Subject : Mathematics II	موضوع: رياضيات II
Weekly Hours : Theoretical: 2 UNITS: 4	الساعات الأسبوعية: نظري :2 الوحدات : 4
Tutorial: 1	مناقشة: 1
Experimental :	عملي:

week	Contents	المحتويات	الأسبوع
1.	Complex function	الدوال المركبة	.1
2.	=	=	.2
3.	Catchy – Remain conditions	معادلة كوشي- ريمان	.3
4.	Ordinary differential equation	المعادلات التفاضلية من المرتبة الاولى	.4
5.	First order .D.E	=	.5
6.	=	=	.6
7.	Second order D.E Applications	المعادلات الخطية من المرتبة الثانية مع التطبيقات	.7
8.	=	=	.8
9.	Higher order .D.E	المعادلات التفاضلية الخطية مع المراتب العليا مع التطبيقات	.9
10.	Simultaneously .D.E	المعادلات التفاضلية الانية	.10
11.	Sequences	متابعة من الاعداد	.11
12.	Infinite series	المتسلسلات اللانهائية	.12
13.	Test for convergence	الاختيارات للتقارب	.13
14.	Alternating series	*المتسلسلات المتناوبة*	.14
15.	Power series	متسلسلات القوى	.15
16.	Taylor and machlurin	متسلسلات تيلر ومكلورين	.16
17.	Application	تطبيقات على المتسلسلات	.17
18.	Fourier series	متسلسلة مورير	.18
19.	Even and odd function	الدوال الزوجية والدوال الفردية	.19
20.	Partial derivatives	المشتقات الجزئية	.20
21.	Directional derivatives	المشتقات الاتجاهية	.21
22.	Min & Max points	القيم العظمى والضغرى	.22
23.	Polar coordinate	نظام الاحداثيات القطبية	.23
24.	=	=	.24
25.	Double Integrals	التكاملات الثنائية	.25
26.	Physical Applications	تطبيقات فيزياوية	.26
27.	Triple integrals	التكاملات الثلاثية	.27
28.	Vector in space	المتجهات في الفضاء	.28
29.	Equation of line – plane	معدلة المستقيم والمستوي	.29
30.	Gradient / divergence / curl	التدرج / التباعد/ التقعر	.30

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Differential Equations

Partial Differential Equations

هي تلك المعادلات التي تحتوي على المشتقة لأكثر من متغير

$$\frac{\partial^2 V}{\partial p^2} + \frac{\partial^2 V}{\partial t^2} + \cdots$$

Ordinary Differential Equations

هي تلك المعادلات التي تحتوي على المشتقة لمتغير واحد فقط

$$\left(\frac{d^2y}{dx^2}\right)^3 + x\frac{dy}{dx} \cdots\cdots$$

degree = 3
order = 2

Ordinary Differential Equations

المعادلات التفاضلية الاعتيادية

non-linear لاخطية

linear خطية

تسمى المعادلة التفاضلية الاعتيادية خطية عند الحالات التالية :-

1. عدم ضرب المشتقة في نفسها او في مشتقة ثانية ، مثل :

$$\dot{y}\,\ddot{y} + 2y = 0 \qquad \& \qquad (\dot{y})^2 + 2xy = \ddot{y}$$

2. عدم ضرب المتغير المعتمد dependent variable (y) بالمشتقة

$$y\,\ddot{y} + 2xy = 0 \qquad \& \qquad y\,\dot{y} + \ddot{y} = x$$

3. عدم ضرب المشتقة الثانية فما فوق في دالة لـ x ، مثل :-

$$\cos x \cdot \dddot{y} + 2\dot{y} = 0$$

Linear Differential Equations

- 1^{st} order المرتبة الأولى
  1. separable منفصلة
  2. homogeneous متجانسة
  3. exact تامة
  4. linear خطية
  5. Bernoulli's eq. معادلة برنولي (non-linear)
- 2^{nd} and higher order

2^{nd} order Differential Eqns.

- Reducible to 1^{st} order قابلة للاختزال إلى الرتبة الأولى
- homogeneous متجانسة $a\ddot{y} + b\dot{y} + cy = 0$
- non-homogeneous غير متجانسة $a\ddot{y} + b\dot{y} + cy = f(x)$

higher order D. E.

- homogeneous
- non-homogeneous

Differential Equations :

المعادلة التفاضلية : هي تلك المعادلة التي تحتوي على المشتقة وحل المعادلة التفاضلية هو التخلص من المشتقة .

1^{st} order Differential Equations :

① Separable : (منفصلة)

هي تلك المعادلة التي يمكن عبرها فصل متغيرات x على حدة ومتغيرات y على حدة بحيث تكون مكتوبة بالشكل التالي :-

$$f(x)\,dx = g(y)\,dy$$

ex-1: Solve $\dfrac{dy}{dx} = \dfrac{x\sqrt{1+y^2}}{2-3x^2}$

Sol$\underline{n}$
$$\int \frac{dy}{\sqrt{1+y^2}} = \int \frac{x\,dx}{2-3x^2}$$

$$\sinh^{-1} y = -\frac{1}{6}\ln|2-3x^2| + c$$

② Homogeneous متجانسة

أي معادلة اذا بدلنا فيها x ب (λx) و y ب (λy) تبقى المعادلة دون تغيير . بحيث يمكن كتابة تلك المعادلة بالشكل التالي

