and density?

(c) Determine the entropy increase of the air.

(d) Assume the same length of duct as computed in part (a), but the initial Mach number is 0.5. Compute the percentage change in temperature, pressure, density, and the entropy increase for this case. Compare the changes in the same length duct for subsonic and supersonic flow.

9.4. Oxygen enters a 6-in.-diameter duct with $T_1 = 600°R$, $p_1 = 50$ psia, and $V_1 = 600$ ft/sec. The friction factor is $f = 0.02$.

(a) What is the maximum length of duct permitted that will not change any of the conditions at the inlet?

(b) Determine T_2, p_2, and V_2 for the maximum duct length found in part (a).

9.5. Air flows in an 8-cm-inside diameter pipe that is 4 m long. The air enters with a Mach number of 0.45 and a temperature of 300 K.

(a) What friction factor would cause sonic velocity at the exit?

(b) If the pipe is made of cast iron, estimate the inlet pressure.

9.6. At one section in a constant-area duct the stagnation pressure is 66.8 psia and the Mach number is 0.80. At another section the pressure is 60 psia and the temperature is 120°F.

(a) Compute the temperature at the first section and the Mach number at the second section if the fluid is air.

(b) Which way is the air flowing?

(c) What is the friction length ($f\ \Delta x / D$) of the duct?

9.7. A 50×50 cm duct is 10 m in length. Nitrogen enters at $M_1 = 3.0$ and leaves at $M_2 = 1.7$, with $T_2 = 280$ K and $p_2 = 7 \times 10^4\ N/m^2$.

(a) Find the static and stagnation conditions at the entrance.

(b) What is the friction factor of the duct?

9.8. A duct of 2 ft × 1 ft cross section is made of riveted steel and is 500 ft long. Air enters with a velocity of 174 ft/sec, $p_1 = 50$ psia, and $T_1 = 100°F$.

(a) Determine the temperature, pressure, and velocity at the exit.

(b) Compute the pressure drop assuming the flow to be incompressible. Use the entering conditions and equation (3.29). Note that equation (3.64) can easily be integrated to evaluate

$$\int T\, ds_i = f \frac{\Delta x}{D_e} \frac{V^2}{2g_c}$$

(c) How do the results of parts (a) and (b) compare? Did you expect this?

9.9. Air enters a duct with a mass flow rate of 35 lbm/sec at $T_1 = 520°R$ and $p_1 = 20$ psia. The duct is square and has an area of 0.64 ft². The outlet Mach number is unity.

(a) Compute the temperature and pressure at the outlet.

(b) Find the length of the duct if it is made of steel.

9.10. Consider the flow of a perfect gas along a Fanno line. Show that the pressure at the * reference state is given by the relation

$$p^* = \frac{\dot{m}}{A}\left[\frac{2RT_t}{\gamma g_c(\gamma + 1)}\right]^{1/2}$$

9.11. A 20-ft duct 12 in. in diameter contains oxygen flowing at the rate of 80 lbm/sec. Measurements at the inlet give $p_1 = 30$ psia and $T_1 = 800°R$. The pressure at the outlet is $p_2 = 23$ psia.

(a) Calculate M_1, M_2, V_2, T_{t2}, and p_{t2}.

(b) Determine the friction factor and estimate the absolute roughness of the duct material.

9.12. At the outlet of a 25-cm-diameter duct, air is traveling at sonic velocity with a temperature of 16°C and a pressure of 1 bar. The duct is very smooth and is 15 m long. There are two possible conditions that could exist at the entrance to the duct.

(a) Find the static and stagnation temperature and pressure for each entrance condition.

(b) Assuming the surrounding air to be at 1 bar pressure, how much horsepower is necessary to get ambient air into the duct for each case? (You may assume no losses in the work process.)

