Block Diagram Reduction

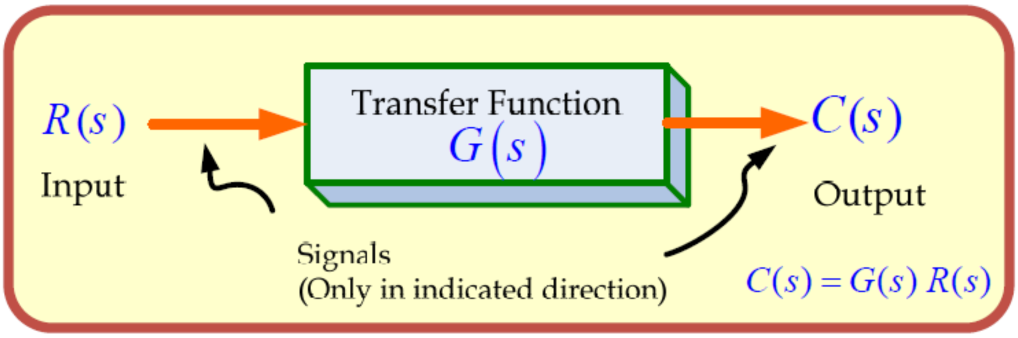


Figure 1: Single block diagram representation

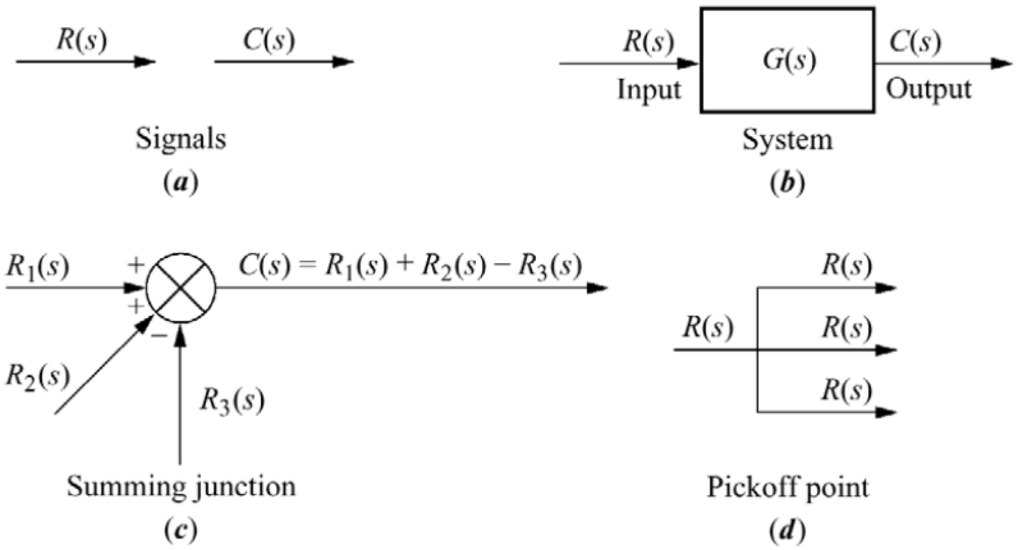


Figure 2: Components of Linear Time Invariant Systems (LTIS)

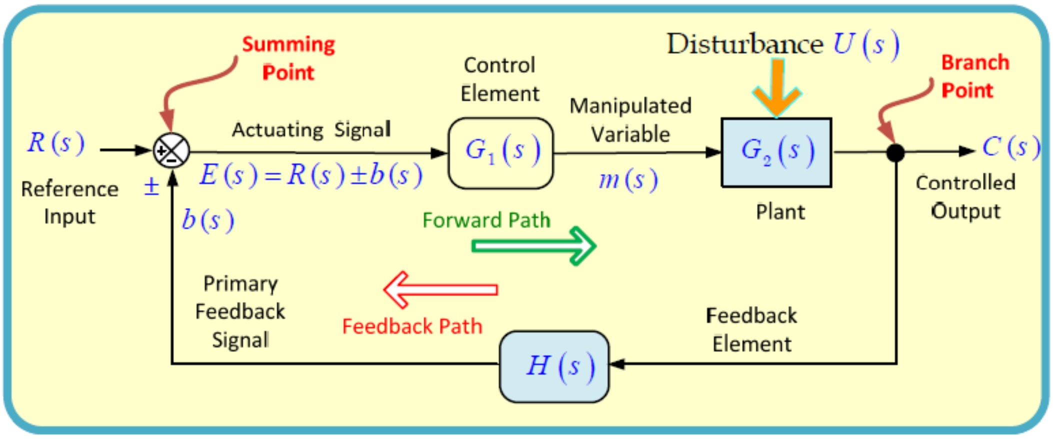


Figure 3: Block diagram components

Definitions

- $G(s)$ $\equiv$ **Direct transfer function = Forward transfer function.**
- $H(s)$ $\equiv$ **Feedback transfer function.**
- $G(s)H(s)$ $\equiv$ **Open-loop transfer function.**
- $C(s)/R(s)$ $\equiv$ Closed-loop transfer function = Control ratio
- $C(s)/E(s)$ $\equiv$ Feed-forward transfer function.

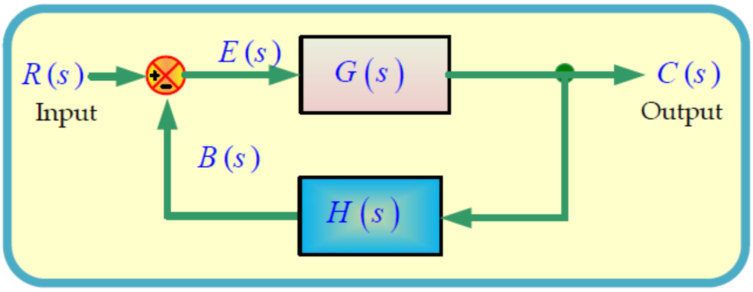


Figure 4: Block diagram of a closed-loop system with a feedback element

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

BLOCK DIAGRAM SIMPLIFICATIONS

Cascade (Series) Connections

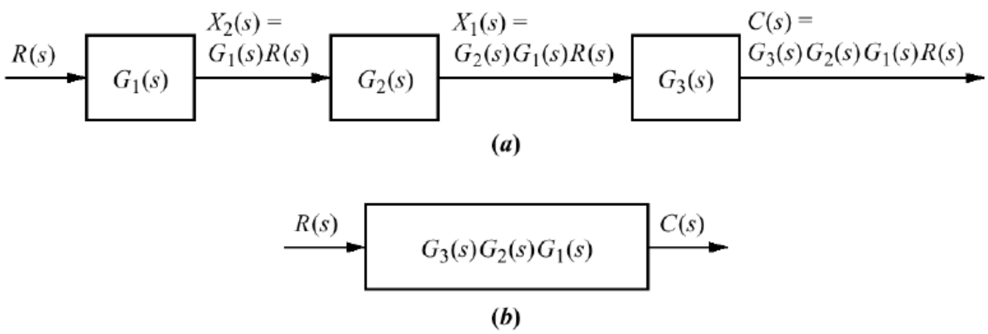


Figure 5: Cascade (Series) Connections

Parallel Connections

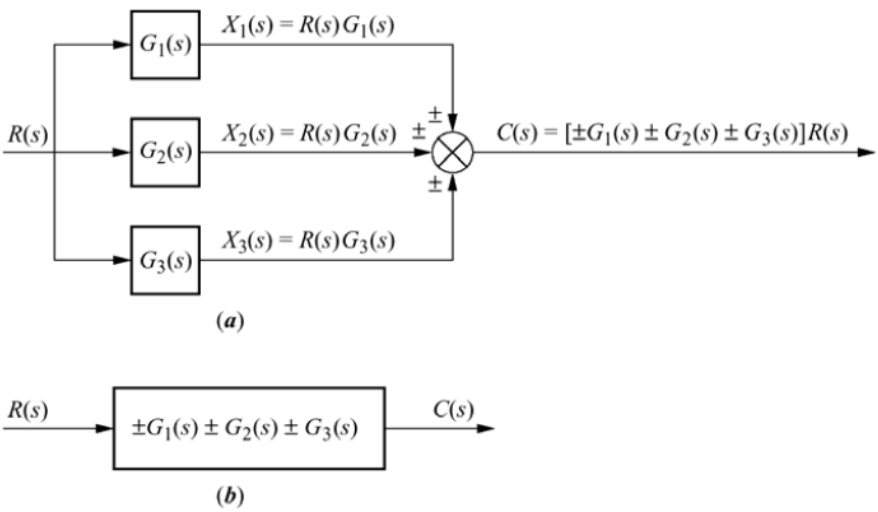


Figure 6: Parallel Connections

Block Diagram Algebra for Summing Junctions

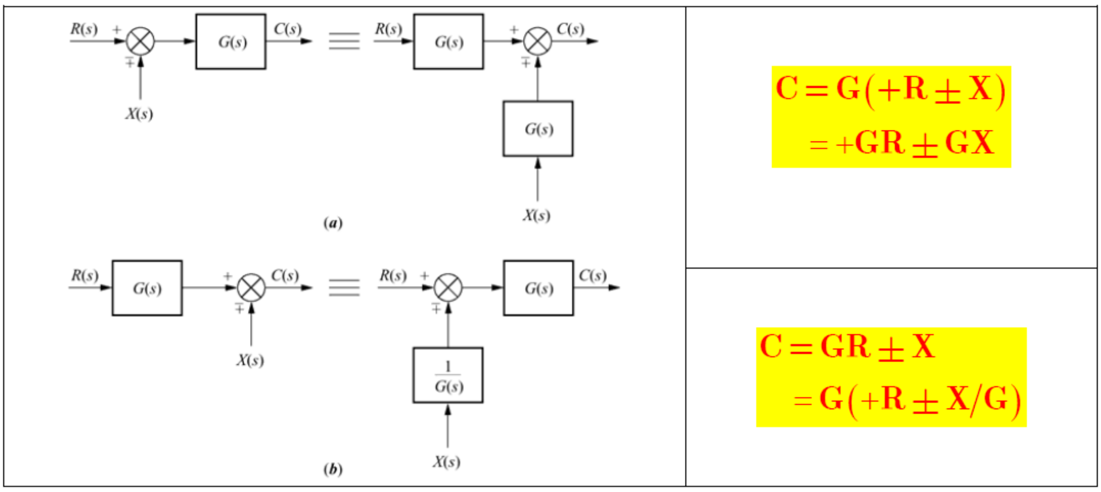


Figure 7: Summing Junctions

Block Diagram Algebra for Branch Point

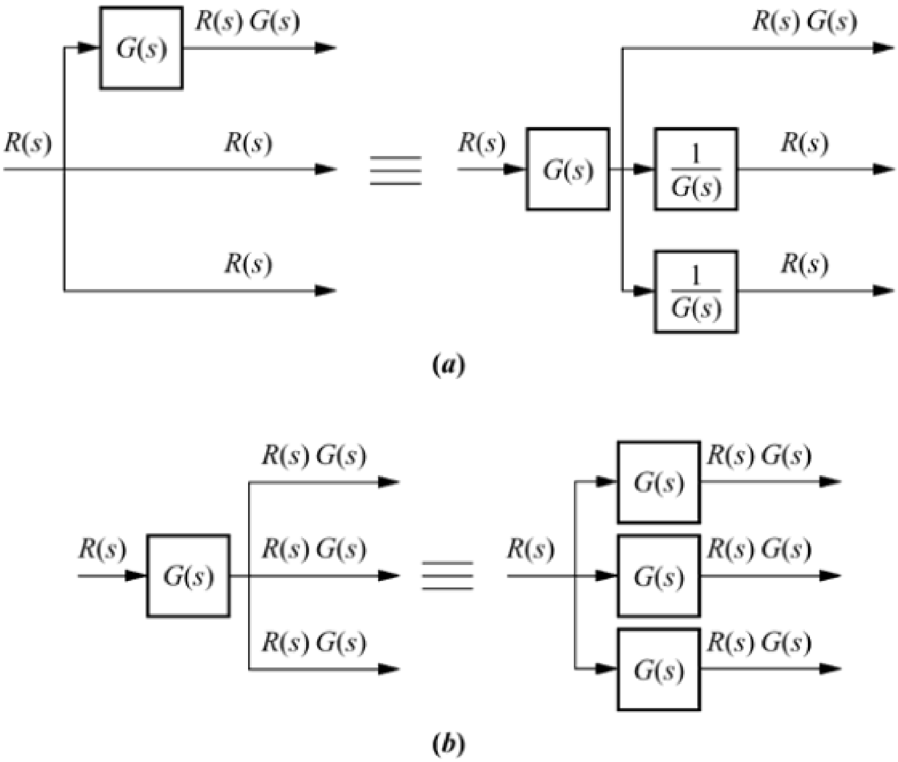


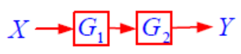
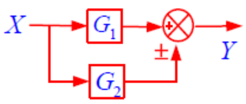
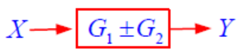
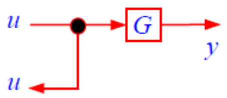
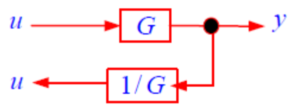
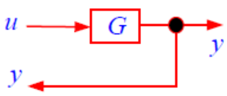
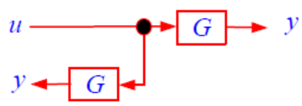
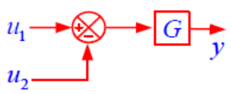
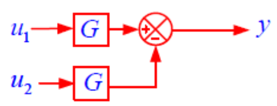
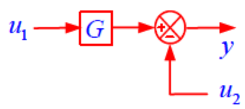
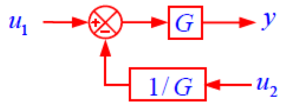
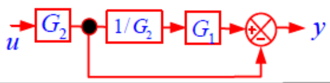
Figure 8: Branch Points

Block Diagram Reduction Rules

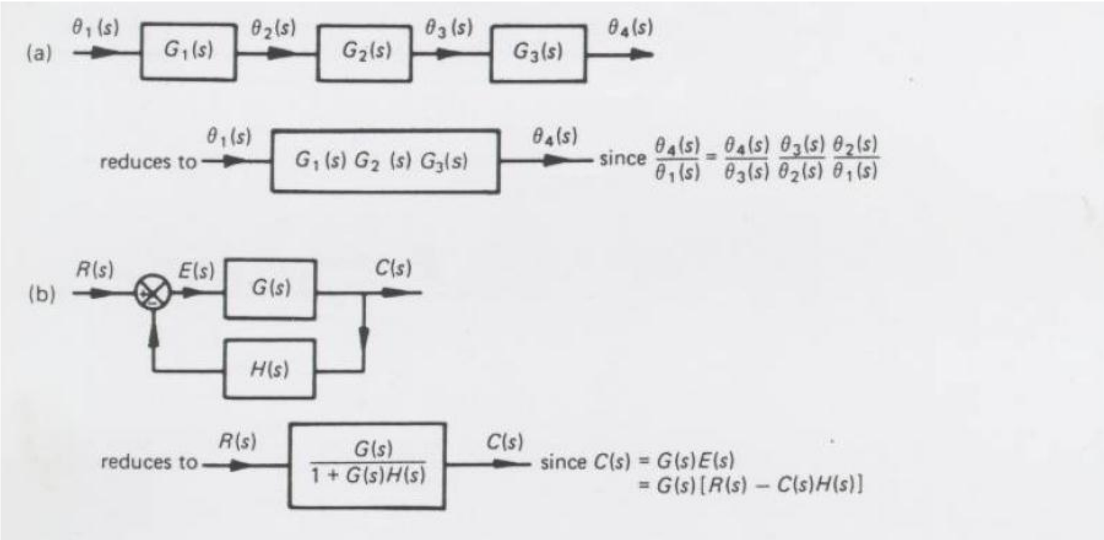
Table 1: Block Diagram Reduction Rules

1.	**Combine all cascade blocks**
2.	**Combine all parallel blocks**
3.	**Eliminate all minor (interior) feedback loops**
4.	**Shift summing points to left**
5.	**Shift takeoff points to the right**
6.	**Repeat Steps 1 to 5 until the canonical form is obtained**