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

تسمى متجانسة ولكي تختزل الى منفصلة نفرض أن

$$\frac{y}{x} = v$$

ex.2: Solve $\frac{dy}{dx} = \frac{x+y}{x-y}$

soln

$x \rightarrow \lambda x \quad \& \quad y \rightarrow \lambda y$

$$\frac{d(\lambda y)}{d(\lambda x)} = \frac{\lambda x + \lambda y}{\lambda x - \lambda y}$$

$\therefore \quad \frac{dy}{dx} = \frac{x+y}{x-y} \Rightarrow$ بقت المعادلة دون تغيير

$\therefore$ It is homogeneous

نقسم المعادلة على x ، فنحصل :

$$\frac{dy}{dx} = \frac{1+\frac{y}{x}}{1-\frac{y}{x}} = f\left(\frac{y}{x}\right) \quad \text{——(1)}$$

let $v = \frac{y}{x} \Rightarrow y = xv \Rightarrow dy = x\,dv + v\,dx$

بالقسمة على dx $\Leftarrow$

$$\therefore \quad \frac{dy}{dx} = x\frac{dv}{dx} + v \quad \text{——(2)}$$

by substituting Eq.(2) into Eq.(1), gets:-

$$x\frac{dv}{dx} + v = \frac{1+v}{1-v} \Rightarrow x\frac{dv}{dx} = \frac{1+v}{1-v} - v$$

$$x\frac{dv}{dx} = \frac{1+v-v+v^2}{1-v} \Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{1-v}$$

It is separable

$$\therefore \int \frac{dx}{x} = \int \frac{1-v}{1+v^2}\,dv \Rightarrow \int \frac{dx}{x} = \int \frac{dv}{1+v^2} - \int \frac{v\,dv}{1+v^2}$$

$$\ln|x| = \tan^{-1} v - \frac{1}{2}\ln|1+v^2| + c$$

$$\ln|x| = \tan^{-1}\frac{y}{x} - \frac{1}{2}\ln\left|1+\left(\frac{y}{x}\right)^2\right| + c$$

ex. 3: Solve $\dfrac{dy}{dx} = \dfrac{x-2y+1}{3x-6y+4}$

soln: $\dfrac{dy}{dx} = \dfrac{(x-2y)+1}{3(x-2y)+4}$

let $x-2y=u \Rightarrow dx-2dy=du$ $\div dx$

$1-2\dfrac{dy}{dx} = \dfrac{du}{dx} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2}\left(1-\dfrac{du}{dx}\right)$

$\therefore \dfrac{1}{2}\left(1-\dfrac{du}{dx}\right) = \dfrac{u+1}{3u+4} \Rightarrow 1-\dfrac{du}{dx} = \dfrac{2u+2}{3u+4}$

$\dfrac{du}{dx} = 1-\dfrac{2u+2}{3u+4} \Rightarrow \dfrac{du}{dx} = \dfrac{3u+4-2u-2}{3u+4}$

$\therefore \dfrac{du}{dx} = \dfrac{u+2}{3u+4}$ separable

$\therefore \displaystyle\int \frac{(3u+4)}{u+2}\,du = \int dx$

let $u+2=t \Rightarrow du=dt$ & $u=t-2$

$\therefore \displaystyle\int \frac{3(t-2)+4}{t}\,dt = \int dx$

$\displaystyle\int \frac{3t-2}{t}\,dt = \int dx \Rightarrow \int 3\,dt - 2\int\frac{dt}{t} = \int dx$

$\therefore 3t-2\ln|t| = x+c$

$\therefore 3(u+2)-2\ln|u+2| = x+c$

$3((x-2y)+2)-2\ln|(x-2y)+2| = x+c$

ex. 4 : Solve $\dfrac{dy}{dx} = \dfrac{x-y-2}{x+y+4}$

sol$\underline{n}$: let

$$\left.\begin{aligned} x &= X+h \\ y &= Y+k \end{aligned}\right\}$$ h & K are constants

$$\frac{dy}{dx} = \frac{dY}{dX}$$

$$\therefore \quad \frac{dY}{dX} = \frac{(X+h)-(Y+k)-2}{(X+h)+(Y+k)+4}$$

$$= \frac{X-Y+(h-k-2)}{X+Y+(h+k+4)} \quad \text{———} \ (*)$$

المعادلة $(*)$ ليست متجانسة لوجود الثابت ولكنها تصبح متجانسة

إذا فرضنا أن :

$$h-k-2=0$$

$$h+k+4=0$$

بالجمع

$$2h+2=0 \Rightarrow h=-1 \quad \therefore \quad k=-3$$

$$\therefore \quad x = X-1 \quad \& \quad y = Y-3$$

وعليه فإن المعادلة $(*)$ تصبح كما يلي :-

$$\frac{dY}{dX} = \frac{X-Y}{X+Y} \quad \text{———} \ (**)$$ It is homogeneous

$$\therefore \quad \frac{dY}{dX} = \frac{1-\frac{Y}{X}}{1+\frac{Y}{X}} \quad \text{———} \ (1)$$

let $\dfrac{Y}{X} = v \Rightarrow \dfrac{dY}{dX} = X\dfrac{dv}{dX} + v$ ——— (2)

by substituting Eq. (2) into Eq. (1), gets :

$$X\frac{dv}{dX} + v = \frac{1-v}{1+v} \Rightarrow X\frac{dv}{dX} = \frac{1-v}{1+v} - v \Rightarrow$$

$$X\frac{dv}{dX} = \frac{1-v-v-v^2}{1+v} \Rightarrow X\frac{dv}{dX} = \frac{1-2v-v^2}{1+v}$$

$\therefore \int \frac{(1+v)\,dv}{1-2v-v^2} = \int \frac{dx}{x} \Rightarrow -\frac{1}{2}\ln|1-2v-v^2| = \ln x + c$