9.13. Ambient air at 60°F and 14.7 psia accelerates isentropically into a 12-in.-diameter duct. After 100 ft the duct transitions into an 8 × 8 in. square section where the Mach number is 0.50. Neglect all frictional effects except in the constant-area duct, where $f = 0.04$.

(a) Determine the Mach number at the duct entrance.

(b) What are the temperature and pressure in the square section?

(c) How much 8 × 8 in. square duct could be added before the flow chokes? (Assume that $f = 0.04$ in this duct also.)

9.14. Nitrogen with $p_t = 7 \times 10^5$ N/m^2 and $T_t = 340$ K enters a frictionless converging–diverging nozzle having an area ratio of 4.0. The nozzle discharges supersonically into a constant-area duct that has a friction length $f\,\Delta x/D = 0.355$. Determine the temperature and pressure at the exit of the duct.

9.15. Conditions before a normal shock are $M_1 = 2.5$, $p_{t1} = 67$ psia, and $T_{t1} = 700°R$. This is followed by a length of Fanno flow and a converging nozzle as shown in Figure P9.15. The area change is such that the system is choked. It is also known that $p_4 = p_{amb} = 14.7$ psia.

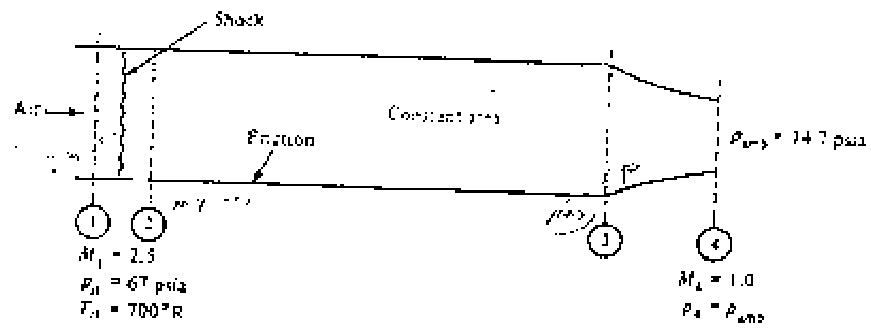

Figure P9.15

(a) Draw a T–s diagram for the system.

(b) Find M_2 and M_3.

(c) What is $f\,\Delta x/D$ for the duct?

9.16. A converging–diverging nozzle (Figure P9.16) has an area ratio of 3.0. The stagnation conditions of the inlet air are 150 psia and 550°R. A constant-area duct with a length of 12 diameters is attached to the nozzle outlet. The friction factor in the duct is 0.025.

(a) Compute the receiver pressure that would place a shock

(i) in the nozzle throat;

(ii) at the nozzle exit;

(iii) at the duct exit.

(b) What receiver pressure would cause supersonic flow throughout the duct with no shocks within the system (or after the duct exit)?

(c) Make a sketch similar to Figure 6.3 showing the pressure distribution for the various operating points of parts (a) and (b).

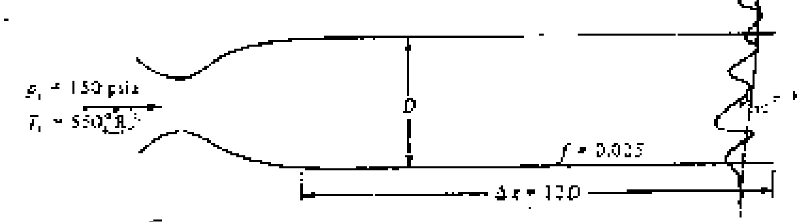

Figure P9.16

9.17. For a nozzle–duct system similar to that of Problem 9.16, the nozzle is designed to produce a Mach number of 2.8 with $\gamma = 1.4$. The inlet conditions are $p_{t1} = 10$ bar and $T_{t1} = 370$ K. The duct is 8 diameters in length, but the duct friction factor is unknown. The receiver pressure is fixed at 3 bar and a normal shock has formed at the duct exit.

(a) Sketch a T–s diagram for the system.