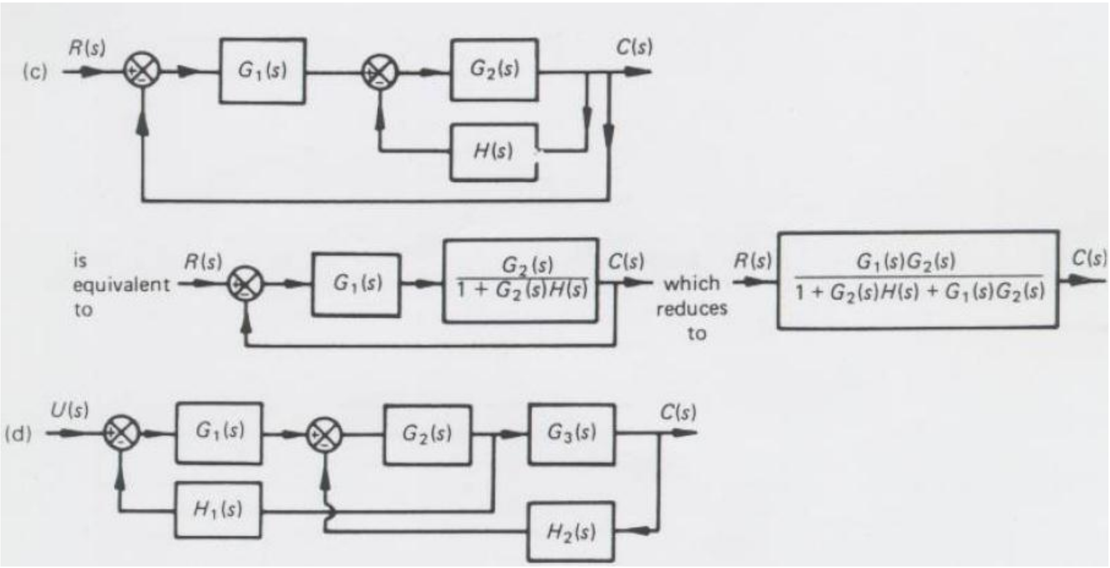
Table 2: Basic rules with block diagram transformation

	Manipulation	Original Block Diagram	Equivalent Block Diagram	Equation
1	Combining Blocks in Cascade			$Y=(G_1G_2)X$
2	Combining Blocks in Parallel; or Eliminating a Forward Loop			$Y=(G_1 \pm G_2)X$
3	Moving a pickoff point behind a block			$y = G u$ $u = \frac{1}{G} y$
4	Moving a pickoff point ahead of a block			$y = Gu$
5	Moving a summing point behind a block			$e_2 = G(u_1 - u_2)$
6	Moving a summing point ahead of a block		 	$y = Gu_1 - u_2$ $y=(G_1 - G_2)u$

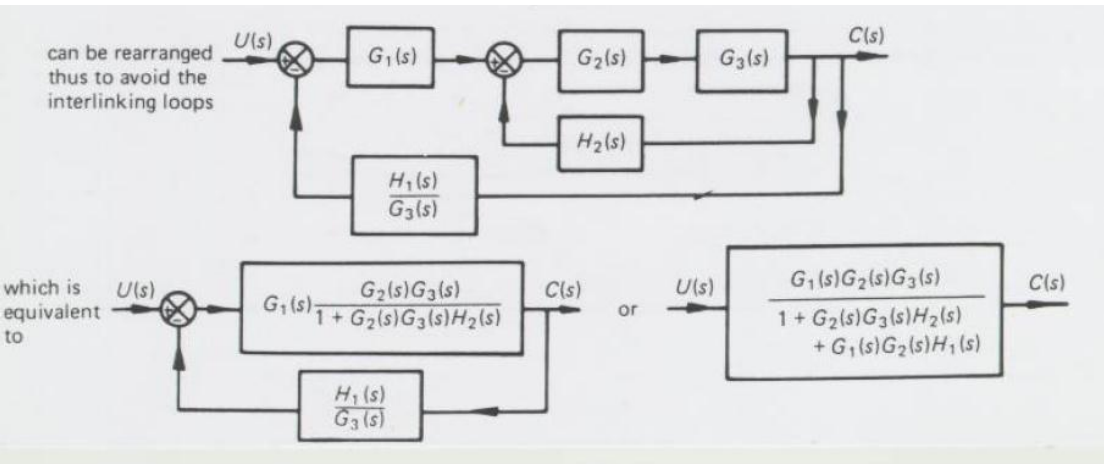
Example 1:



(a)
θ1(s)
G1(s)
θ2(s)
G2(s)
θ3(s)
G3(s)
θ4(s)
reduces to
G1(s) G2(s) G3(s)
since $\frac{\theta_4(s)}{\theta_1(s)} = \frac{\theta_4(s)}{\theta_3(s)}\frac{\theta_3(s)}{\theta_2(s)}\frac{\theta_2(s)}{\theta_1(s)}$
(b)
R(s)
E(s)
G(s)
C(s)
H(s)
reduces to
$\frac{G(s)}{1 + G(s)H(s)}$
since $C(s) = G(s)E(s)$
$= G(s)[R(s) - C(s)H(s)]$

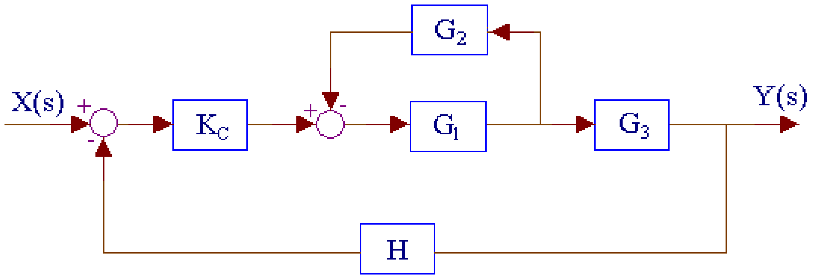


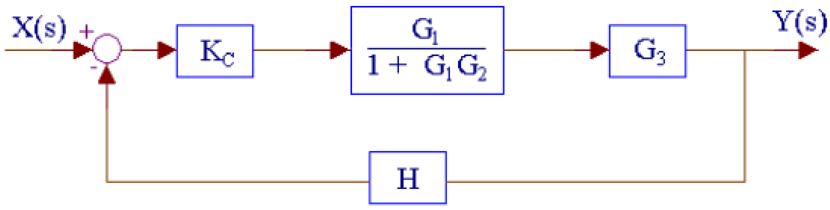
(c)
R(s)
G1(s)
G2(s)
C(s)
H(s)
is equivalent to
$\frac{G_2(s)}{1 + G_2(s)H(s)}$
which reduces to
$\frac{G_1(s)G_2(s)}{1 + G_2(s)H(s) + G_1(s)G_2(s)}$
(d)
U(s)
G1(s)
G2(s)
G3(s)
C(s)
H1(s)
H2(s)

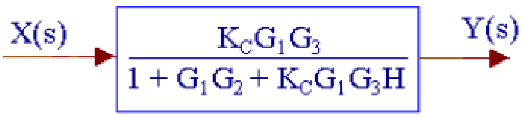


can be rearranged thus to avoid the interlinking loops
U(s)
G1(s)
G2(s)
G3(s)
C(s)
H2(s)
$\frac{H_1(s)}{G_3(s)}$
which is equivalent to
$G_1(s)\frac{G_2(s)G_3(s)}{1 + G_2(s)G_3(s)H_2(s)}$
or
$\frac{G_1(s)G_2(s)G_3(s)}{1 + G_2(s)G_3(s)H_2(s) + G_1(s)G_2(s)H_1(s)}$

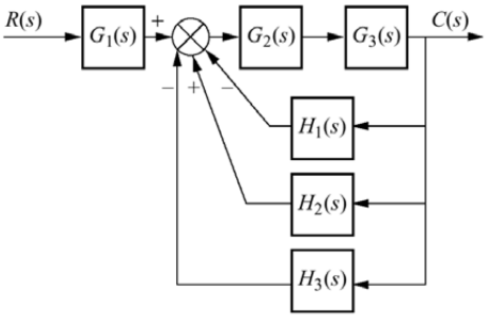
Example 2:



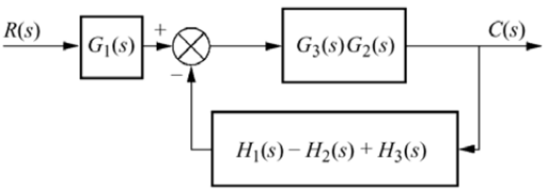




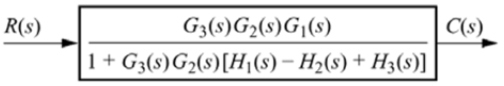
Example 3:



(*a*)



(*b*)



(*c*)

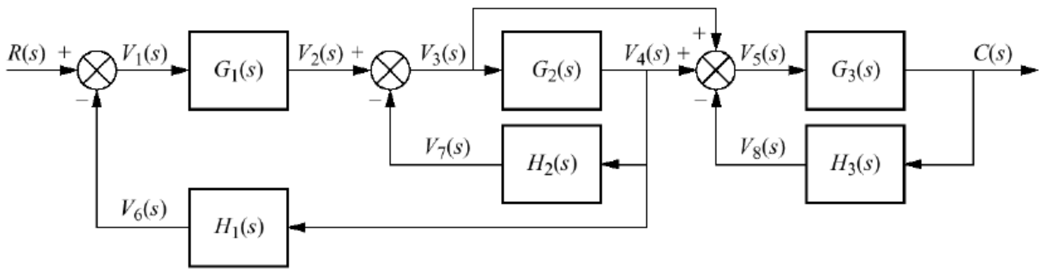
G_1 and G_2 are in series

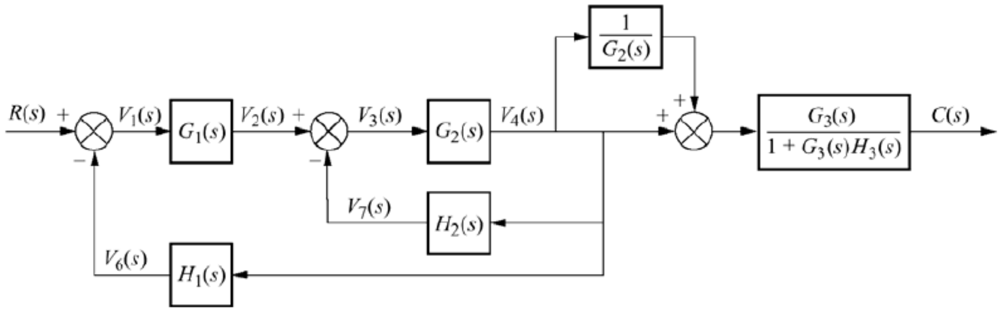
H_1 and H_2 and H_3 are in parallel

G_1 is in series with the feedback configuration.

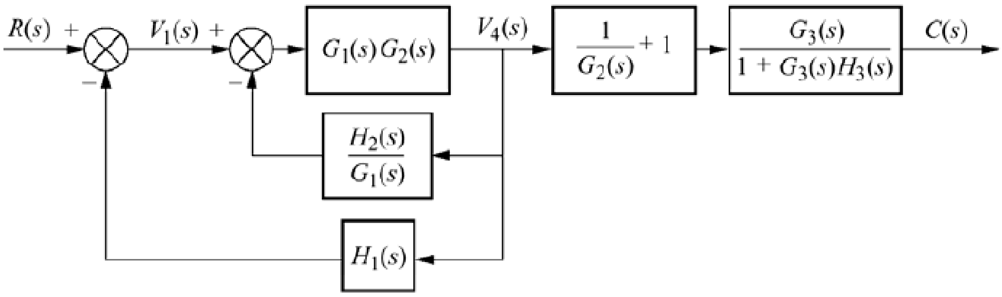
$$\frac{C(s)}{R(s)} = G_1\left[\frac{G_3G_2}{1+G_3G_2\left(H_1 - H_2 + H_3\right)}\right]$$

Example 4:

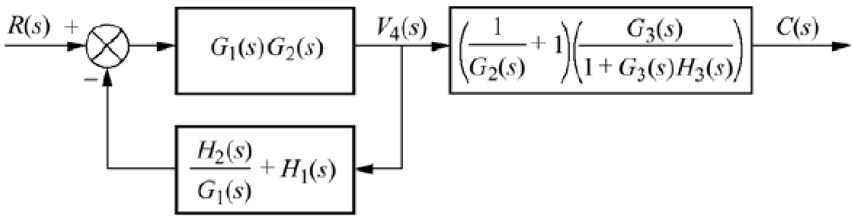




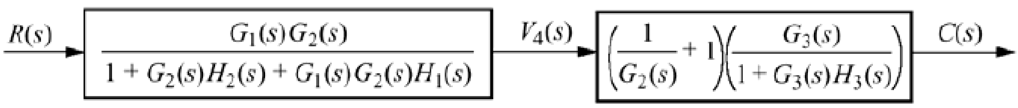
(*a*)



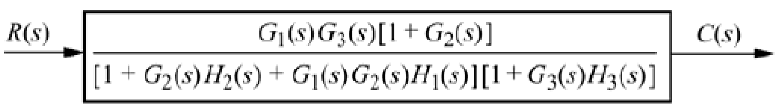
(*b*)



(*c*)

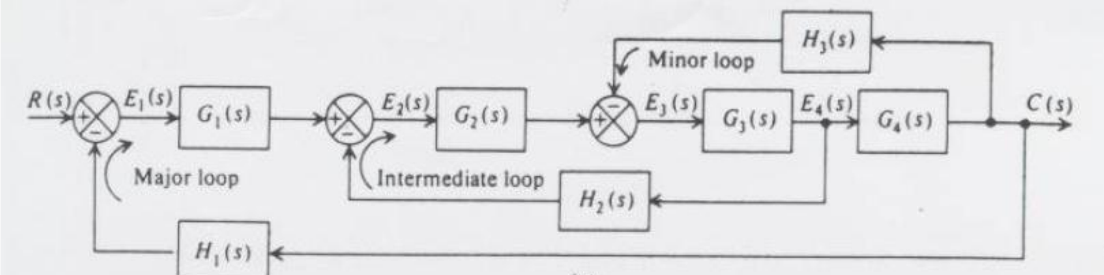


(*d*)

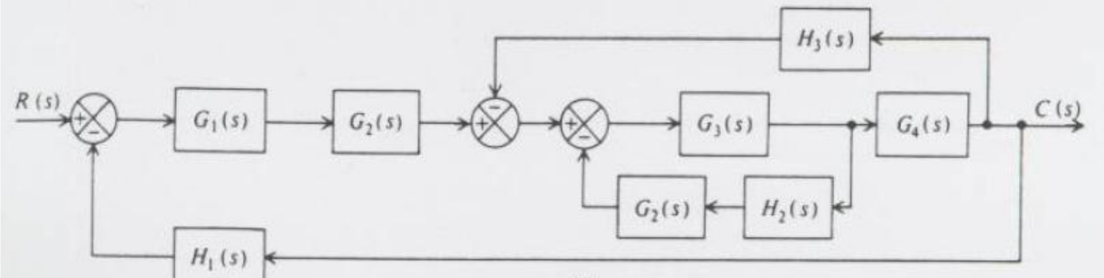


(*e*)

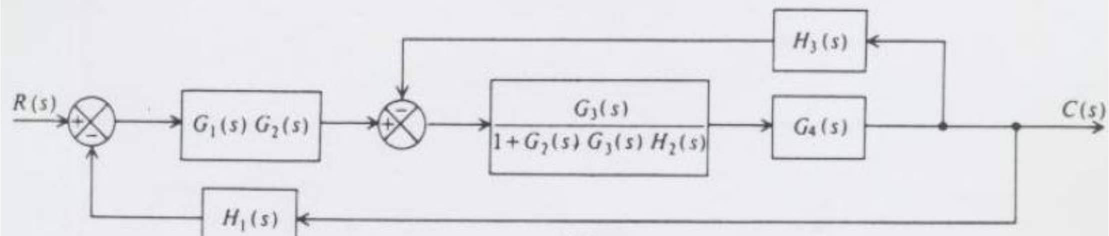
Example5:



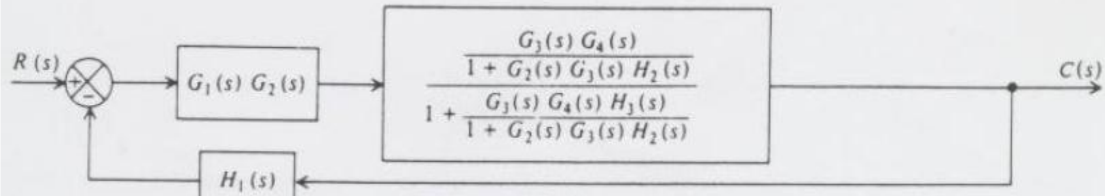
(a)



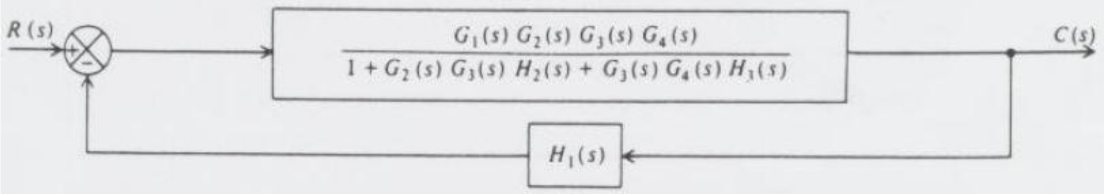
(b)



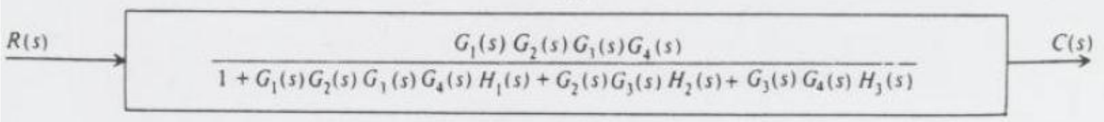
(c)



(d)



(e)



(f)

ROUTH'S STABILITY CRITERION

Consider a closed-loop transfer function

$$H(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} = \frac{B(s)}{A(s)} \tag{1}$$

where the a_i's and b_i's are real constants and $m \leq n$. An alternative to factoring the denominator polynomial, Routh's stability criterion, determines the number of closed-loop poles in the right-half s plane.