$\therefore -\frac{1}{2}\ln\left|1-2\left(\frac{Y}{X}\right)-\left(\frac{Y}{X}\right)^2\right| = \ln|X| + c$

$-\frac{1}{2}\ln\left|1-2\left(\frac{y+3}{x+1}\right)-\left(\frac{y+3}{x+1}\right)^2\right| = \ln|x+1| + c$

ملاحظة مهمة : أن حل المعادلة $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$

يعتمد على قيمة M حيث أن :-

$$M = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

فاذا كانت قيمة $M=0$ نستعمل طريقة حل ex.3

واذا كانت $M \neq 0$ فنستعمل طريقة ex.1

ex.3 $\frac{dy}{dx} = \frac{x-2y+1}{3x-6y+1} \Rightarrow M = \begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix} = -6+6=0$

ex.1 $\frac{dy}{dx} = \frac{x-y-2}{x+y+1} \Rightarrow M = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1+1=2 \neq 0$

③ Exact تامة

اية معادلة تكتب بالشكل التالي :-

$M(x,y)\,dx + N(x,y)\,dy = 0$

دائما موجبة

بحيث أن :-

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

فأن المعادلة تسمى تامة

ملاحظة : اية معادلة تحتوي على دالة مثلثية او دالة زائدية او لوغاريتمية او أسية فانها ليست متجانسة .

ex. 5 : Solve $\dfrac{dy}{dx} = \dfrac{x^3 - 5x^4y^3}{3x^5y^2 - \sin y}$

Sol$\underline{n}$:

المعادلة ليست منفصلة لأنه لا يمكن فصل متغيرات x
على حدة ومتغيرات y على حدة وليست متجانسة لاحتوائها
على دالة مثلثية .

$\therefore \; (3x^5y^2 - \sin y)\,dy = (x^3 - 5x^4y^3)\,dx$

$\therefore \; (3x^5y^2 - \sin y)\,dy + (-x^3 + 5x^4y^3)\,dx = 0$

$$\left.\begin{aligned} \frac{\partial}{\partial x}(3x^5y^2 - \sin y) &= 15x^4y^2 \\ \frac{\partial}{\partial y}(-x^3 + 5x^4y^3) &= 15x^4y^2 \end{aligned}\right\} \text{متساويات}$$

$\therefore$ It is exact

الآن، نفتح الأقواس

$\underbrace{3x^5y^2\,dy} - \sin y\,dy - x^3\,dx + \underbrace{5x^4y^3\,dx} = 0$

صعب التكامل $\Rightarrow$ لاحتوائهما على x و y معاً $\Leftarrow$ صعب التكامل

الحدود التي لا نستطيع أن نكاملها نحصرها بين أقواس

$(3x^5y^2\,dy + 5x^4y^3\,dx) - \sin y\,dy - x^3\,dx = 0$

$\int d(x^5y^3) - \int \sin y\,dy - \int x^3\,dx = 0$

دائماً نجده
إذا كانت مختلطة تامة
وهو عبارة عن مشتقة
حاصل ضرب دالتين

$x^5y^3 + \cos y - \dfrac{x^4}{4} = C$

Integrating Factor العامل التكاملي

بعض المعادلات ليست تامة ولكنها تصبح تامة بعد ضربها بعامل تكاملي مناسب

* مشتقة حاصل ضرب دالتين $x\,dy + y\,dx = d(xy)$

$x\,dy - y\,dx = d(\,?\,)$

احتمالات العامل التكاملي عديدة جداً

$x\,dy - y\,dx = d(\,?\,)$

* x^2 نقسم على $\quad \dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$ مشتقة قسمة دالتين *

* y^2 نقسم على $\quad \dfrac{x\,dy - y\,dx}{y^2} = \dfrac{-(y\,dx - x\,dy)}{y^2} = -d\left(\dfrac{x}{y}\right)$

* xy نقسم على $\quad \dfrac{x\,dy - y\,dx}{xy} = \dfrac{dy}{y} - \dfrac{dx}{x} = d(\ln y) - d(\ln x)$

$= d(\ln y - \ln x) = d\left(\ln \dfrac{y}{x}\right)$

ex. 6: solve $\dfrac{dy}{dx} = \dfrac{y - 5x^4 y^7}{x}$

Sol$\underline{n}$

هذه المعادلة ليست متجانسة ولا منفصلة

$x\,dy = (y - 5x^4 y^7)\,dx$

$x\,dy + (-y + 5x^4 y^7)\,dx = 0$

$$\left.\begin{aligned} &\frac{\partial}{\partial x}(x) = 1 \\ &\frac{\partial}{\partial y}(-y + 5x^4 y^7) = -1 + 35x^4 y^6 \end{aligned}\right\} \text{غير متساويين}$$