(b) Determine the friction factor of the duct.

(c) What is the total change in entropy for the system?

9.18. A large chamber contains air at 65 bar pressure and 400 K. The air passes through a converging-only nozzle and then into a constant-area duct. The friction length of the duct is $f\,\Delta x/D = 1.067$ and the Mach number at the duct exit is 0.96.

(a) Draw a T–s diagram for the system.

(b) Determine conditions at the duct entrance.

(c) What is the pressure in the receiver? (*Hint:* How is this related to the duct exit pressure?)

(d) If the length of the duct is doubled and the chamber and receiver conditions remain unchanged, what are the new Mach numbers at the entrance and exit of the duct?

9.19. A constant-area duct is fed by a converging-only nozzle as shown in Figure P9.19. The nozzle receives oxygen from a large chamber at $p_1 = 100$ psia and $T_1 = 1000°$R. The duct has a friction length of 5.3 and it is choked at the exit. The receiver pressure is exactly the same as the pressure at the duct exit.

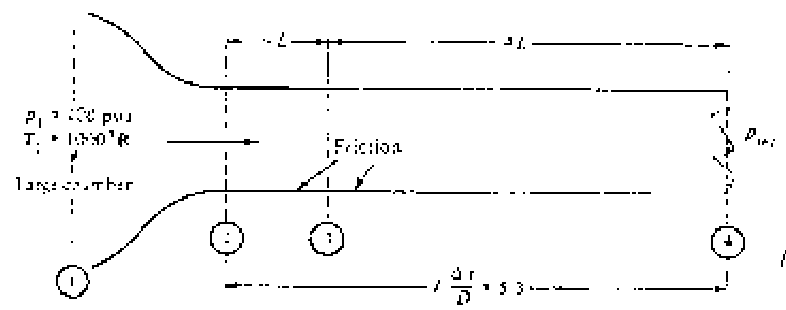

Figure P9.19

(a) What is the pressure at the end of the duct?

(b) Four-fifths of the duct is removed. (The end of the duct is now at 3.) The chamber pressure, receiver pressure, and friction factor remain unchanged. Now what is the pressure at the exit of the duct?

(c) Sketch both of the cases above on the same T–s diagram.

9.20. (a) Plot a Fanno line to scale in the T–s plane for air entering a duct with a Mach number of 0.20, a static pressure of 100 psia, and a static temperature of 540°R. Indicate the Mach number at various points along the curve.

(b) On the same diagram, plot another Fanno line for a flow with the same total enthalpy, the same entering entropy, but double the mass velocity.

9.21. Which, if any, of the ratios tabulated in the Fanno table (T/T^*, p/p^*, p_t/p_t^*, etc.) could also be listed in the Isentropic table with the same numerical values?

9.22. A contractor is to connect an air supply from a compressor to test apparatus 21 ft away. The exit diameter of the compressor is 2 in. and the entrance to the test equipment has a 1-in.-diameter pipe. The contractor has the choice of putting a reducer at the compressor followed by 1-in. tubing or using 2-in. tubing and putting the reducer at the entrance to the test equipment. Since smaller tubing is cheaper and less obtrusive, the contractor is leaning toward the first possibility, but just to be sure, he sends the problem to the engineering personnel. The air coming out of the compressor is at 520°R and the pressure is 40 psia. The flow rate is 0.7 lbm/sec. Consider that each size of tubing has an effective $f = 0.02$. What would be the conditions at the entrance to the test equipment for each tubing size? (You may assume isentropic flow everywhere but in the 21 ft of tubing.)

9.23. (*Optional*) (a) Introduce the * reference condition into equation (9.27) and develop an expression for $(s^* - s)/R$.

(b) Write a computer program for the expression developed in part (a) and compute a table of $(s^* - s)/R$ versus Mach number. Also include other entries of the Fanno table.

CHECK TEST

You should be able to complete this test without reference to material in the chapter.