Algorithm for applying Routh's stability criterion

The algorithm described below, like the stability criterion, requires the order of $A(s)$ to be finite.

1. Factor out any roots at the origin to obtain the polynomial, and multiply by -1 if necessary, to obtain

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n = 0 \tag{2}$$

   where $a_0 \neq 0$ and $a_n > 0$.

2. If the order of the resulting polynomial is at least two and any coefficient a_i is zero or negative, the polynomial has at least one root with nonnegative real part. To obtain the precise number of roots with nonnegative real part, proceed as follows. Arrange the coefficients of the polynomial, and values subsequently calculated from them as shown below:

$$\begin{array}{llllll} s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\ s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\ s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\ s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\ s^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdots \\ \vdots & \vdots & \vdots & & & \\ s^2 & e_1 & e_2 & & & \\ s^1 & f_1 & & & & \\ s^0 & g_0 & & & & \end{array} \tag{3}$$

   where the coefficients b_i are

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \tag{4}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} \tag{5}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1} \tag{6}$$

$$\vdots$$

generated until all subsequent coefficients are zero. Similarly, cross multiply the coefficients of the two previous rows to obtain the c_i, d_i, etc.

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} \tag{7}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1} \tag{8}$$

$$c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1} \tag{9}$$

$$\vdots$$

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1} \tag{10}$$

$$d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1} \tag{11}$$

$$\vdots$$

until the nth row of the array has been completed[1] Missing coefficients are replaced by zeros. The resulting array is called the Routh array. The powers of s are not considered to be part of the array. We can think of them as labels. The column beginning with a_0 is considered to be the first column of the array.

The Routh array is seen to be triangular. It can be shown that multiplying a row by a positive number to simplify the calculation of the next row does not affect the outcome of the application of the Routh criterion.

3. Count the number of sign changes in the first column of the array. It can be shown that a necessary and sufficient condition for all roots of (2) to be located in the left-half plane is that all the a_i are positive and all of the coefficients in the first column be positive.

Example: Generic Quadratic Polynomial.

Consider the quadratic polynomial:

$$a_0 s^2 + a_1 s + a_2 = 0 \tag{12}$$

where all the a_i are positive. The array of coefficients becomes

$$\begin{array}{lll} s^2 & a_0 & a_2 \\ s^1 & a_1 & 0 \\ s^0 & a_2 & \end{array} \tag{13}$$

[1]There is one important detail that we have not yet mentioned. If an element of the first column becomes zero, we must alter the procedure. Since this altered procedure is requires some explanation, we postpone discussion of it to a pair of subsections below.

where the coefficient a_1 is the result of multiplying a_1 by a_2 and subtracting $a_0(0)$ then dividing the result by a_2. In the case of a second order polynomial, we see that Routh's stability criterion reduces to the condition that all a_i be positive.

Example: Generic Cubic Polynomial.

Consider the generic cubic polynomial:

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0 \tag{14}$$

where all the a_i are positive. The Routh array is

$$\begin{array}{lcc} s^3 & a_0 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & \frac{a_1 a_2 - a_0 a_3}{a_1} & \\ s^0 & a_3 & \end{array} \tag{15}$$

so the condition that all roots have negative real parts is

$$a_1 a_2 > a_0 a_3. \tag{16}$$

Example: A Quartic Polynomial.

Next we consider the fourth-order polynomial:

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0. \tag{17}$$

Here we illustrate the fact that multiplying a row by a positive constant does not change the result. One possible Routh array is given at left, and an alternative is given at right,

$$\begin{array}{lccc||lcccl} s^4 & 1 & 3 & 5 & s^4 & 1 & 3 & 5 & \\ s^3 & 2 & 4 & 0 & s^3 & \cancel{2} & \cancel{4} & \cancel{0} & \text{Divide this row by two to get} \\ & & & & & 1 & 2 & 0 & \\ s^2 & 1 & 5 & & s^2 & 1 & 5 & & \\ s^1 & -6 & & & s^1 & -3 & & & \\ s^0 & 5 & & & s^0 & 5 & & & \end{array}$$

In this example, the sign changes twice in the first column so the polynomial equation $A(s) = 0$ has two roots with positive real parts.

Necessity of all coefficients being positive.

In stating the algorithm above, we did not justify the stated conditions. Here we show that all coefficients being positive is necessary for all roots to be located in the left half-plane. It can be shown that any polynomial in s, all of whose coefficients are real, can

be factored into a product of a maximal number linear and quadratic factors also having real coefficients. Clearly a linear factor $(s + a)$ has nonnegative real root iff a is positive. For both roots of a quadratic factor $(s^2 + bs + c)$ to have negative real parts both b and c must be positive. (If c is negative, the square root of $b^2 - 4c$ is real and the quadratic factor can be factored into two linear factors so the number of factors was not maximal.) It is easy to see that if all coefficients of the factors are positive, those of the original polynomial must be as well. To see that the condition is not sufficient, we can refer to several examples above.

Example: Determining Acceptable Gain Values

So far we have discussed only one possible application of the Routh criterion, namely determining the number of roots with nonnegative real parts. In fact, it can be used to determine limits on design parameters, as shown below.

Consider a system whose closed-loop transfer function is

$$H(s) = \frac{K}{s(s^2 + s + 1)(s + 2) + K}. \tag{18}$$

The characteristic equation is

$$s^4 + 3s^3 + 3s^2 + 2s^4 + K = 0. \tag{19}$$

The Routh array is

$$\begin{array}{lccc} s^4 & 1 & 3 & K \\ s^3 & 3 & 2 & 0 \\ s^2 & 7/3 & K & \\ s^1 & 2 - 9K/7 & & \\ s^0 & K & & \end{array} \tag{20}$$

so the s^1 row yields the condition that, for stability,

$$14/9 > K > 0. \tag{21}$$

Special Case: Zero First-Column Element.

If the first term in a row is zero, but the remaining terms are not, the zero is replaced by a small, positive value of ϵ and the calculation continues as described above. Here's an example:

$$s^3 + 2s^2 + s + 2 = 0 \tag{22}$$

has Routh array

$$\begin{array}{lcc} s^3 & 1 & 1 \\ s^2 & 2 & 2 \\ s^1 & 0 \cong \epsilon & \\ s^0 & 2 & \end{array} \tag{23}$$

where the last element of the first column is equal $2 = (\epsilon 2 - 0)/\epsilon$. In counting changes of sign, the row beginning with ϵ is not counted.

If the elements above and below the ϵ in the first column have the same sign, a pair of imaginary roots is indicated. Here, for example, (22) has two roots at $s = \pm j$.

On the other hand, if the elements above and below the ϵ have opposite signs, this counts as a sign change. For example,

$$s^3 - 3s + 2 = (s^2 - 1)(s + 2) = 0 \tag{24}$$

has Routh array

$$\begin{array}{lcc} s^3 & 1 & -3 \\ s^2 & 0 \cong \epsilon & 2 \\ s^1 & -3 - 2/\epsilon & \\ s^0 & 2 & \end{array} \tag{25}$$

with two sign changes in the first column.

Special Case: Zero Row. If all the coefficients in a row are zero, a pair of roots of equal magnitude and opposite sign is indicated. These could be two real roots with equal magnitudes and opposite signs or two conjugate imaginary roots. The zero row is replaced by taking the coefficients of $dP(s)/ds$, where $P(s)$, called the auxiliary polynomial, is obtained from the values in the row above the zero row. The pair of roots can be found by solving $dP(s)/ds = 0$.

Note that the auxiliary polynomial always has even degree. It can be shown that an auxiliary polynomial of degree $2n$ has n pairs of roots of equal magnitude and opposite sign.

Example: Use of Auxiliary Polynomial

Consider the quintic equation $A(s) = 0$ where $A(s)$ is

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 50. \tag{26}$$

The Routh array starts off as

$$\begin{array}{llll} s^5 & 1 & 24 & -25 \\ s^4 & 2 & 48 & -50 \quad \longleftarrow \quad \text{auxiliary polynomial } P(s) \\ s^3 & 0 & 0 & \end{array} \tag{27}$$

The auxiliary polynomial $P(s)$ is

$$P(s) = 2s^4 + 48s^2 - 50 \tag{28}$$

which indicates that $A(s) = 0$ must have two pairs of roots of equal magnitude and opposite sign, which are also roots of the auxiliary polynomial equation $P(s) = 0$. Taking

the derivative of $P(s)$ with respect to s we obtain

$$\frac{dP(s)}{ds} = 8s^3 + 96s. \tag{29}$$

so the s^3 row is as shown below and the Routh array is

$$\begin{array}{lrrrl}
s^5 & 1 & 24 & -25 & \\
s^4 & 2 & 48 & -50 & \\
s^3 & 8 & 96 & & \longleftarrow \text{ Coefficients of } dP(s)/ds \\
s^2 & 24 & -50 & & \\
s^1 & 112.7 & 0 & & \\
s^0 & -50 & & &
\end{array} \tag{30}$$

There is a single change of sign in the first column of the resulting array, indicating that there $A(s) = 0$ has one root with positive real part. Solving the auxiliary polynomial equation,

$$2s^4 + 48s^2 - 50 = 0 \tag{31}$$

yields the remaining roots, namely, from

$$s^2 = 1, \quad s^2 = -25, \tag{32}$$

$$s = \pm 1, \quad s = \pm j5. \tag{33}$$

so the original equation can be factored as

$$(s + 1)(s - 1)(s + j5)(s - j5)(s + 2) = 0. \tag{34}$$

Relative stability analysis. *Routh's stability criterion provides the answer to the question of absolute stability. This, in many practical cases, is not sufficient. We usually require information about the relative stability of the system. A useful approach for examining relative stability is to shift the s-plane axis and apply Routh's stability criterion. Namely, we substitute $s = z - \sigma$ (σ = constant) into the characteristic equation of the system, write the polynomial in terms of z, and apply Routh's stability criterion to the new polynomial in z. The number of changes of sign in the first column of the array developed for the polynomial in z is equal to the number of roots which are located to the right of the vertical line $s = -\sigma$. Thus, this test reveals the number of roots which lie to the right of the vertical line $s = -\sigma$.* [2]

[2]This italicized text and most of the numerical examples are from Section 6-6 of Ogata, Katsuhiko, *Modern Control Engineering*, Englewood Cliffs, NJ: Prentice-Hall, 1970, pp. 252–258. The rest of the text, including the descriptions of the examples is mine.

Similarly, the program for the fourth-order transfer function approximation with $T = 0.1$ sec is

```
[num,den] = pade(0.1, 4);
printsys(num, den, 's')

  num/den =

  s^4 - 200s^3 + 18000s^2 - 840000s + 16800000
  ---------------------------------------------
  s^4 + 200s^3 + 18000s^2 + 840000s + 16800000
```

Notice that the pade approximation depends on the dead time T and the desired order for the approximating transfer function.

EXAMPLE PROBLEMS AND SOLUTIONS

A–6–1. Sketch the root loci for the system shown in Figure 6–39(a). (The gain K is assumed to be positive.) Observe that for small or large values of K the system is overdamped and for medium values of K it is underdamped.

Solution. The procedure for plotting the root loci is as follows:

1. Locate the open-loop poles and zeros on the complex plane. Root loci exist on the negative real axis between 0 and -1 and between -2 and -3.
2. The number of open-loop poles and that of finite zeros are the same. This means that there are no asymptotes in the complex region of the s plane.

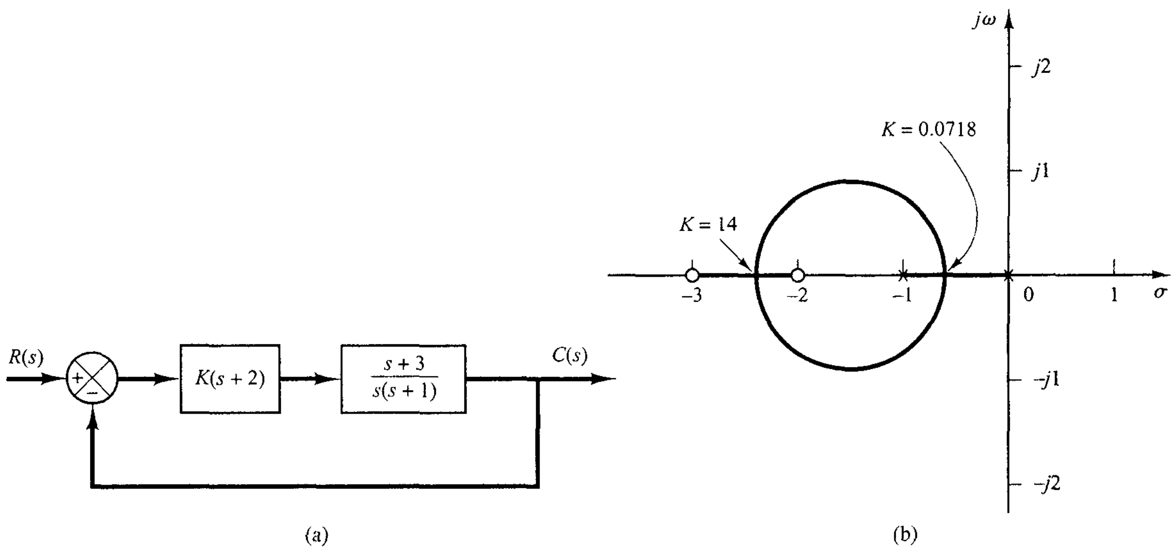


Figure 6–39
(a) Control system; (b) root-locus plot.

3. Determine the breakaway and break-in points. The characteristic equation for the system is

$$1 + \frac{K(s+2)(s+3)}{s(s+1)} = 0$$

or

$$K = -\frac{s(s+1)}{(s+2)(s+3)}$$

The breakaway and break-in points are determined from

$$\begin{aligned}\frac{dK}{ds} &= -\frac{(2s+1)(s+2)(s+3) - s(s+1)(2s+5)}{\left[(s+2)(s+3)\right]^2}\\ &= -\frac{4(s+0.634)(s+2.366)}{\left[(s+2)(s+3)\right]^2}\\ &= 0\end{aligned}$$

as follows:

$$s = -0.634, \qquad s = -2.366$$

Notice that both points are on root loci. Therefore, they are actual breakaway or break-in points. At point $s = -0.634$, the value of K is

$$K = -\frac{(-0.634)(0.366)}{(1.366)(2.366)} = 0.0718$$

Similarly, at $s = -2.366$,

$$K = -\frac{(-2.366)(-1.366)}{(-0.366)(0.634)} = 14$$

(Because point $s = -0.634$ lies between two poles, it is a breakaway point, and because point $s = -2.366$ lies between two zeros, it is a break-in point.)