$\therefore\ x\,dy - y\,dx = -5x^4 y^7\,dx$ not exact

الشيء الذي يجعلنا نستخدم العامل التكاملي هو قسمة المعادلة السابقة على x^2

$$\therefore \quad \frac{x\,dy - y\,dx}{x^2} = -5x^2 y^7 dx$$

يجب التخلص من y بالضرب بـ $\left(\frac{x^7}{x^7}\right)$

$$\therefore \quad d\left(\frac{y}{x}\right) = -5x^9 \left(\frac{y}{x}\right)^7 dx$$

$$\frac{d\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)^7} = -5x^9\,dx$$

$$\therefore \int \left(\frac{y}{x}\right)^{-7} d\left(\frac{y}{x}\right) = -5\int x^9\,dx$$

$$-\frac{1}{6}\left(\frac{y}{x}\right)^{-6} = -5\,\frac{x^{10}}{10} + c$$

④ Linear الخطية

أية معادلة تكتب بالشكل التالي

$$\boxed{\frac{dy}{dx} + p(x)\,\overset{1}{y} = Q(x)}$$

تسمى خطية بـ y (linear in y) ولكي تكون تامة تضرب بالعامل التكاملي

$$I.F. = e^{\int p(x)\,dx}$$

أو تكون مكتوبة بالشكل التالي

$$\frac{dx}{dy} + P(y)\,\overset{1}{x} = Q(y) \qquad \text{linear in } x$$

$$I.F. = e^{\int P(y)\,dy}$$

ex-4 : solve $\frac{dy}{dx} = \frac{3y - 4x^5}{x}$

sol$\underline{n}$: المعادلة ليست منفصلة ولا متجانسة ولا تامة

$\therefore \quad \frac{dy}{dx} = \frac{3}{x} y - 4x^4$

or $\quad \frac{dy}{dx} + \left(-\frac{3}{x}\right) y^1 = -4x^4$ ——— (*)

linear in y , $p(x) = -\frac{3}{x}$

$\therefore$ I.F. $= e^{\int -\frac{3}{x} dx} = e^{-3 \ln x} = e^{\ln x^{-3}} = x^{-3}$

Eq.(*) is multiplied by $(x^{-3}\, dx)$

$x^{-3}\, dy - 3x^{-4} y\, dx = -4x\, dx$

It is exact

$\therefore \quad \int d(x^{-3} y) = \int -4x\, dx$

$\therefore \quad x^{-3} y = -2x^2 + c$

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(5) Bernoulli Equation معادلة برنولي

أية معادلة تكتب بالشكل التالي

$$\boxed{\frac{dy}{dx} + p(x)\, y^1 = Q(x)\, y^n}$$ (Bernoulli in y)

ولكي تكون خطية بـ z نفرض ان

$y^{1-n} = z$

$\underline{or}$ $\quad \frac{dx}{dy} + p(y)\, x^1 = Q(y)\, x^n$ (Bernoulli in x)

$x^{1-n} = z$

ex. 8 : Solve $\frac{dy}{dx} = \frac{2x^5y^3 - 4y}{x}$

Soln :

ليست منفصلة ، غير تامة ، غير متجانسة

$$\frac{dy}{dx} = 2x^4y^3 - \frac{4}{x}y \Rightarrow \frac{dy}{dx} + \left(\frac{4}{x}\right)y = 2x^4y^3 \quad --- (*)$$

Bernouli in y $\left(\frac{dy}{dx} + P(x)y = Q(x)y^n\right)$

let $y^{1-3} = z$

$$\therefore \quad z = y^{-2} \Rightarrow \frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}$$

Eq. (*) is multiplied by $(-2y^{-3})$, gets

$$\underbrace{-2y^{-3}\frac{dy}{dx}} - \frac{8}{x}y^{-2} = -4x^4$$

or $\frac{dz}{dx} + \left(-\frac{8}{x}\right)z = -4x^4 \quad --- (**)$

linear in z , $p(x) = -\frac{8}{x}$

$$\text{I.F.} = e^{\int -\frac{8}{x}dx}$$

$$= e^{-8\ln x} = e^{\ln x^{-8}} = x^{-8}$$

Eq. (**) is multiplied by $(x^{-8}\,dx)$, gets :

$$(x^{-8}\,dz - 8x^{-9}z\,dx) = -4x^{-4}\,dx$$

$$\int d(x^{-8}\cdot z) = \int -4x^{-4}\,dx$$

$$x^{-8}z = -\frac{4}{-3}x^{-3} + c$$

$$x^{-8}y^{-2} = \frac{4}{3}x^{-3} + c$$

Second Order Differential Equations :

1- Reducible to 1^{st} order قابلة للاختزال الى الرتبة الأولى

2- Homogeneous $a\ddot{y} + b\dot{y} + cy = 0$ متجانسة

3- non-homogeneous غير متجانسة

① Reducible to 1^{st} order :

$\dot{y} = P$ وهي حالة خاصة من المعادلات التي عندها نفرض أن

ثم نجري عملية التكامل ان أمكن

ex-9 : Solve $\ddot{y} - x(\dot{y})^2 = 0$.

Soln : put $\dot{y} = P \Rightarrow \ddot{y} = \frac{dP}{dx}$

$\therefore \frac{dP}{dx} - xP^2 = 0$ separable

$\therefore \int \frac{dP}{P^2} = \int x\,dx \Rightarrow -\frac{1}{P} = \frac{x^2}{2} + c$

$\therefore P = \frac{-1}{\frac{x^2}{2} + c} \Rightarrow \frac{dy}{dx} = \frac{-1}{\frac{x^2}{2} + c}$ separable

$\therefore \int dy = -\frac{1}{c} \int \frac{\sqrt{2c} * \frac{dx}{\sqrt{2c}}}{1 + (\frac{x}{\sqrt{2c}})^2}$

$\therefore y = -\sqrt{\frac{2}{c}} \tan^{-1}(\frac{x}{\sqrt{2c}}) + k$

2. **Homogeneous :** أية معادلة تكتب بالشكل التالي

$$a\ddot{y} + b\dot{y} + cy = 0 \quad \text{——(1)}$$

or $$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

or $$aD^2y + bDy + cy = 0 \qquad \text{where } D = \frac{d}{dx}$$

let $y = e^{mx}$, $\dot{y} = me^{mx}$, $\ddot{y} = m^2 e^{mx}$

نعوض في معادلة (1) نحصل

$$am^2 e^{mx} + bme^{mx} + ce^{mx} = 0$$

$$(am^2 + bm + c)e^{mx} = 0$$

$$e^{mx} \neq 0 \quad \therefore \quad am^2 + bm + c = 0 \quad \text{——(2)}$$

معادلة (2) تسمى المعادلة المميزة (characteristic eq.)