9.1. Sketch a Fanno line in the h–s plane. Include enough additional information as necessary to locate the sonic point and then identify the regions of subsonic and supersonic flow.

9.2. Fill in the blanks in Table CT9.2 to indicate whether the quantities *increase*, *decrease*, or *remain constant* in the case of Fanno flow.

Table CT9.2 Analysis of Fanno Flow

Property	Subsonic Regime	Supersonic Regime
Velocity		
Temperature		
Pressure		
Thrust function $(p + \rho V^2/g_c)$		

9.3. In the system shown in Figure CT9.3, the friction length of the duct is $f\,\Delta x/D = 12.40$ and the Mach number at the exit is 0.8. $A_3 = 1.5$ in^2 and $A_4 = 1.0$ in^2. What is the air pressure in the tank if the receiver is at 15 psia?

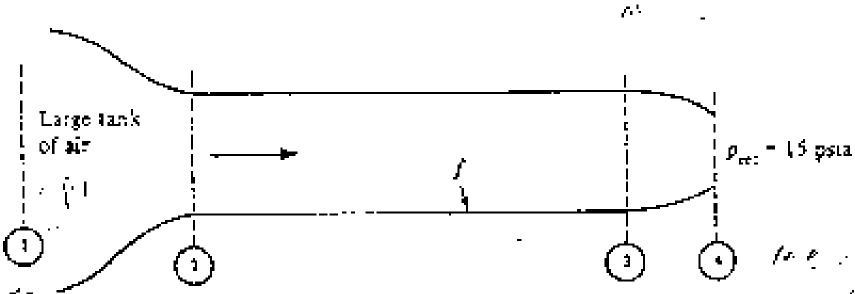

Figure CT9.3

9.4. Over what range of receiver pressures will normal shocks occur someplace within the system shown in Figure CT9.4? The area ratio of the nozzle is $A_3/A_2 = 2.403$ and the duct $f\,\Delta x/D = 0.30$.

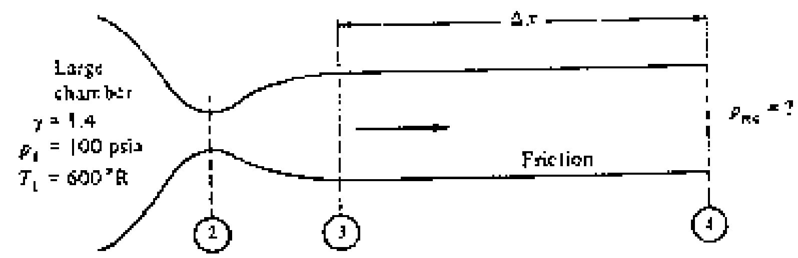

Figure CT9.4

9.5. There is no friction in the system shown in Figure CT9.5 except in the constant-area ducts from 3 to 4 and from 6 to 7. Sketch the T–s diagram for the entire system.

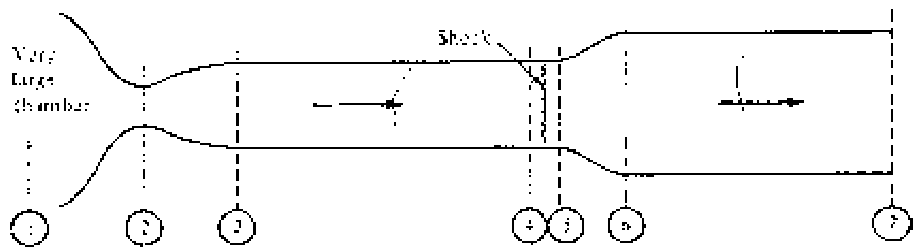

Figure CT9.5

9.6. Starting with the basic principles of continuity, energy, and so on, derive an expression for the property ratio p_{t2}/p_{t1} in terms of Mach numbers and the specific heat ratio for Fanno flow with a perfect gas.

9.7. Work Problem 9.16.