4. Determine a sufficient number of points that satisfy the angle condition. (It can be found that the root loci involve a circle with center at -1.5 that passes through the breakaway and break-in points.) The root-locus plot for this system is shown in Figure 6–39(b).

Note that this system is stable for any positive value of K since all the root loci lie in the left-half s plane.

Small values of K ($0 < K < 0.0718$) correspond to an overdamped system. Medium values of K ($0.0718 < K < 14$) correspond to an underdamped system. Finally, large values of K ($14 < K$) correspond to an overdamped system. With a large value of K, the steady state can be reached in much shorter time than with a small value of K.

The value of K should be adjusted so that system performance is optimum according to a given performance index.

A–6–2. Sketch the root loci of the control system shown in Figure 6–40(a).

Solution. The open-loop poles are located at $s = 0$, $s = -3 + j4$, and $s = -3 - j4$. A root locus branch exists on the real axis between the origin and $-\infty$. There are three asymptotes for the root loci. The angles of asymptotes are

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k+1)}{3} = 60°, -60°, 180°$$

Referring to Equation (6–13), the intersection of the asymptotes and the real axis is obtained as

$$s = -\frac{0+3+3}{3} = -2$$

Next we check the breakaway and break-in points. For this system we have

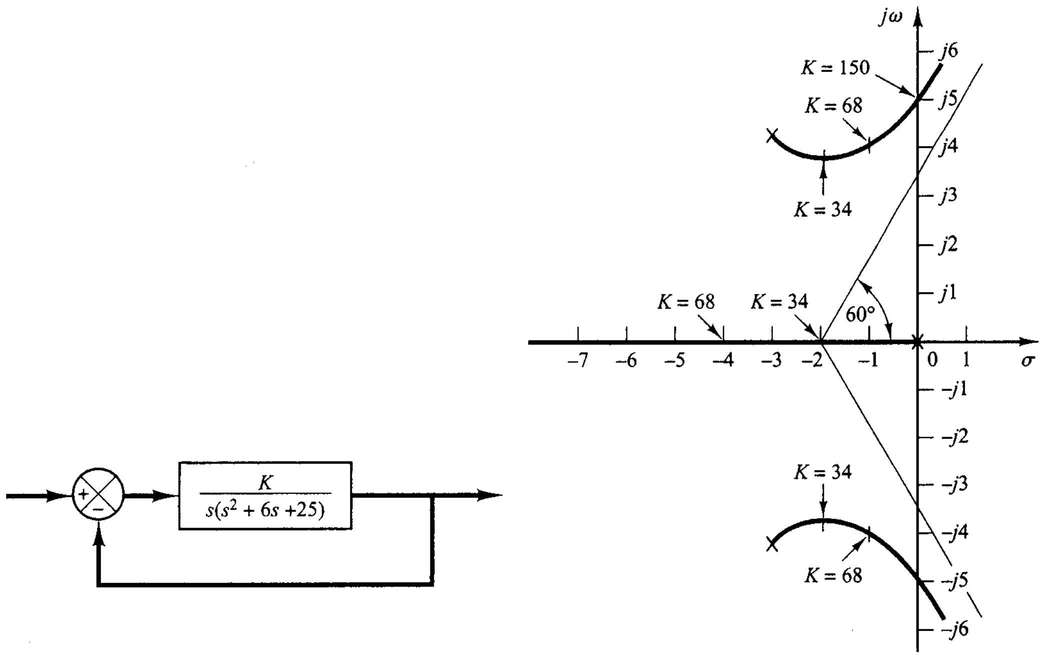
$$K = -s\left(s^2 + 6s + 25\right)$$

Now we set

$$\frac{dK}{ds} = -\left(3s^2 + 12s + 25\right) = 0$$

which yields

$$s = -2 + j2.0817, \qquad s = -2 - j2.0817$$



(a) (b)

Figure 6–40
(a) Control system; (b) root-locus plot.

Notice that at points $s = -2 \pm j2.0817$ the angle condition is not satisfied. Hence, they are neither breakaway nor break-in points. In fact, if we calculate the value of K, we obtain

$$K = -s(s^2 + 6s + 25)\Big|_{s=-2\pm j2.0817} = 34 \pm j18.04$$

(To be an actual breakaway or break-in point, the corresponding value of K must be real and positive.)

The angle of departure from the complex pole in the upper half s plane is

$$\theta = 180° - 126.87° - 90°$$

or

$$\theta = -36.87°$$

The points where root-locus branches cross the imaginary axis may be found by substituting $s = j\omega$ into the characteristic equation and solving the equation for ω and K as follows: Noting that the characteristic equation is

$$s^3 + 6s^2 + 25s + K = 0$$

we have

$$(j\omega)^3 + 6(j\omega)^2 + 25(j\omega) + K = (-6\omega^2 + K) + j\omega(25 - \omega^2) = 0$$

which yields

$$\omega = \pm 5, \qquad K = 150 \qquad \text{or} \qquad \omega = 0, \qquad K = 0$$

Root-locus branches cross the imaginary axis at $\omega = 5$ and $\omega = -5$. The value of gain K at the crossing points is 150. Also, the root-locus branch on the real axis touches the imaginary axis at $\omega = 0$. Figure 6–40(b) shows a root-locus plot for the system.

It is noted that if the order of the numerator of $G(s)H(s)$ is lower than that of the denominator by two or more, and if some of the closed-loop poles move on the root locus toward the right as gain K is increased, then other closed-loop poles must move toward the left as gain K is increased. This fact can be seen clearly in this problem. If the gain K is increased from $K = 34$ to $K = 68$, the complex-conjugate closed-loop poles are moved from $s = -2 + j3.65$ to $s = -1 + j4$; the third pole is moved from $s = -2$ (which corresponds to $K = 34$) to $s = -4$ (which corresponds to $K = 68$). Thus, the movements of two complex-conjugate closed-loop poles to the right by one unit cause the remaining closed-loop pole (real pole in this case) to move to the left by two units.

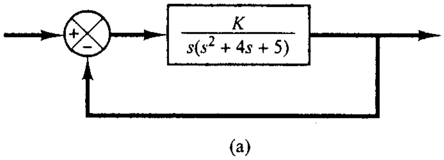
A–6–3. Consider the system shown in Figure 6–41(a). Sketch the root loci for the system. Observe that for small or large values of K the system is underdamped and for medium values of K it is overdamped.

Solution. A root locus exists on the real axis between the origin and $-\infty$. The angles of asymptotes of the root-locus branches are obtained as

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k + 1)}{3} = 60°, -60°, -180°$$

The intersection of the asymptotes and the real axis is located on the real axis at

$$s = -\frac{0 + 2 + 2}{3} = -1.3333$$



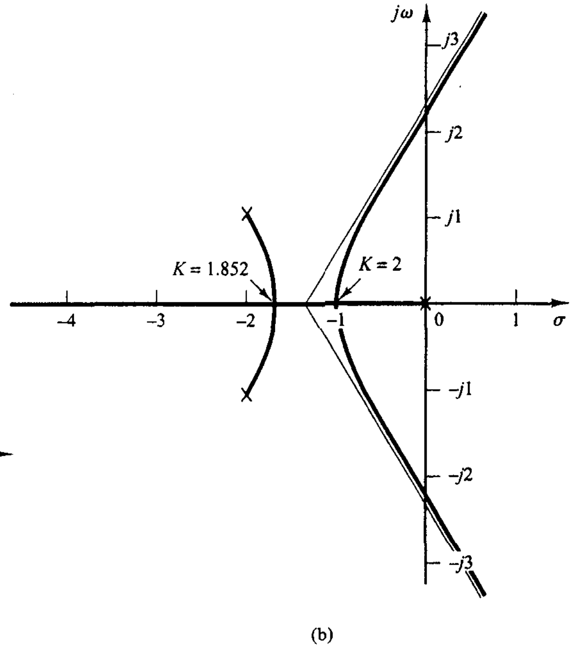


Figure 6–41
(a) Control system;
(b) root-locus plot.

The breakaway and break-in points are found from $dK/ds = 0$. Since the characteristic equation is

$$s^3 + 4s^2 + 5s + K = 0$$

we have

$$K = -(s^3 + 4s^2 + 5s)$$

Now we set

$$\frac{dK}{ds} = -(3s^2 + 8s + 5) = 0$$

which yields

$$s = -1, \qquad s = -1.6667$$

Since these points are on root loci, they are actual breakaway or break-in points. (At point $s = -1$, the value of K is 2, and at point $s = -1.6667$, the value of K is 1.852.)

The angle of departure from a complex pole in the upper half s plane is obtained from

$$\theta = 180° - 153.43° - 90°$$

or

$$\theta = -63.43°$$

The root-locus branch from the complex pole in the upper half s plane breaks into the real axis at $s = -1.6667$.

Next we determine the points where root-locus branches cross the imaginary axis. By substituting $s = j\omega$ into the characteristic equation, we have

$$(j\omega)^3 + 4(j\omega)^2 + 5(j\omega) + K = 0$$

or

$$(K - 4\omega^2) + j\omega(5 - \omega^2) = 0$$

from which we obtain

$$\omega = \pm\sqrt{5}, \qquad K = 20 \qquad \text{or} \qquad \omega = 0, \qquad K = 0$$

Root-locus branches cross the imaginary axis at $\omega = \sqrt{5}$ and $\omega = -\sqrt{5}$. The root-locus branch on the real axis touches the $j\omega$ axis at $\omega = 0$. A sketch of the root loci for the system is shown in Figure 6–41(b).

Note that since this system is of third order, there are three closed-loop poles. The nature of the system response to a given input depends on the locations of the closed-loop poles.

For $0 < K < 1.852$, there are a set of complex-conjugate closed-loop poles and a real closed-loop pole. For $1.852 \leq K \leq 2$, there are three real closed-loop poles. For example, the closed-loop poles are located at

$$s = -1.667, \quad s = -1.667, \quad s = -0.667, \quad \text{for } K = 1.852$$

$$s = -1, \quad s = -1, \quad s = -2, \quad \text{for } K = 2$$

For $2 < K$, there are a set of complex-conjugate closed-loop poles and a real closed-loop pole. Thus, small values of K $(0 < K < 1.852)$ correspond to an underdamped system. (Since the real closed-loop pole dominates, only a small ripple may show up in the transient response.) Medium values of K $(1.852 \leq K \leq 2)$ correspond to an overdamped system. Large values of K $(2 < K)$ correspond to an underdamped system. With a large value of K, the system responds much faster than with a smaller value of K.

A–6–4. Sketch the root loci for the system shown in Figure 6–42(a).

Solution. The open-loop poles are located at $s = 0, s = -1, s = -2 + j3$, and $s = -2 - j3$. A root locus exists on the real axis between points $s = 0$ and $s = -1$. The angles of the asymptotes are found as follows:

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k + 1)}{4} = 45°, -45°, 135°, -135°$$

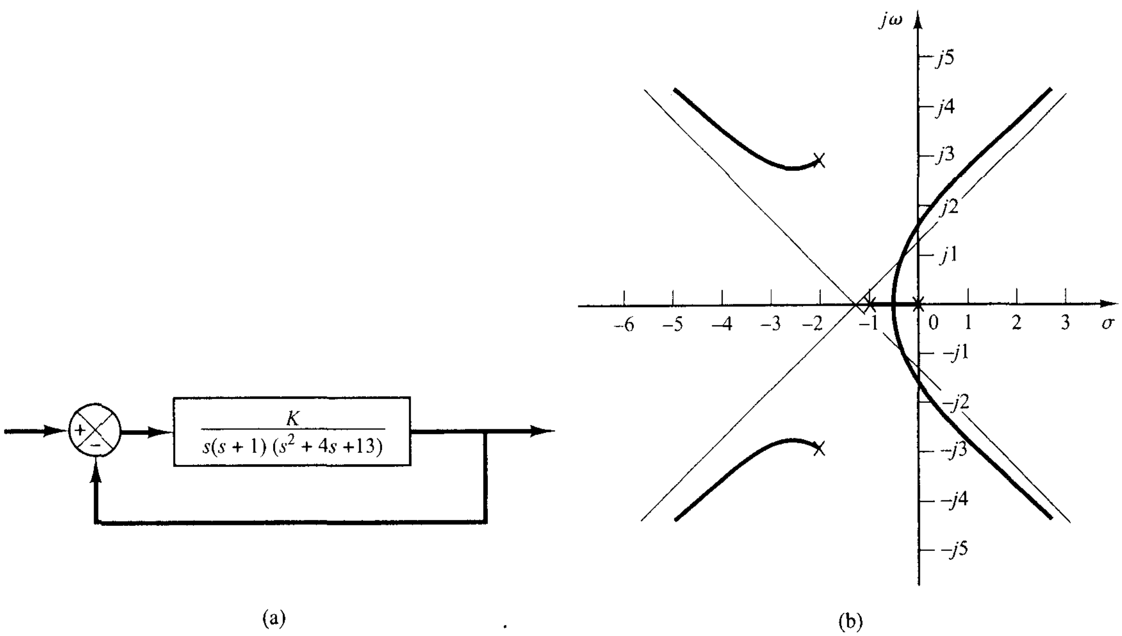


Figure 6–42
(a) Control system; (b) root-locus plot.

The intersection of the asymptotes and the real axis is found from

$$s = -\frac{0 + 1 + 2 + 2}{4} = -1.25$$

The breakaway and break-in points are found from $dK/ds = 0$. Noting that

$$K = -s(s + 1)(s^2 + 4s + 13) = -(s^4 + 5s^3 + 17s^2 + 13s)$$

we have

$$\frac{dK}{ds} = -(4s^3 + 15s^2 + 34s + 13) = 0$$

from which we get

$$s = -0.467, \qquad s = -1.642 + j2.067, \qquad s = -1.642 - j2.067$$

Point $s = -0.467$ is on a root locus. Therefore, it is an actual breakaway point. The gain values K corresponding to points $s = -1.642 \pm j2.067$ are complex quantities. Since the gain values are not real positive, these points are neither breakaway nor break-in points.

The angle of departure from the complex pole in the upper half s plane is

$$\theta = 180° - 123.69° - 108.44° - 90°$$

or

$$\theta = -142.13°$$

Next we shall find the points where root loci may cross the $j\omega$ axis. Since the characteristic equation is

$$s^4 + 5s^3 + 17s^2 + 13s + K = 0$$

by substituting $s = j\omega$ into it we obtain

$$(j\omega)^4 + 5(j\omega)^3 + 17(j\omega)^2 + 13(j\omega) + K = 0$$

or

$$(K + \omega^4 - 17\omega^2) + j\omega(13 - 5\omega^2) = 0$$

from which we obtain

$$\omega = \pm\, 1.6125, \qquad K = 37.44 \qquad \text{or} \qquad \omega = 0, \qquad K = 0$$

The root-locus branches that extend to the right-half s plane cross the imaginary axis at $\omega = \pm 1.6125$. Also, the root-locus branch on the real axis touches the imaginary axis at $\omega = 0$. Figure 6–42(b) shows a sketch of the root loci for the system. Notice that each root-locus branch that extends to the right half s plane crosses its own asymptote.

A–6–5. Sketch the root loci for the system shown in Figure 6–43(a).