و للمعادلة المميزة جذران هما m_1 و m_2 و للجذرين m_1, m_2

ثلاث حالات :-

① If $m_1 \neq m_2 \Rightarrow y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

② If $m_1 = m_2 = m \Rightarrow y = c_1 e^{mx} + c_2 x e^{mx}$

③ If $\left.\begin{matrix} m_1 = \alpha + i\beta \\ m_2 = \alpha - i\beta \end{matrix}\right\} \Rightarrow y = e^{\alpha x}\{c_1 \sin\beta x + c_2 \cos\beta x\}$

where $i = \sqrt{-1}$

ex. 10: solve $\ddot{y} + 4\dot{y} + 3y = 0$

sol$\underline{n}$ let $y = e^{mx}$ $\Rightarrow m^2 + 4m + 3 = 0$

$(m+3)(m+1) = 0 \Rightarrow m_1 = -3$ & $m_2 = -1$

$\therefore \quad y = c_1 e^{-3x} + c_2 e^{-x}$

ex-11: Solve $\ddot{y}-4\dot{y}+4y=0$

soln: let $y=e^{mx} \Longrightarrow m^2-4m+4=0$

$(m-2)(m-2)=0 \Longrightarrow m_1=m_2=2$

$y=c_1 e^{2x}+c_2 x e^{2x}$

ex-12: Solve $\ddot{y}+2\dot{y}+5y=0$

characteristic eq. $m^2+2m+5=0$

$m_{1,2}=\frac{-b\mp\sqrt{b^2-4ac}}{2a}=\frac{-2\mp\sqrt{4-4\times1\times5}}{2}$

$=-1\mp 2i \qquad \alpha=-1 \; , \; \beta=2$

$\therefore \quad y=e^{(-1)x}\{c_1 \sin 2x+c_2 \cos 2x\}$

ex-13: solve $\ddot{y}-6\dot{y}+9y=0$

soln $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=0$

or $D^2y-6Dy+9y=0$

or $(D^2-6D+9)y=0$

$(D-3)^2y=0$

نفس الحل — نكتب m بدل D

let $y=e^{mx} \Longrightarrow m^2-6m+9=0$

$(m-3)^2=0 \Longrightarrow m_1=m_2=3$

$\therefore \quad y=c_1 e^{3x}+c_2 x e^{3x}$

3- Non-homogeneous :

أي معادلة تكتب بالشكل التالي :

$$a\ddot{y} + b\dot{y} + cy = f(x)$$

لهذه المعادلة حلان :-

① الحل التجانس (y_h) homogeneous solution

let $a\ddot{y} + b\dot{y} + cy = 0$

② كل الخاص (y_p) particular solution

$$y = y_h + y_p$$

طريقة أيجاد الحل الخاص

1. Undetermined Coefficients Method

* طريقة المعاملات غير المعنية :

أذا أحتوت المعادلة على أحد هذه الدوال في الطرف الثاني فأن y_p تكون كما موضح في الجدول

$f(x)$	y_p
① e^{ax}	$K e^{ax}$ ⇒ اذا كانت e^{ax} غير موجودة في y_h $Kx e^{ax}$ ⇒ اذا كانت e^{ax} موجودة في y_h مرة واحدة $Kx^2 e^{ax}$ ⇒ اذا كانت e^{ax} موجودة في y_h مرتان
② x^n	$Ax^n + Bx^{n-1} + \dots + K$
$2x^2+1$	$Ax^2 + Bx + C$
③ $\sin ax$ $\cos ax$ $2\sin ax \mp 3\cos ax$	$A\sin ax + B\cos ax$
④	$x\{A\sin ax + B\cos ax\}$ ⇒ اذا كانت دالة الـ $\sin ax$ او الـ $\cos ax$ موجودة في الـ y_h

ex. 14 : solve $y'' - 2y' + y = 3e^{2x} - 5e^{4x}$

soln : characteristics eq. is

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0 \implies m_1 = m_2 = 1$$

$$y_h = c_1 e^x + c_2 x e^x$$

let

$$\left.\begin{aligned} y_p &= K e^{2x} + H e^{4x} \\ y_p' &= 2K e^{2x} + 4H e^{4x} \\ y_p'' &= 4K e^{2x} + 16H e^{4x} \end{aligned}\right\}$$ نعوض في المعادلة الأصلية

$$(4Ke^{2x} + 16He^{4x}) - 2(2Ke^{2x} + 4He^{4x}) + (Ke^{2x} + He^{4x}) = 3e^{2x} - 5e^{4x}$$

$$\therefore 4Ke^{2x} + 16He^{4x} - 4Ke^{2x} - 8He^{4x} + Ke^{2x} + He^{4x} = 3e^{2x} - 5e^{4x}$$