Solution. A root locus exists on the real axis between points $s = -1$ and $s = -3.6$. The asymptotes can be determined as follows:

$$\text{Angles of asymptotes} = \frac{\pm 180^\circ(2k+1)}{3-1} = 90^\circ, -90^\circ$$

The intersection of the asymptotes and the real axis is found from

$$s = -\frac{0 + 0 + 3.6 - 1}{3 - 1} = -1.3$$

Since the characteristic equation is

$$s^3 + 3.6s^2 + K(s+1) = 0$$

we have

$$K = -\frac{s^3 + 3.6s^2}{s+1}$$

The breakaway and break-in points are found from

$$\frac{dK}{ds} = -\frac{(3s^2 + 7.2s)(s+1) - (s^3 + 3.6s^2)}{(s+1)^2} = 0$$

or

$$s^3 + 3.3s^2 + 3.6s = 0$$

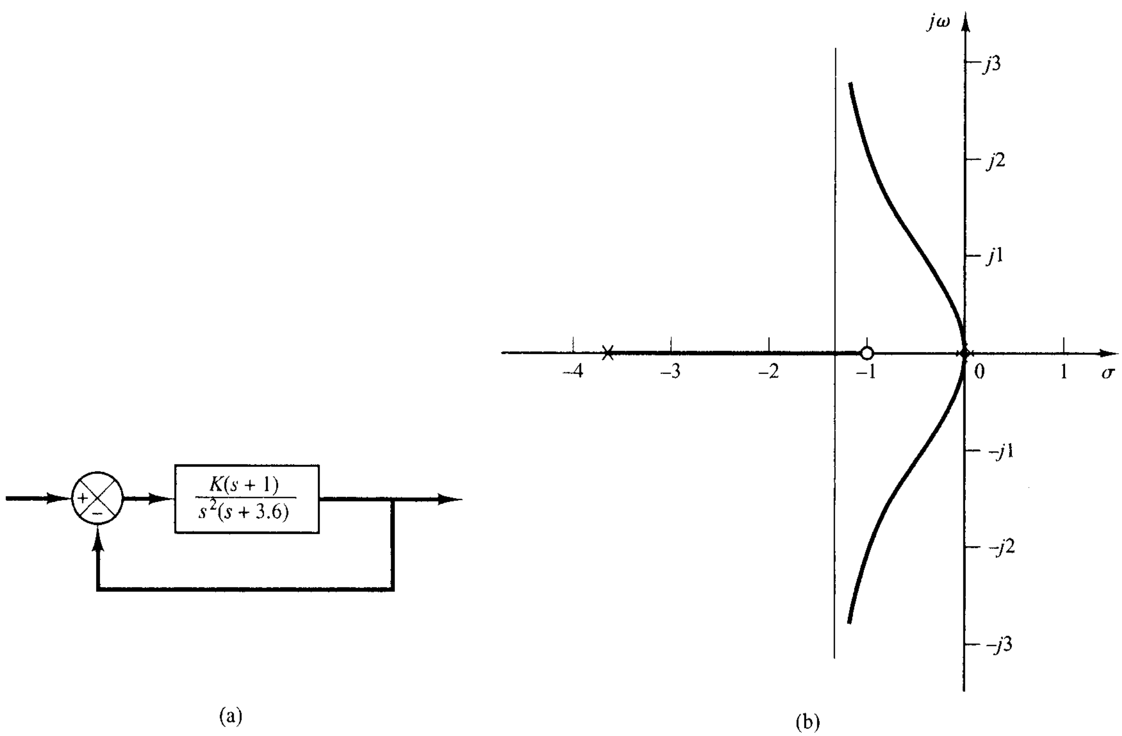


Figure 6–43
(a) Control system; (b) root-locus plot.

from which we get

$$s = 0, \qquad s = -1.65 + j0.9367, \qquad s = -1.65 - j0.9367$$

Point $s = 0$ corresponds to the actual breakaway point. But points $s = 1.65 \pm j0.9367$ are neither breakaway nor break-in points, because the corresponding gain values K become complex quantities.

To check the points where root-locus branches may cross the imaginary axis, substitute $s = j\omega$ into the characteristic equation, yielding.

$$(j\omega)^3 + 3.6(j\omega)^2 + Kj\omega + K = 0$$

or

$$\left(K - 3.6\omega^2\right) + j\omega\left(K - \omega^2\right) = 0$$

Notice that this equation can be satisfied only if $\omega = 0, K = 0$. Because of the presence of a double pole at the origin, the root locus is tangent to the $j\omega$ axis at $\omega = 0$. The root-locus branches do not cross the $j\omega$ axis. Figure 6–43(b) is a sketch of the root loci for this system.

A–6–6. Sketch the root loci for the system shown in Figure 6–44(a).

Solution. A root locus exists on the real axis between point $s = -0.4$ and $s = -3.6$. The angles of asymptotes can be found as follows:

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k + 1)}{3 - 1} = 90°, -90°$$

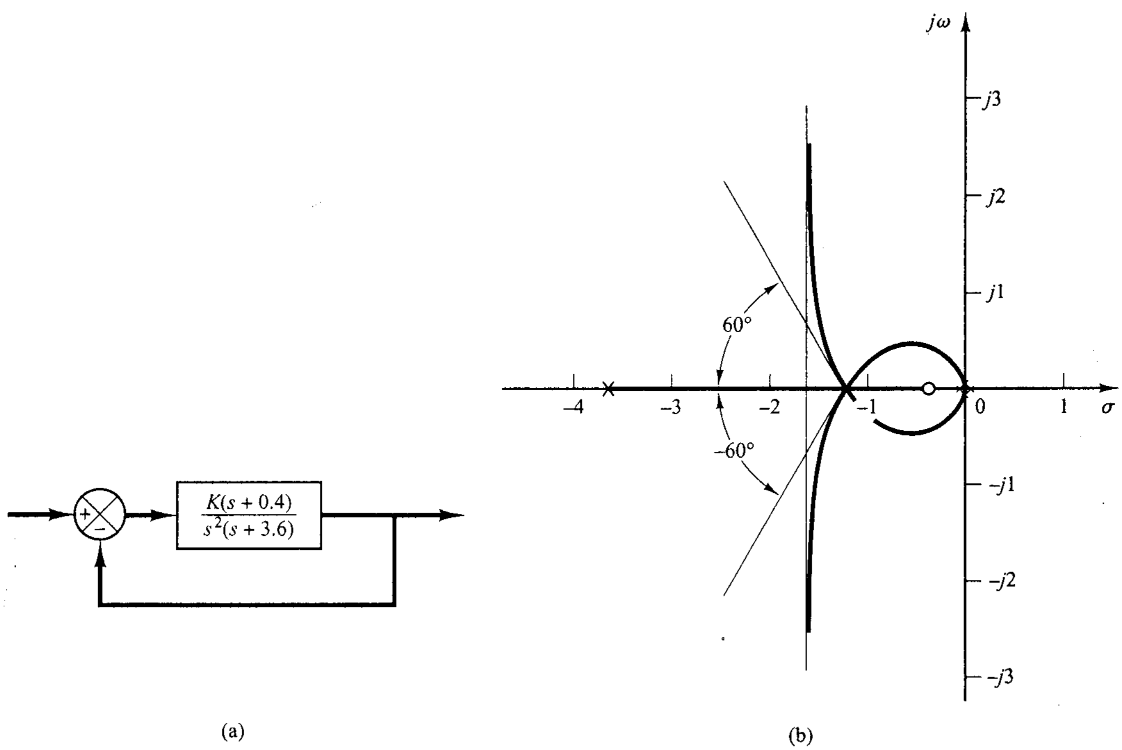


Figure 6–44
(a) Control system; (b) root-locus plot.

The intersection of the asymptotes and the real axis is obtained from

$$s = -\frac{0 + 0 + 3.6 - 0.4}{3 - 1} = -1.6$$

Next we shall find the breakaway points. Since the characteristic equation is

$$s^3 + 3.6s^2 + Ks + 0.4K = 0$$

we have

$$K = -\frac{s^3 + 3.6s^2}{s + 0.4}$$

The breakaway and break-in points are found from

$$\frac{dK}{ds} = -\frac{(3s^2 + 7.2s)(s + 0.4) - (s^3 + 3.6s^2)}{(s + 0.4)^2} = 0$$

from which we get

$$s^3 + 2.4s^2 + 1.44s = 0$$

or

$$s(s + 1.2)^2 = 0$$

Thus, the breakaway or break-in points are at $s = 0$ and $s = -1.2$. Note that $s = -1.2$ is a double root. When a double root occurs in $dK/ds = 0$ at point $s = -1.2$, $d^2K/(ds^2) = 0$ at this point. The value of gain K at point $s = -1.2$ is

$$K = -\left.\frac{s^3 + 3.6s^2}{s + 4}\right|_{s=-1.2} = 4.32$$

This means that with $K = 4.32$ the characteristic equation has a triple root at point $s = -1.2$. This can be easily verified as follows:

$$s^3 + 3.6s^2 + 4.32s + 1.728 = (s + 1.2)^3 = 0$$

Hence, three root-locus branches meet at point $s = -1.2$. The angles of departures at point $s = -1.2$ of the root locus branches that approach the asymptotes are $\pm 180°/3$, that is, 60° and −60°. (See Problem **A–6–7**.)

Finally, we shall examine if root-locus branches cross the imaginary axis. By substituting $s = j\omega$ into the characteristic equation, we have

$$(j\omega)^3 + 3.6(j\omega)^2 + K(j\omega) + 0.4K = 0$$

or

$$(0.4K - 3.6\omega^2) + j\omega(K - \omega^2) = 0$$

This equation can be satisfied only if $\omega = 0$, $K = 0$. At point $\omega = 0$, the root locus is tangent to the $j\omega$ axis because of the presence of a double pole at the origin. There are no points that root-locus branches cross the imaginary axis.

A sketch of the root loci for this system is shown in Figure 6–44(b).

A–6–7. Referring to Problem **A–6–6**, obtain the equations for the root-locus branches for the system shown in Figure 6–44(a). Show that the root-locus branches cross the real axis at the breakaway point at angles $\pm 60°$.

Solution. The equations for the root-locus branches can be obtained from the angle condition

$$\angle \frac{K(s+0.4)}{s^2(s+3.6)} = \pm 180°(2k+1)$$

which can be rewritten as

$$\angle s+0.4 - 2\angle s - \angle s+3.6 = \pm 180°(2k+1)$$

By substituting $s = \sigma + j\omega$, we obtain

$$\angle \sigma + j\omega + 0.4 - 2\angle \sigma + j\omega - \angle \sigma + j\omega + 3.6 = \pm 180°(2k+1)$$

or

$$\tan^{-1}\left(\frac{\omega}{\sigma+0.4}\right) - 2\tan^{-1}\left(\frac{\omega}{\sigma}\right) - \tan^{-1}\left(\frac{\omega}{\sigma+3.6}\right) = \pm 180°(2k+1)$$

By rearranging, we have

$$\tan^{-1}\left(\frac{\omega}{\sigma+0.4}\right) - \tan^{-1}\left(\frac{\omega}{\sigma}\right) = \tan^{-1}\left(\frac{\omega}{\sigma}\right) + \tan^{-1}\left(\frac{\omega}{\sigma+3.6}\right) \pm 180°(2k+1)$$

Taking tangents of both sides of this last equation, and noting that

$$\tan\left[\tan^{-1}\left(\frac{\omega}{\sigma+3.6}\right) \pm 180°(2k+1)\right] = \frac{\omega}{\sigma+3.6}$$

we obtain

$$\frac{\dfrac{\omega}{\sigma+0.4} - \dfrac{\omega}{\sigma}}{1 + \dfrac{\omega}{\sigma+0.4}\dfrac{\omega}{\sigma}} = \frac{\dfrac{\omega}{\sigma} + \dfrac{\omega}{\sigma+3.6}}{1 - \dfrac{\omega}{\sigma}\dfrac{\omega}{\sigma+3.6}}$$

which can be simplified to

$$\frac{\omega\sigma - \omega(\sigma+0.4)}{(\sigma+0.4)\sigma + \omega^2} = \frac{\omega(\sigma+3.6) + \omega\sigma}{\sigma(\sigma+3.6) - \omega^2}$$

or

$$\omega(\sigma^3 + 2.4\sigma^2 + 1.44\sigma + 1.6\omega^2 + \sigma\omega^2) = 0$$

which can be further simplified to

$$\omega\left[\sigma(\sigma+1.2)^2 + (\sigma+1.6)\omega^2\right] = 0$$

For $\sigma \neq -1.6$, we may write this last equation as

$$\omega\left[\omega - (\sigma+1.2)\sqrt{\frac{-\sigma}{\sigma+1.6}}\right]\left[\omega + (\sigma+1.2)\sqrt{\frac{-\sigma}{\sigma+1.6}}\right] = 0$$

which gives the equations for the root-locus as follows:

$$\omega = 0$$

$$\omega = (\sigma + 1.2)\sqrt{\frac{-\sigma}{\sigma + 1.6}}$$

$$\omega = -(\sigma + 1.2)\sqrt{\frac{-\sigma}{\sigma + 1.6}}$$

The equation $\omega = 0$ represents the real axis. The root locus for $0 \leq K \leq \infty$ is between points $s = -0.4$ and $s = -3.6$. (The real axis other than this line segment and the origin $s = 0$ corresponds to the root locus for $-\infty \leq K < 0$.)

The equations

$$\omega = \pm(\sigma + 1.2)\sqrt{\frac{-\sigma}{\sigma + 1.6}} \tag{6–21}$$

represent the complex branches for $0 \leq K \leq \infty$. These two branches lie between $\sigma = -1.6$ and $\sigma = 0$. [See Figure 6–44(b).] The slopes of the complex root-locus branches at the breakaway point ($\sigma = -1.2$) can be found by evaluating $d\omega/d\sigma$ of Equation (6–21) at point $\sigma = -1.2$.

$$\left.\frac{d\omega}{d\sigma}\right|_{\sigma=-1.2} = \pm\left.\sqrt{\frac{-\sigma}{\sigma + 1.6}}\right|_{\sigma=-1.2} = \pm\sqrt{\frac{1.2}{0.4}} = \pm\sqrt{3}$$

Since $\tan^{-1}\sqrt{3} = 60°$, the root-locus branches intersect the real axis with angles $\pm 60°$.

A–6–8. Consider the system shown in Figure 6–45(a), which has an unstable feedforward transfer function. Sketch the root-locus plot and locate the closed-loop poles. Show that, although the closed-loop poles lie on the negative real axis and the system is not oscillatory, the unit-step response curve will exhibit overshoot.

Solution. The root-locus plot for this system is shown in Figure 6–45(b). The closed-loop poles are located at $s = -2$ and $s = -5$.

The closed-loop transfer function becomes

$$\frac{C(s)}{R(s)} = \frac{10(s + 1)}{s^2 + 7s + 10}$$

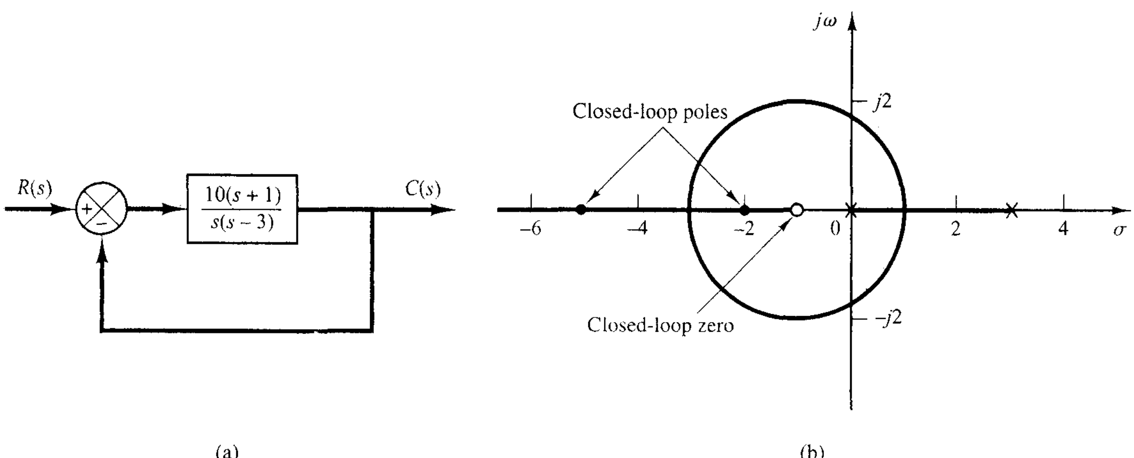


Figure 6–45
(a) Control system; (b) root-locus plot.

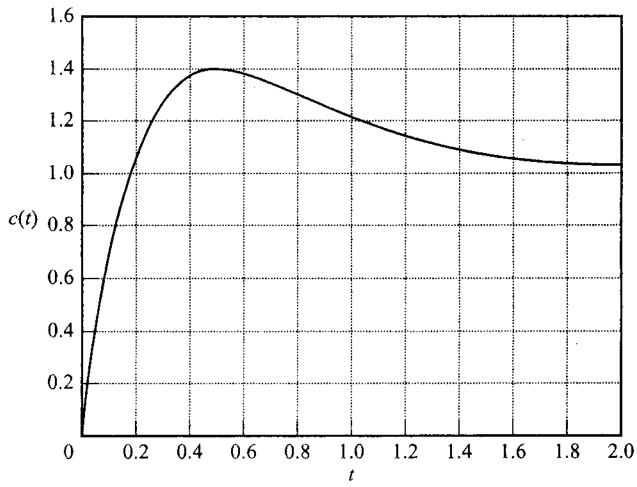


Figure 6–46 Unit-step response curve for the system shown in Figure 6–45(a).

The unit-step response of this system is

$$C(s) = \frac{10(s + 1)}{s(s + 2)(s + 5)}$$

The inverse Laplace transform of $C(s)$ gives

$$c(t) = 1 + 1.666e^{-2t} - 2.666e^{-5t}, \qquad \text{for } t \geq 0$$

The unit-step response curve is shown in Figure 6–46. Although the system is not oscillatory, the unit-step response curve exhibits overshoot. (This is due to the presence of a zero at $s = -1$.)

A–6–9. Sketch the root loci of the control system shown in Figure 6–47(a). Determine the range of gain K for stability.

Solution. Open-loop poles are located at $s = 1$, $s = -2 + j\sqrt{3}$, and $s = -2 - j\sqrt{3}$. A root locus exists on the real axis between points $s = 1$ and $s = -\infty$. The asymptotes of the root-locus branches are found as follows:

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k + 1)}{3} = 60°, -60°, 180°$$

The intersection of the asymptotes and the real axis is obtained as

$$s = -\frac{-1 + 2 + 2}{3} = -1$$

The breakaway and break-in points can be located from $dK/ds = 0$. Since

$$K = -(s - 1)(s^2 + 4s + 7) = -(s^3 + 3s^2 + 3s - 7)$$

we have

$$\frac{dK}{ds} = -(3s^2 + 6s + 3) = 0$$

which yields

$$(s + 1)^2 = 0$$

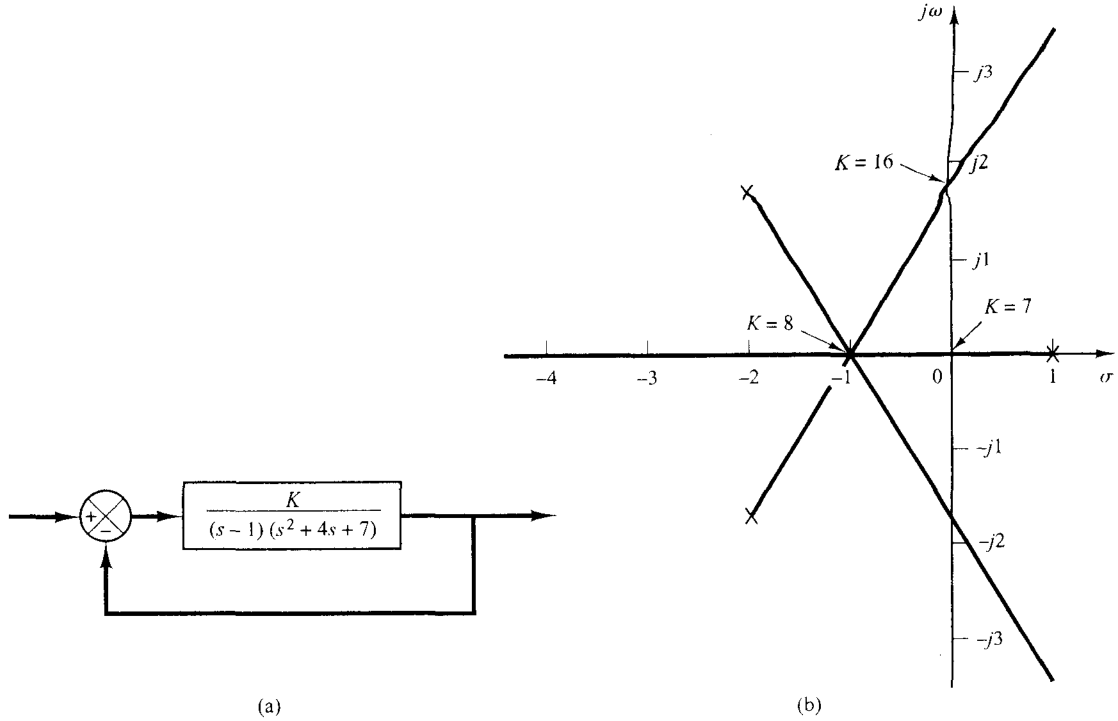


Figure 6–47
(a) Control system; (b) root-locus plot.

Thus the equation $dK/ds = 0$ has a double root at $s = -1$. (This means that the characteristic equation has a triple root at $s = -1$.) The breakaway point is located at $s = -1$. Three root-locus branches meet at this breakaway point. The angles of departure of the branches at the breakaway point are $\pm 180°/3$, that is, $60°$ and $-60°$.

We shall next determine the points where root-locus branches may cross the imaginary axis. Noting that the characteristic equation is

$$(s - 1)(s^2 + 4s + 7) + K = 0$$

or

$$s^3 + 3s^2 + 3s - 7 + K = 0$$

we substitute $s = j\omega$ into it and obtain

$$(j\omega)^3 + 3(j\omega)^2 + 3(j\omega) - 7 + K = 0$$

By rewriting this last equation, we have

$$(K - 7 - 3\omega^2) + j\omega(3 - \omega^2) = 0$$

This equation is satisfied when

$$\omega = \pm\sqrt{3}, \qquad K = 7 + 3\omega^2 = 16 \qquad \text{or} \qquad \omega = 0, \qquad K = 7$$

The root-locus branches cross the imaginary axis at $\omega = \pm\sqrt{3}$ (where $K = 16$) and $\omega = 0$ (where $K = 7$). Since the value of gain K at the origin is 7, the range of gain value K for stability is

$$7 < K < 16$$

Figure 6–47(b) shows a sketch of the root loci for the system. Notice that all branches consist of parts of straight lines.

The fact that the root-locus branches consist of straight lines can be verified as follows: Since the angle condition is

$$\angle \frac{K}{(s-1)(s+2+j\sqrt{3})(s+2-j\sqrt{3})} = \pm 180°(2k+1)$$

we have

$$-\angle s-1 - \angle s+2+j\sqrt{3} - \angle s+2-j\sqrt{3} = \pm 180°(2k+1)$$

By substituting $s = \sigma + j\omega$ into this last equation,

$$\angle \sigma - 1 + j\omega + \angle \sigma + 2 + j\omega + j\sqrt{3} + \angle \sigma + 2 + j\omega - j\sqrt{3} = \pm 180°(2k+1)$$

or

$$\angle \sigma + 2 + j(\omega + \sqrt{3}) + \angle \sigma + 2 + j(\omega - \sqrt{3}) = -\angle \sigma - 1 + j\omega \pm 180°(2k+1)$$

which can be rewritten as

$$\tan^{-1}\left(\frac{\omega + \sqrt{3}}{\sigma + 2}\right) + \tan^{-1}\left(\frac{\omega - \sqrt{3}}{\sigma + 2}\right) = -\tan^{-1}\left(\frac{\omega}{\sigma - 1}\right) \pm 180°(2k+1)$$

Taking tangents of both sides of this last equation, we obtain

$$\frac{\dfrac{\omega + \sqrt{3}}{\sigma + 2} + \dfrac{\omega - \sqrt{3}}{\sigma + 2}}{1 - \left(\dfrac{\omega + \sqrt{3}}{\sigma + 2}\right)\left(\dfrac{\omega - \sqrt{3}}{\sigma + 2}\right)} = -\frac{\omega}{\sigma - 1}$$

or

$$\frac{2\omega(\sigma + 2)}{\sigma^2 + 4\sigma + 4 - \omega^2 + 3} = -\frac{\omega}{\sigma - 1}$$

which can be simplified to

$$2\omega(\sigma + 2)(\sigma - 1) = -\omega(\sigma^2 + 4\sigma + 7 - \omega^2)$$

or

$$\omega(3\sigma^2 + 6\sigma + 3 - \omega^2) = 0$$

Further simplification of this last equation yields

$$\omega\left(\sigma + 1 + \frac{1}{\sqrt{3}}\omega\right)\left(\sigma + 1 - \frac{1}{\sqrt{3}}\omega\right) = 0$$

which defines three lines:

$$\omega = 0, \qquad \sigma + 1 + \frac{1}{\sqrt{3}}\omega = 0, \qquad \sigma + 1 - \frac{1}{\sqrt{3}}\omega = 0$$

Thus the root-locus branches consist of three lines. Note that the root loci for $K > 0$ consist of portions of the straight lines as shown in Figure 6–47(b). (Note that each straight line starts from an open-loop pole and extends to infinity in the direction of 180°, 60°, or −60° measured from the real axis.) The remaining portion of each straight line corresponds to $K < 0$.

A–6–10. Consider the system shown in Figure 6–48(a). Sketch the root loci.

Solution. The open-loop zeros of the system are located at $s = \pm j$. The open-loop poles are located at $s = 0$ and $s = -2$. This system involves two poles and two zeros. Hence, there is a possibility that a circular root-locus branch exists. In fact, such a circular root locus exists in this case, as shown in the following. The angle condition is

$$\angle \frac{K(s + j)(s - j)}{s(s + 2)} = \pm 180^\circ(2k + 1)$$

or

$$\angle s + j + \angle s - j - \angle s - \angle s + 2 = \pm 180^\circ(2k + 1)$$

By substituting $s = \sigma + j\omega$ into this last equation, we obtain

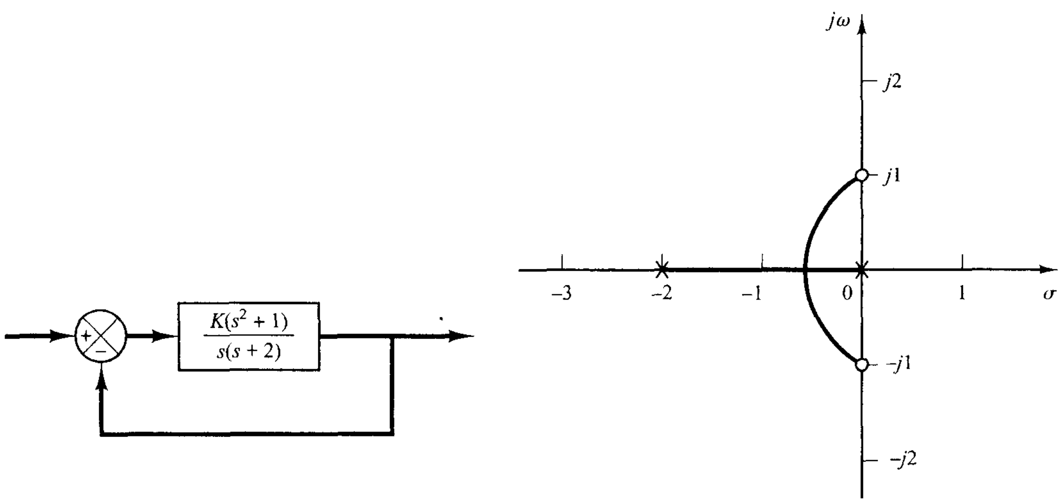
$$\angle \sigma + j\omega + j + \angle \sigma + j\omega - j = \angle \sigma + j\omega + \angle \sigma + 2 + j\omega \pm 180^\circ(2k + 1)$$

or

$$\tan^{-1}\left(\frac{\omega + 1}{\sigma}\right) + \tan^{-1}\left(\frac{\omega - 1}{\sigma}\right) = \tan^{-1}\left(\frac{\omega}{\sigma}\right) + \tan^{-1}\left(\frac{\omega}{\sigma + 2}\right) \pm 180^\circ(2k + 1)$$

Taking tangents of both sides of this equation and noting that

$$\tan\left[\tan^{-1}\left(\frac{\omega}{\sigma + 2}\right) \pm 180^\circ\right] = \frac{\omega}{\sigma + 2}$$



(a) (b)

Figure 6–48
(a) Control system; (b) root-locus plot.

we obtain

$$\frac{\dfrac{\omega+1}{\sigma}+\dfrac{\omega-1}{\sigma}}{1-\dfrac{\omega+1}{\sigma}\dfrac{\omega-1}{\sigma}}=\frac{\dfrac{\omega}{\sigma}+\dfrac{\omega}{\sigma+2}}{1-\dfrac{\omega}{\sigma}\dfrac{\omega}{\sigma+2}}$$

or

$$\omega\left[\left(\sigma-\frac{1}{2}\right)^2+\omega^2-\frac{5}{4}\right]=0$$

which is equivalent to

$$\omega=0 \quad \text{or} \quad \left(\sigma-\frac{1}{2}\right)^2+\omega^2=\frac{5}{4}$$

These two equations are equations for the root loci. The first equation corresponds to the root locus on the real axis. (The segment between $s = 0$ and $s = -2$ corresponds to the root locus for $0 \le K < \infty$. The remaining parts of the real axis correspond to the root locus for $K < 0$.) The second equation is an equation for a circle. Thus, there exists a circular root locus with center at $\sigma = \frac{1}{2}$, $\omega = 0$ and the radius equal to $\sqrt{5}/2$. The root loci are sketched in Figure 6–48(b). [That part of the circular locus to the left of the imaginary zeros corresponds to $K > 0$. The portion of the circular locus not shown in Figure 6–48(b) corresponds to $K < 0$.]

A–6–11. Consider the control system shown in Figure 6–49. Plot the root loci with MATLAB.

Solution. MATLAB Program 6–11 generates a root-locus plot as shown in Figure 6–50. The root loci must be symmetric about the real axis. However, Figure 6–50 shows otherwise.

MATLAB supplies its own set of gain values that are used to calculate a root-locus plot. It does so by an internal adaptive step-size routine. However, in certain systems, very small changes in the gain cause drastic changes in root locations within a certain range of gains. Thus, MATLAB takes too big a jump in its gain values when calculating the roots, and root locations change by a relatively large amount. When plotting, MATLAB connects these points and causes a strange-looking graph at the location of sensitive gains. Such erroneous root-locus plots typically occur when the loci approach a double pole (or triple or higher pole), since the locus is very sensitive to small gain changes.

MATLAB Program 6–11

```
% ---------- Root-locus plot ----------

num = [0  0  1  0.4];
den = [1  3.6  0  0];
rlocus(num,den);
v = [-5  1  -3  3]; axis(v)
grid
title('Root-Locus Plot of G(s) = K(s + 0.4)/[s^2(s + 3.6)]')
```

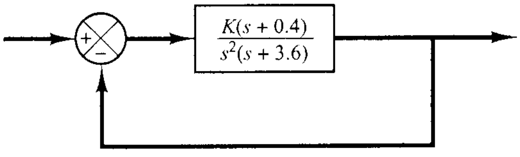


Figure 6–49 Control system.

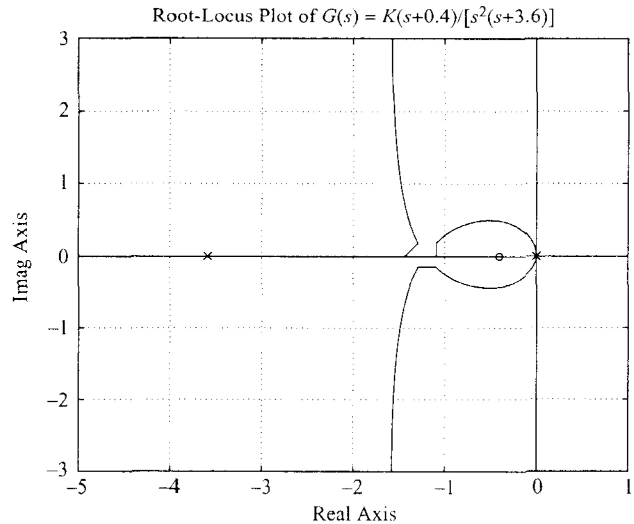


Figure 6–50
Root-locus plot.

In the problem considered here, the critical region of gain K is between 4.2 and 4.4. Thus we need to set the step size small enough in this region. We may divide the region for K as follows:

```
K1 = [0:0.2:4.2];
K2 = [4.2:0.002:4.4];
K3 = [4.4:0.2:10];
K4 = [10:5:200];
K = [K1  K2  K3  K4];
```

Entering MATLAB Program 6–12 into the computer, we obtain the plot as shown in Figure 6–51. If we change the plot command plot(r,'o') in MATLAB Program 6–12 to plot(r,'-'), we obtain Figure 6–52. Figures 6–51 and 6–52 respectively, show satisfactory root-locus plots.