$$\implies 9H e^{4x} + K e^{2x} = 3e^{2x} - 5e^{4x}$$

$$\therefore 9H = -5 \implies H = -\frac{5}{9}$$

$$\& \quad K = 3$$

$$\therefore y_p = 3e^{2x} - \frac{5}{9} e^{4x}$$

$$\therefore y = c_1 e^x + c_2 x e^x + 3e^{2x} - \frac{5}{9} e^{4x}$$

ex.15: Solve $y'' - 6y' + 9y = 2e^{3x}$

Sol$\underline{\underline{n}}$

$$y_h'' - 6y_h' + 9y_h = 0$$

$$m^2 - 6m + 9 = 0$$

$$(m-3)(m-3) = 0 \Rightarrow m_1 = m_2 = 3$$

$$\therefore\ y_h = c_1 e^{3x} + c_2 x e^{3x}$$

$$f(x) = 2e^{3x} \longrightarrow \quad (\text{مكررتان في } y_h)$$

$$\therefore\ \text{let}\quad y_p = Kx^2 e^{3x}$$

$$y_p' = 2Kx e^{3x} + 3Kx^2 e^{3x}$$

$$y_p'' = 2K(3xe^{3x} + e^{3x}) + 3K(3x^2 e^{3x} + 2xe^{3x})$$

نعوض عن y_p , y_p' , y_p'' في المعادلة الأصلية

$$(9Kx^2 e^{3x} + 12Kxe^{3x} + 2Ke^{3x}) - 6(2Kxe^{3x} + 3Kx^2 e^{3x}) + 9(Kx^2 e^{3x}) = 2e^{3x}$$

$$2Ke^{3x} = 2e^{3x} \Rightarrow K = 1$$

$$\therefore\ y_p = x^2 e^{3x}$$

$$\therefore\ y = c_1 e^{3x} + c_2 x e^{3x} + x^2 e^{3x}$$

H.W.

ex.16: Solve $y'' - 4y' + 3y = 5e^{3x}$

Ans.: $y = c_1 e^{3x} + c_2 e^{x} + \frac{5}{2} x e^{3x}$

ex.17: Solve $y'' - 4y' - 5y = 2\sin 2x$

Soln: $y'' - 4y' - 5y = 0$

ch. eq. $m^2 - 4m - 5 = 0$

$(m-5)(m+1) = 0 \Rightarrow m_1 = 5 \;\&\; m_2 = -1$

$\therefore \; y_h = c_1 e^{5x} + c_2 e^{-x}$

let $y_p = A\sin 2x + B\cos 2x$

$y_p' = 2A\cos 2x - 2B\sin 2x$

$y_p'' = -4A\sin 2x - 4B\cos 2x$

نعوضها في المعادلة الأصلية

$(-4A\sin 2x - 4B\cos 2x) - 4(2A\cos 2x - 2B\sin 2x)$

$-5(A\sin 2x + B\cos 2x) = 2\sin 2x$

$\Rightarrow (-4A + 8B - 5A)\sin 2x + (-4B - 8A - 5B)\cos 2x$

$= 2\sin 2x$

نقارن المعاملات

$\sin 2x: \quad 8B - 9A = 2$

$\cos 2x: \quad -9B - 8A = 0$

$\Rightarrow A = -\frac{18}{145} \;\&\; B = \frac{16}{145}$

$\therefore \; y = y_h + y_p$

$= c_1 e^{5x} + c_2 e^{-x} - \frac{18}{145}\sin 2x + \frac{16}{145}\cos 2x$

ex.18: Solve $y'' + y' - 2y = 2x - 5x^3$

Soln $y'' + y' - 2y = 0$

ch. eq. $m^2 + m - 2 = 0 \Rightarrow (m+2)(m-1) = 0$

$\therefore \; m_1 = 1 \;\&\; m_2 = -2$

$\therefore \; y_h = c_1 e^{x} + c_2 e^{-2x}$

let $y_p = Ax^3 + Bx^2 + Cx + D$

$\bar{y}_p = 3Ax^2 + 2Bx + C$

$\bar{\bar{y}}_p = 6Ax + 2B$

تعوض في المعادلة الأصلية

$$(6Ax + 2B) + (3Ax^2 + 2Bx + C) - 2(Ax^3 + Bx^2 + Cx + D) = 2x - 5x^3$$

نقارن المعاملات

x^3 : $-2A = -5 \Rightarrow A = \frac{5}{2}$

x^2 : $3A - 2B = 0 \Rightarrow B = \frac{15}{4}$

x : $6A + 2B - 2C = 2 \Rightarrow C = \frac{41}{4}$

x^0 : $2B + C + 2D = 0 \Rightarrow D = \frac{71}{8}$

$\therefore \quad y_p = \frac{5}{2}x^3 + \frac{15}{4}x^2 + \frac{41}{4}x + \frac{71}{8}$

$\therefore \quad y = y_h + y_p$

$$= c_1 e^{x} + c_2 e^{-2x} + \frac{5}{2}x^3 + \frac{15}{4}x^2 + \frac{31}{4}x + \frac{71}{8}$$

2. Variation of Parameters : طريقة تغيير الثوابت

و هي طريقة عامة إلا أنها تكون في بعض الأحيان صعبة بسبب صعوبة التكامل .

$$a\bar{\bar{y}} + b\bar{y} + cy = f(x)$$

$y_h = c_1 y_1 + c_2 y_2$

let $y_p = v_1(x)\, y_1 + v_2(x)\, y_2$

شرط أساسية لإيجاد v_2 & v_1

$$\boxed{\begin{aligned} \bar{v}_1 y_1 + \bar{v}_2 y_2 &= 0 \quad (A) \\ \bar{v}_1 \bar{y}_1 + \bar{v}_2 \bar{y}_2 &= f(x) \quad (B) \end{aligned}}$$

نستخدم قاعدة كرامر (Cramer Rule) لإيجاد $V_1(x)$ & $V_2(x)$ من معادلتي Ⓐ و Ⓑ

$$V_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \quad \& \quad V_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

ex. 19 : Solve $y'' + y = \sec x$

Sol$\underline{n}$: لهذه المعادلة لايمكن إيجاد الحل باستخدام طريقة المعاملات غير المعينة.