MATLAB Program 6–12

```
% ---------- Root-locus plot ----------
num = [0  0  1  0.4];
den = [1  3.6  0  0];
K1 = [0:0.2:4.2];
K2 = [4.2:0.002:4.4];
K3 = [4.4:0.2:10];
K4 = [10:5:200];
K = [K1  K2  K3  K4];
r = rlocus(num,den,K);
plot(r,'o')
v = [-5  1  -5  5]; axis(v)
grid
title('Root-Locus Plot of G(s) = K(s + 0.4)/[s^2(s + 3.6)]')
xlabel('Real Axis')
ylabel('Imag Axis')
```

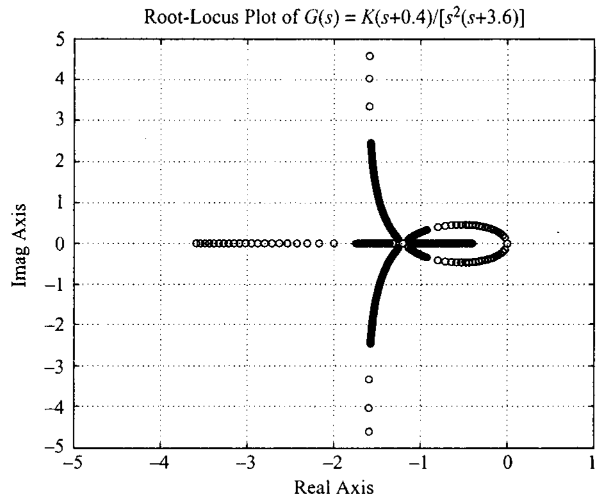


Figure 6–51
Root-locus plot.

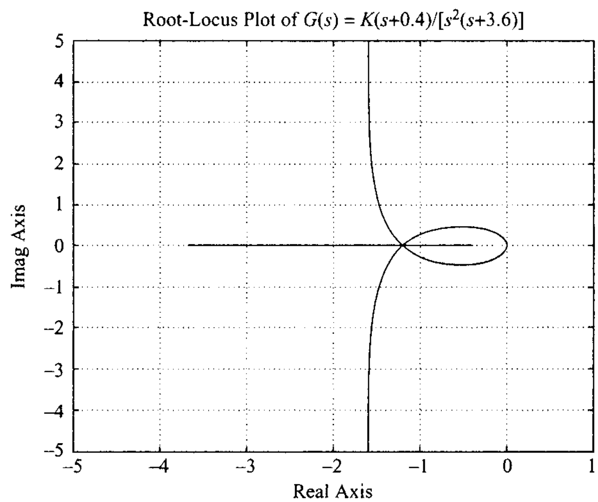


Figure 6–52
Root-locus plot.

A–6–12. Consider the system whose open-loop transfer function $G(s)H(s)$ is given by

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

Using MATLAB, plot root loci and their asymptotes.

Solution. We shall plot the root loci and asymptotes on one diagram. Since the open-loop transfer function is given by

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)} = \frac{K}{s^3 + 3s^2 + 2s}$$

the equation for the asymptotes may be obtained as follows: Noting that

$$\lim_{s\to\infty} \frac{K}{s^3 + 3s^2 + 2s} \doteq \lim_{s\to\infty} \frac{K}{s^3 + 3s^2 + 3s + 1} = \frac{K}{(s+1)^3}$$

the equation for the asymptotes may be given by

$$G_a(s)H_a(s) = \frac{K}{(s+1)^3}$$

Hence, for the system we have

```
num = [0  0  0  1]
den = [1  3  2  0]
```

and for the asymptotes,

```
numa = [0  0  0  1]
dena = [1  3  3  1]
```

In using the following root-locus and plot commands

```
r = rlocus(num,den)
a = rlocus(numa,dena)
plot([r  a])
```

the number of rows of r and that of a must be the same. To ensure this, we include the gain constant K in the commands. For example,

```
K1 = 0:0.1:0.3;
K2 = 0.3:0.005:0.5:
K3 = 0.5:0.5:10;
K4 = 10:5:100;
K = [K1  K2  K3  K4]
r = rlocus(num,den,K)
a = rlocus(numa,dena,K)
y = [r    a]
plot(y, '-')
```

MATLAB Program 6–13

```
% ---------- Root-Locus Plots ----------
num = [0  0  0  1];
den = [1  3  2  0];
numa = [0  0  0  1];
dena = [1  3  3  1];
K1 = 0:0.1:0.3;
K2 = 0.3:0.005:0.5;
K3 = 0.5:0.5:10;
K4 = 10:5:100;
K = [K1  K2  K3  K4];
r = rlocus(num,den,K);
a = rlocus(numa,dena,K);
y = [r    a];
plot(y,'-')
v = [-4  4  -4  4]; axis(v)
grid
title('Root-Locus Plot of G(s) = K/[s(s + 1)(s + 2)] and Asymptotes')
xlabel('Real Axis')
ylabel('Imag Axis')
% ***** Manually draw open-loop poles in the hard copy *****
```

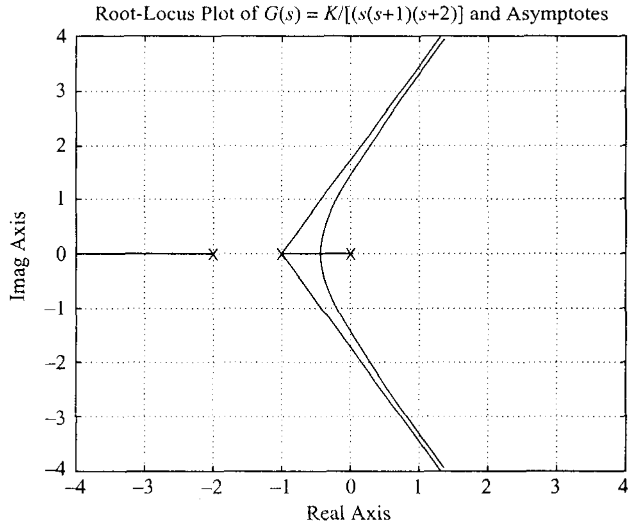


Figure 6–53
Root-locus plot.

Including gain K in rlocus command ensures that the r matrix and a matrix have the same number of rows. MATLAB Program 6–13 will generate a plot of root loci and their asymptotes. See Figure 6–53.

Drawing two or more plots in one diagram can also be accomplished by using the hold command. MATLAB Program 6–14 uses the hold command. The resulting root-locus plot is shown in Figure 6–54.

MATLAB Program 6–14

```
% ------------ Root-Locus Plots ------------
num = [0  0  0  1];
den = [1  3  2  0];
numa = [0  0  0  1];
dena = [1  3  3  1];
K1 = 0:0.1:0.3;
K2 = 0.3:0.005:0.5;
K3 = 0.5:0.5:10;
K4 = 10:5:100;
K = [K1  K2  K3  K4];
r = rlocus(num,den,K);
a = rlocus(numa,dena,K);
plot(r,'o')
hold
Current plot held
plot(a,'-')
v = [-4  4  -4  4]; axis(v)
grid
title('Root-Locus Plot of G(s) = K/[s(s+1)(s+2)] and Asymptotes')
xlabel('Real Axis')
ylabel('Imag Axis')
```

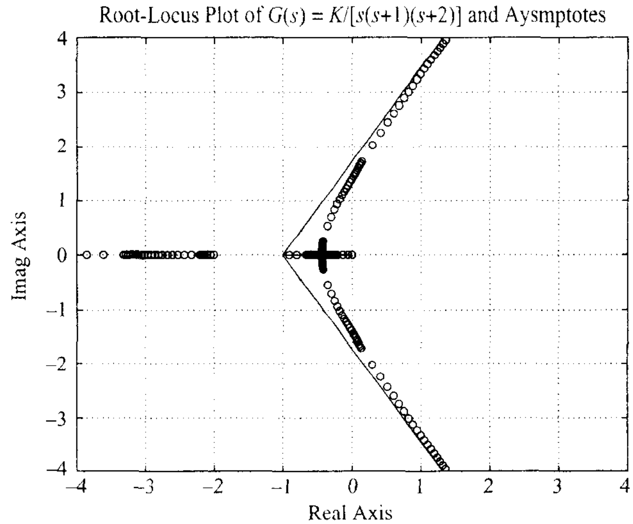


Figure 6–54
Root-locus plot.

A–6–13. Consider a unity-feedback system with the following feedforward transfer function $G(s)$:

$$G(s) = \frac{K(s+2)^2}{(s^2+4)(s+5)^2}$$

Plot root loci for the system with MATLAB.

Solution. A MATLAB program to plot the root loci is given as MATLAB Program 6–15. The resulting root-locus plot is shown in Figure 6–55.

Notice that this is a special case where no root locus exists on the real axis. This means that for any value of $K > 0$ the closed-loop poles of the system are two sets of complex-conjugate poles. (No real closed-loop poles exist.) For example, with $K = 25$, the characteristic equation for the system becomes

$$\begin{aligned} &s^4 + 10s^3 + 54s^2 + 140s + 200 \\ &= (s^2 + 4s + 10)(s^2 + 6s + 20) \\ &= (s + 2 + j2.4495)(s + 2 - j2.4495)(s + 3 + j3.3166)(s + 3 - j3.3166) \end{aligned}$$

MATLAB Program 6–15

```
% ------------ Root-Locus Plot ------------
num = [0  0  1  4  4];
den = [1  10  29  40  100];
r = rlocus(num,den);
plot(r,'o')
hold
current plot held
plot(r,'-')
v = [-8  4  -6  6]; axis(v); axis('square')
grid
title('Root-Locus Plot of G(s) = (s + 2)^2/[(s^2 + 4)(s + 5)^2]')
xlabel('Real Axis')
ylabel('Imag Axis')
```

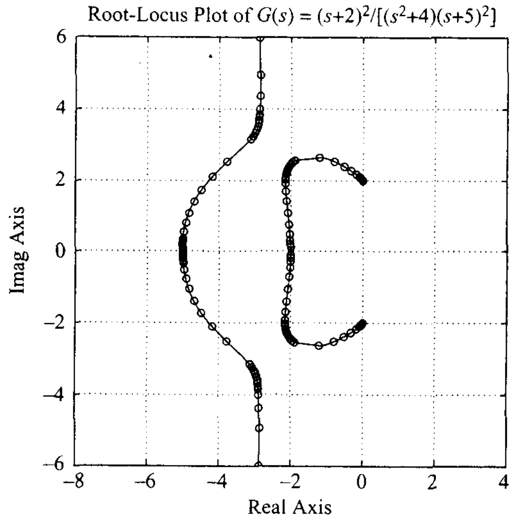


Figure 6–55
Root-locus plot.

Since no closed-loop poles exist in the right-half s plane, the system is stable for all values of $K > 0$.

A–6–14. Consider a unity-feedback control system with the following feedforward transfer function:

$$G(s) = \frac{s + 2}{s^3 + 9s^2 + 8s}$$

Plot a root-locus diagram with MATLAB. Superimpose on the s plane constant ζ lines and constant ω_n circles.

Solution. MATLAB Program 6–16 produces the desired plot as shown in Figure 6–56.

MATLAB Program 6–16

```
num = [0 0 1 2];
den = [1 9 8 0];
K = 0:0.2:200;
rlocus(num,den,K)
v = [-10 2 -6 6]; axis(v); axis('square')
sgrid
title('Root-Locus Plot with Constant \zeta Lines and Constant \omega_n Circles')
gtext('\zeta = 0.9')
gtext('0.7')
gtext('0.5')
gtext('0.3')
gtext('\omega_n = 10')
gtext('8')
gtext('6')
gtext('4')
gtext('2')
```

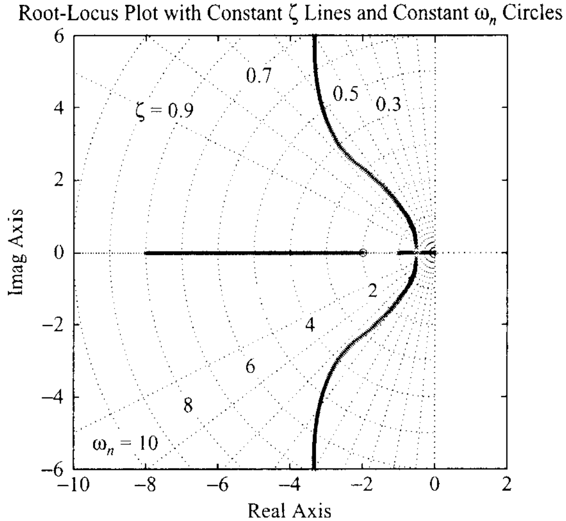


Figure 6–56 Root-locus plot with constant ζ lines and constant ω_n circles.

A–6–15. Consider a unity-feedback control system with the following feedforward transfer function:

$$G(s) = \frac{K(s^2 + 25)s}{s^4 + 404s^2 + 1600}$$

Plot root loci for the system with MATLAB. Show that the system is stable for all values of $K > 0$.

Solution. MATLAB Program 6–17 gives a plot of root loci as shown in Figure 6–57. Since the root loci are entirely in the left-half s plane, the system is stable for all $K > 0$.

MATLAB Program 6–17

```
num = [0 1 0 25 0];
den = [1 0 404 0 1600];
K = 0:0.4:1000;
rlocus(num,den,K)
v = [-30 20 -25 25]; axis(v); axis('square')
grid
title('Root-Locus Plot of G(s) = K(s^2 + 25)s/(s^4 + 404s^2 + 1600)')
```

A–6–16. A simplified form of the open-loop transfer function of an airplane with an autopilot in the longitudinal mode is

$$G(s)H(s) = \frac{K(s + a)}{s(s - b)(s^2 + 2\zeta\omega_n s + \omega_n^2)}, \qquad a > 0, \qquad b > 0$$

Such a system involving an open-loop pole in the right-half s plane may be conditionally stable. Sketch the root loci when $a = b = 1$, $\zeta = 0.5$, and $\omega_n = 4$. Find the range of gain K for stability.

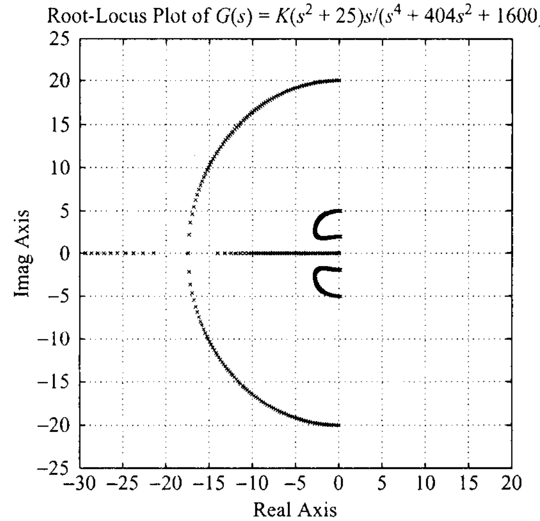


Figure 6–57
Root-locus plot.

Solution. The open-loop transfer function for the system is

$$G(s)H(s) = \frac{K(s+1)}{s(s-1)(s^2+4s+16)}$$

To sketch the root loci, we follow this procedure:

1. Locate the open-loop poles and zero in the complex plane. Root loci exist on the real axis between 1 and 0 and between -1 and $-\infty$.
2. Determine the asymptotes of the root loci. There are three asymptotes whose angles can be determined as

$$\text{Angles of asymptotes} = \frac{180°(2k+1)}{4-1} = 60°, -60°, 180°$$

Referring to Equation (6–13), the abscissa of the intersection of the asymptotes and the real axis is

$$s = -\frac{(0 - 1 + 2 + j2\sqrt{3} + 2 - j2\sqrt{3}) - 1}{4-1} = -\frac{2}{3}$$

3. Determine the breakaway and break-in points. Since the characteristic equation is

$$1 + \frac{K(s+1)}{s(s-1)(s^2+4s+16)} = 0$$

we obtain

$$K = -\frac{s(s-1)(s^2+4s+16)}{s+1}$$

By differentiating K with respect to s, we get

$$\frac{dK}{ds} = -\frac{3s^4 + 10s^3 + 21s^2 + 24s - 16}{(s+1)^2}$$

The numerator can be factored as follows:

$$3s^4 + 10s^3 + 21s^2 + 24s - 16$$

$$= 3(s + 0.76 + j2.16)(s + 0.76 - j2.16)(s + 2.26)(s - 0.45)$$

Points $s = 0.45$ and $s = -2.26$ are on root loci on the real axis. Hence, these points are actual breakaway and break-in points, respectively. Points $s = -0.76 \pm j2.16$ do not satisfy the angle condition. Hence, they are neither breakaway nor break-in points.

4. Using Routh's stability criterion, determine the value of K at which the root loci cross the imaginary axis. Since the characteristic equation is

$$s^4 + 3s^3 + 12s^2 + (K - 16)s + K = 0$$

the Routh array becomes

$$\begin{array}{llll}
s^4 & 1 & 12 & K \\
s^3 & 3 & K - 16 & 0 \\
s^2 & \dfrac{52 - K}{3} & K & 0 \\
s^1 & \dfrac{-K^2 + 59K - 832}{52 - K} & 0 & \\
s^0 & K & &
\end{array}$$

The values of K that make the s^1 term in the first column equal zero are $K = 35.7$ and $K = 23.3$.

The crossing points on the imaginary axis can be found by solving the auxiliary equation obtained from the s^2 row, that is, by solving the following equation for s:

$$\frac{52 - K}{3} s^2 + K = 0$$

The results are

$$s = \pm j2.56, \qquad \text{for } K = 35.7$$

$$s = \pm j1.56, \qquad \text{for } K = 23.3$$

The crossing points on the imaginary axis are thus $s = \pm j2.56$ and $s = \pm j1.56$.

5. Find the angles of departure of the root loci from the complex poles. For the open-loop pole at $s = -2 + j2\sqrt{3}$, the angle of departure θ is

$$\theta = 180° - 120° - 130.5° - 90° + 106°$$

or

$$\theta = -54.5°$$

(The angle of departure from the open-loop pole at $s = -2 - j2\sqrt{3}$ is 54.5°.)

6. Choose a test point in the broad neighborhood of the $j\omega$ axis and the origin, and apply the angle condition. If the test point does not satisfy the angle condition, select another test point until it does. Continue the same process and locate a sufficient number of points that satisfy the angle condition.

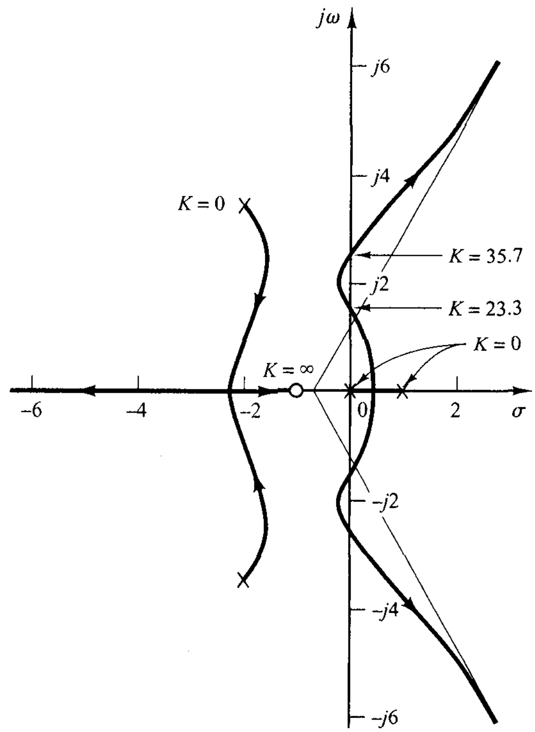


Figure 6–58
Root-locus plot.

Figure 6–58 shows the root loci for this system. From step 4, the system is stable for $23.3 < K < 35.7$. Otherwise, it is unstable. Thus, the system is conditionally stable.

A–6–17. Consider the system shown in Figure 6–59, where the dead time T is 1 sec. Suppose that we approximate the dead time by the second-order pade approximation. The expression for this approximation can be obtained with MATLAB as follows:

```
[num,den] = pade(1, 2);
printsys(num, den)
num/den =
   s^2 - 6s + 12
   -------------
   s^2 + 6s + 12
```

Hence

$$e^{-s} = \frac{s^2 - 6s + 12}{s^2 + 6s + 12} \tag{6–22}$$

Using this approximation, determine the critical value of K (where $K > 0$) for stability.

Solution. Since the characteristic equation for the system is

$$s + 1 + Ke^{-s} = 0$$

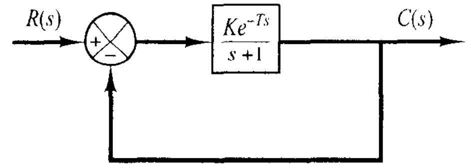


Figure 6–59
A control system with dead time.

by substituting Equation (6–22) into this characteristic equation, we obtain

$$s + 1 + K\frac{s^2 - 6s + 12}{s^2 + 6s + 12} = 0$$

or

$$s^3 + (7 + K)s^2 + (18 - 6K)s + 12(1 + K) = 0$$

Applying the Routh stability criterion, we get the Routh table as follows:

$$\begin{array}{lcc} s^3 & 1 & 18 - 6K \\ s^2 & 7 + K & 12(1 + K) \\ s^1 & \dfrac{-6K^2 - 36K + 114}{7 + K} & 0 \\ s^0 & 12(1 + K) & \end{array}$$

Hence, for stability we require

$$-6K^2 - 36K + 114 > 0$$

which can be written as

$$(K + 8.2915)(K - 2.2915) < 0$$

or

$$K < 2.2915$$

Since K must be positive, the range of K for stability is

$$0 < K < 2.2915$$

Notice that according to the present analysis, the upper limit of K for stability is 2.2915. This value is greater than the exact upper limit of K. (Earlier, we obtained the exact upper limit of K to be 2, as shown in Figure 6–38.) This is because we approximated e^{-s} by the second-order pade approximation. A higher-order pade approximation will improve the accuracy. However, the computations involved increase considerably.

A–6–18. Consider the system shown in Figure 6–60. The plant involves the dead time of T sec. Design a suitable controller $G_c(s)$ for the system.

Solution. We shall present the Smith predictor approach to design a controller. The first step to design the controller $G_c(s)$ is to design a suitable controller $\hat{G}_c(s)$ when the system has no dead time. Otto J. M. Smith designed an innovative controller scheme, now called the "Smith predictor,"

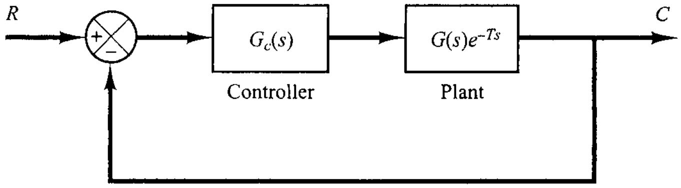


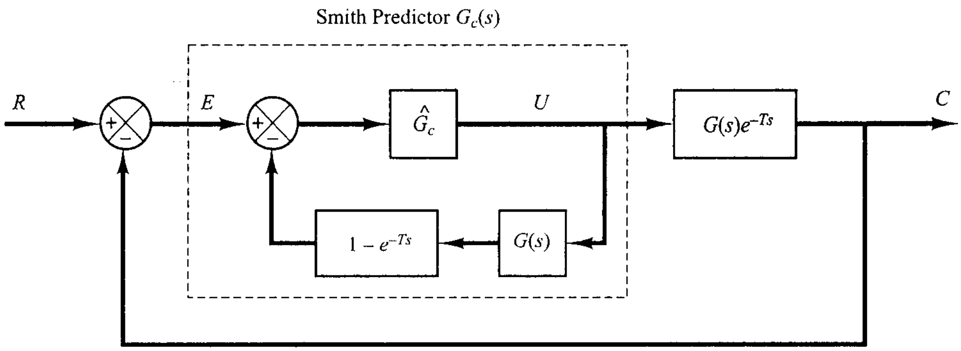
Figure 6–60
Control system with plant with dead time.

to control the plant with dead time. The Smith predictor consists of $G_c(s)$, dead time e^{-Ts}, and the plant transfer function $G(s)$. It has the form

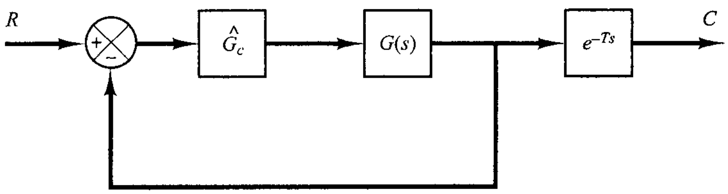
$$G_c(s) = \frac{\hat{G}_c(s)}{1 + (1 - e^{-Ts})\hat{G}_c(s)G(s)}$$

Figure 6-61(a) shows the Smith predictor as a minor loop in the block diagram. The transfer function between $U(s)$ and $E(s)$ is

$$\frac{U(s)}{E(s)} = \frac{\hat{G}_c(s)}{1 + (1 - e^{-Ts})\hat{G}_c(s)G(s)}$$



(a)



(b)

Figure 6–61
(a) Control system with Smith predictor; (b) equivalent block diagram for Smith predictor controlled system shown in (a).

Then the closed-loop transfer function $C(s)/R(s)$ can be given by

$$\frac{C(s)}{R(s)} = \frac{\hat{G}_c(s)G(s)e^{-Ts}}{1 + (1 - e^{-Ts})\hat{G}_c(s)G(s) + \hat{G}_c(s)G(s)e^{-Ts}}$$

$$= \frac{\hat{G}_c(s)G(s)}{1 + \hat{G}_c(s)G(s)} e^{-Ts}$$

Hence, the block diagram of Figure 6–61(a) can be modified to that of Figure 6–61(b). The closed-loop response of the system with dead time e^{-Ts} is the same as the response of the system without dead time e^{-Ts}, except that the response is delayed by T sec.

Typical step-response curves of the system without dead time controlled by the controller $\hat{G}_c(s)$ and of the system with dead time controlled by the Smith predictor type controller are shown in Figure 6–62.

It is noted that implementing the Smith predictor in digital form is not difficult, because dead time can be handled easily in digital control. However, implementing the Smith predictor in an analog form creates some difficulty.

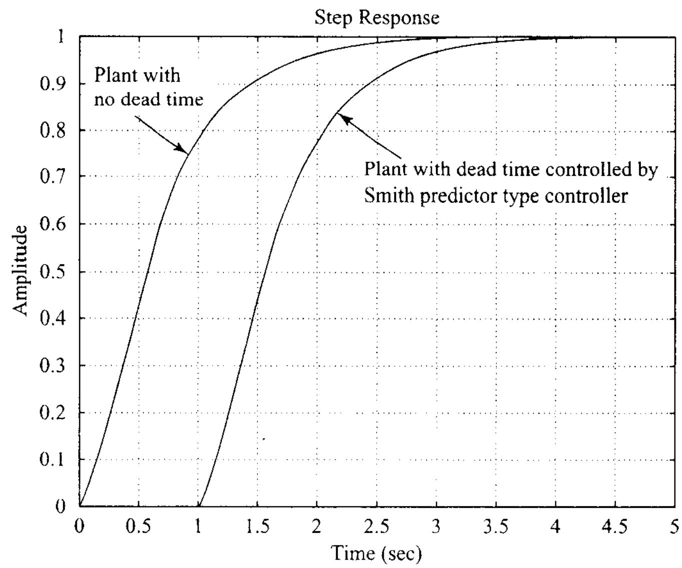


Figure 6–62 Step-response curves.

PROBLEMS

B–6–1. Plot the root loci for the closed-loop control system with

$$G(s) = \frac{K(s + 1)}{s^2}, \qquad H(s) = 1$$

B–6–2. Plot the root loci for the closed-loop control system with

$$G(s) = \frac{K(s + 4)}{(s + 1)^2}, \qquad H(s) = 1$$

B–6–3. Plot the root loci for the closed-loop control system with

$$G(s) = \frac{K}{s(s + 1)(s^2 + 4s + 5)}, \qquad H(s) = 1$$

B–6–4. Plot the root loci for the system with

$$G(s) = \frac{K}{s(s + 0.5)(s^2 + 0.6s + 10)}, \qquad H(s) = 1$$

B–6–5. Plot the root loci for a system with

$$G(s) = \frac{K}{(s^2 + 2s + 2)(s^2 + 2s + 5)}, \qquad H(s) = 1$$

Determine the exact points where the root loci cross the $j\omega$ axis.

B–6–6. Show that the root loci for a control system with

$$G(s) = \frac{K(s^2 + 6s + 10)}{s^2 + 2s + 10}, \qquad H(s) = 1$$

are arcs of the circle centered at the origin with radius equal to $\sqrt{10}$.

B–6–7. Plot the root loci for a closed-loop control system with

$$G(s) = \frac{K(s + 0.2)}{s^2(s + 3.6)}, \qquad H(s) = 1$$

B–6–8. Plot the root loci for a closed-loop control system with

$$G(s) = \frac{K(s + 0.5)}{s^3 + s^2 + 1}, \qquad H(s) = 1$$

B–6–9. Plot the root loci for a closed-loop control system with

$$G(s) = \frac{K(s + 9)}{s(s^2 + 4s + 11)}, \qquad H(s) = 1$$

Locate the closed-loop poles on the root loci such that the dominant closed-loop poles have a damping ratio equal to 0.5. Determine the corresponding value of gain K.

B–6–10. Plot the root loci for the system shown in Figure 6–63. Determine the range of gain K for stability.

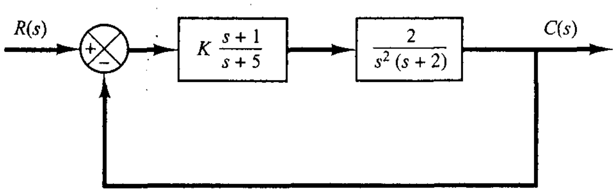


Figure 6–63
Control system.

B–6–11. Consider a unity-feedback control system with the following feedforward transfer function:

$$G(s) = \frac{K}{s(s^2 + 4s + 8)}$$

Plot the root loci for the system. If the value of gain K is set equal to 2, where are the closed-loop poles located?

B–6–12. Consider the system whose open-loop transfer function $G(s)H(s)$ is given by

$$G(s)H(s) = \frac{K(s + 1)}{(s^2 + 2s + 2)(s^2 + 2s + 5)}$$

Plot a root-locus diagram with MATLAB.

B–6–13. Consider the system whose open-loop transfer function is given by

$$G(s)H(s) = \frac{K(s - 0.6667)}{s^4 + 3.3401s^3 + 7.0325s^2}$$

Show that the equation for the asymptotes is given by

$$G_a(s)H_a(s) = \frac{K}{s^3 + 4.0068s^2 + 5.3515s + 2.3825}$$

Using MATLAB, plot the root loci and asymptotes for the system.

B–6–14. Consider the unity-feedback system whose feedforward transfer function is

$$G(s) = \frac{K}{s(s + 1)}$$

The constant-gain locus for the system for a given value of K is defined by the following equation:

$$\left|\frac{K}{s(s + 1)}\right| = 1$$

Show that the constant-gain loci for $0 \le K \le \infty$ may be given by

$$[\sigma(\sigma + 1) + \omega^2]^2 + \omega^2 = K^2$$

Sketch the constant-gain loci for $K = 1, 2, 5, 10$, and 20 on the s plane.

B–6–15. Consider the system shown in Figure 6–64. Plot the root loci with MATLAB. Locate the closed-loop poles when the gain K is set equal to 2.

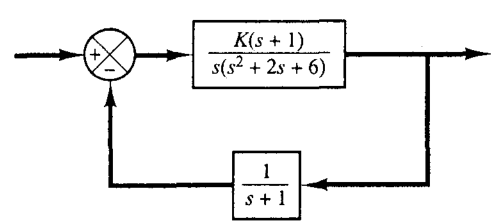


Figure 6–64
Control system.

B–6–16. Plot root-locus diagrams for the nonminimum-phase systems shown in Figures 6–65(a) and (b), respectively.

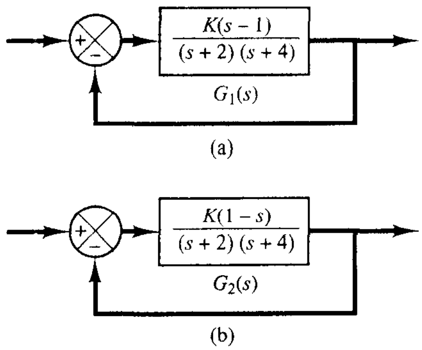


Figure 6–65
(a) and (b) Nonminimum-phase systems.

B–6–17. Consider the closed-loop system with transport lag shown in Figure 6–66. Determine the stability range for gain K.

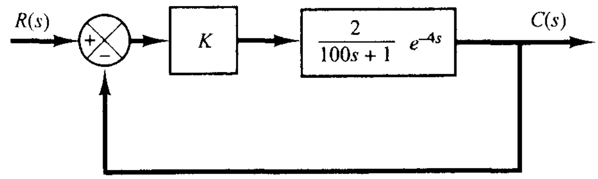


Figure 6–66
Control system.