نجد الحل المتجانس $\Rightarrow \quad y'' + y = 0 \quad \Rightarrow$ ch. eq. $m^2 + 1 = 0$

$\therefore \quad m_1 = 0 + i \quad , \quad m_2 = 0 - i$

$\alpha = 0 \quad , \quad \beta = 1$

$\therefore \quad y_h = e^{0 \cdot x} \{ c_1 \sin x + c_2 \cos x \} \quad \therefore$

$y_h = c_1 \sin x + c_2 \cos x$

i.e. $y_h = c_1 y_1 + c_2 y_2$

let $y_p = V_1 \sin x + V_2 \cos x$

$\therefore \quad V_1' \sin x + V_2' \cos x = 0$ —— Ⓐ

$V_1' \cos x + V_2' (-\sin x) = \sec x$ —— Ⓑ

$$V_1' = \frac{\begin{vmatrix} 0 & \cos x \\ \sec x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{-\cos x \cdot \sec x}{-\sin^2 x - \cos^2 x} = \frac{-1}{-1} = 1$$

$\therefore \quad V_1' = 1 \quad \therefore \quad V_1 = x$

$$V_2' = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \sec x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{\sin x \cdot \sec x}{-1} = -\sin x \cdot \frac{1}{\cos x}$$

$$\therefore V_2 = \int \frac{-\sin x}{\cos x}\, dx = \ln|\cos x| + c$$

$$\therefore \quad y_p = V_1 \sin x + V_2 \cos x$$
$$= x \sin x + (\ln|\cos x|)\cos x$$

$$\therefore \quad y = y_h + y_p$$

** —————————— **

<u>Ex</u>. 20 : Solve $y'' - 6y' + 9y = 2e^{3x}$

Sol<u>n</u> : $y_h \Rightarrow y'' - 6y' + 9y = 0$

ch. eq. $\Rightarrow m^2 - 6m + 9 = 0$

$(m-3)(m-3) = 0 \Rightarrow m_1 = m_2 = 3$

$$\therefore \quad y_h = c_1 e^{3x} + c_2 x e^{3x}$$

let $y_p = V_1 e^{3x} + V_2 x e^{3x}$

$$\therefore \quad V_1' e^{3x} + V_2' x e^{3x} = 0 \quad \text{—— (1)}$$
$$V_1'(3e^{3x}) + V_2'(3xe^{3x} + e^{3x}) = 2e^{3x} \quad \text{—— (2)}$$

بقسمة معادلتي (1) و (2) على e^{3x} نحصل :

$$V_1' + xV_2' = 0 \quad \text{—— (A)}$$
$$V_1' 3 + (3x+1)V_2' = 2 \quad \text{—— (B)}$$

$$\bar{V}_1 = \frac{\begin{vmatrix} 0 & x \\ 2 & 3x+1 \end{vmatrix}}{\begin{vmatrix} 1 & x \\ 3 & 3x+1 \end{vmatrix}} = \frac{0-2x}{3x+1-3x} = -2x$$

$$\therefore V_1 = \int -2x\,dx \Rightarrow V_1 = -x^2$$

$$\bar{V}_2 = \frac{\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & x \\ 3 & 3x+1 \end{vmatrix}} = \frac{2-0}{1} = 2 \Rightarrow V_2 = 2x$$

$$\begin{aligned} \therefore y_p &= V_1 e^{3x} + V_2\, x e^{3x} \\ &= -x^2 e^{3x} + (2x)\, x e^{3x} \\ &= -x^2 e^{3x} + 2x^2 e^{3x} \\ &= x^2 e^{3x} \end{aligned}$$

$$\begin{aligned} \therefore y &= y_h + y_p \\ &= c_1 e^{3x} + c_2 x e^{3x} + x^2 e^{3x} \end{aligned}$$

3. Laplace Transformation method
4. D. operator method
5. Series solution

Higher Order Differential Equations :

ex. 21 : Solve $(D-3)(D^2-3D+2)y=0$

Soln ch. eq. $(m-3)(m^2-3m+2)=0$

$(m-3)(m-2)(m-1)=0$

$\therefore \quad m_1 = 1 \;,\; m_2 = 2 \;,\; m_3 = 3$

$\therefore \quad y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$

ex. 22 : Solve $(D-2)(D+4)^3 y = 0$

Soln ch. eq. $(m-2)(m+4)^3 = 0$

$\therefore \quad m_1 = 2 \;\&\; m_2 = m_3 = m_4 = -4$

$\therefore \quad y = c_1 e^{2x} + c_2 e^{-4x} + c_3 x e^{-4x} + c_4 x^2 e^{-4x}$

$= c_1 e^{2x} + (c_4 x^2 + c_3 x + c_2) e^{-4x}$

ex. 23 : Solve $(D-4)(D^2+4)y = 0$

Soln ch. eq. $(m-4)(m^2+4) = 0$

$\therefore \quad m_1 = 4 \;\&\; m_{2,3} = 0 \mp 2i$

$\therefore \quad y = c_1 e^{4x} + e^{0 \cdot x}\left\{ c_2 \sin 2x + c_3 \cos 2x \right\}$

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Mechanical Engineering Department
Advance Engineering Mathematics
Chapter ( ): Partial Differentiation
Dr. Akeel Abdullah Mohammed

Partial Differentiation :

$$w = f(x, y)$$

$$\frac{\partial w}{\partial x} = w_x = f_x = \frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

$$\frac{\partial w}{\partial y} = w_y = f_y = \frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

ex1: If $w = x^3 + 2x^4 y^5 + 5y^3 + \sin\left(\frac{x}{y}\right)$ then find w_x & w_y.

Soln

$$w_x = 3x^2 + 2y^5(4x^3) + 0 + \cos\left(\frac{x}{y}\right) * \frac{1}{y}$$

$$w_y = 0 + 2x^4(5y^4) + 15y^2 + \cos\left(\frac{x}{y}\right) * \left(-\frac{x}{y^2}\right)$$

ex.2: If $w = f\left(\frac{x}{y}\right)$ then show that $x w_x + y w_y = 0$

Soln

$$w_x = \acute{f}\left(\frac{x}{y}\right) \cdot \frac{1}{y} = \frac{1}{y} \acute{f}\left(\frac{x}{y}\right)$$

$$w_y = \acute{f}\left(\frac{x}{y}\right) \cdot \left(-\frac{x}{y^2}\right) = -\frac{x}{y^2} \acute{f}\left(\frac{x}{y}\right)$$

$$\therefore x w_x + y w_y = x \cdot \left(\frac{1}{y} \acute{f}\left(\frac{x}{y}\right)\right) + y\left(-\frac{x}{y^2} \acute{f}\left(\frac{x}{y}\right)\right)$$

$$= \frac{x}{y} \acute{f}\left(\frac{x}{y}\right) - \frac{x}{y} \acute{f}\left(\frac{x}{y}\right) = 0$$

$\underline{\underline{ex}}$. 3 : If $w = x^n f\left(\frac{x^2}{x^2+y^2}\right)$ then show that :

$$x\, w_x + y\, w_y = n\, w$$

Sol$\underline{n}$: $w_x = x^n \bar{f}\left(\frac{x^2}{x^2+y^2}\right) * \frac{(x^2+y^2)2x - x^2(2x)}{(x^2+y^2)^2} +$

$$f\left(\frac{x^2}{x^2+y^2}\right) * n\, x^{n-1}$$

$$= \frac{2y^2 x^{n+1}}{(x^2+y^2)^2} \bar{f}\left(\frac{x^2}{x^2+y^2}\right) + n\, x^{n-1} f\left(\frac{x^2}{x^2+y^2}\right) \quad \text{—(}$$

$$w_y = x^n \bar{f}\left(\frac{x^2}{x^2+y^2}\right) * x^2(-1)(x^2+y^2)^{-2} * 2y$$

$$= -\frac{2y\, x^{n+2}}{(x^2+y^2)^2} \bar{f}\left(\frac{x^2}{x^2+y^2}\right) \quad \text{———— (2}$$

$$x\, w_x + y\, w_y = x \cdot \left\{\frac{2y^2 x^{n+1}}{(x^2+y^2)^2} \bar{f}\left(\frac{x^2}{x^2+y^2}\right) + n\, x^{n-1} f\left(\frac{x^2}{x^2+y^2}\right)\right.$$

$$+ y \cdot \left\{ \frac{-2y\, x^{n+2}}{(x^2+y^2)^2} \bar{f}\left(\frac{x^2}{x^2+y^2}\right) \right\}$$

$$= \frac{2y^2 x^{n+2}}{(x^2+y^2)^2} \bar{f}\left(\frac{x^2}{x^2+y^2}\right) + n\, x^n f\left(\frac{x^2}{x^2+y^2}\right)$$

$$- \frac{2y^2 x^{n+2}}{(x^2+y^2)^2} \bar{f}\left(\frac{x^2}{x^2+y^2}\right)$$

$$= n\, x^n f\left(\frac{x^2}{x^2+y^2}\right)$$

$$= n\, w$$

ex. 4 : If $xy^2z^3 + x^3z + y^3z^2 = 2$ then find $\frac{\partial z}{\partial y}, \frac{\partial y}{\partial x}$

and show that $\frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x} \cdot \frac{\partial x}{\partial z} = -1$

Sol$\underline{n}$:

$$x\left[y^2 \cdot 3z^2 \cdot \frac{\partial z}{\partial y} + z^3 \cdot 2y\right] + x^3 \cdot \frac{\partial z}{\partial y} + \left\{y^3 \cdot 2z \cdot \frac{\partial z}{\partial y} + z^2 \cdot 3y^2\right\} = 0$$

$$\therefore \frac{\partial z}{\partial y} = -\frac{2xyz^3 + 3y^2z^2}{3xz^2y^2 + x^3 + 2zy^3} \qquad \text{———— (1)}$$

$$z^3\left[x \cdot 2y \cdot \frac{\partial y}{\partial x} + y^2 \cdot 1\right] + z \cdot 3x^2 + z^2 \cdot 3y^2 \cdot \frac{\partial y}{\partial x} = 0$$

$$\therefore \frac{\partial y}{\partial x} = -\frac{y^2z^3 + 3x^2z}{2xyz^3 + 3z^2y^2} \qquad \text{———— (2)}$$

$$y^2\left\{x \cdot 3z^2 + z^3 \cdot \frac{\partial x}{\partial z}\right\} + \left\{x^3 + z \cdot 3x^2 \cdot \frac{\partial x}{\partial z}\right\} + y^3 \cdot 2z = 0$$

$$\therefore \frac{\partial x}{\partial z} = -\frac{2y^3z + x^3 + 3xy^2z^2}{z^3y^2 + 3x^2z} \qquad \text{———— (3)}$$

by multiplying Eqns (1), (2) & (3) in each to other, gets:

$$\frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x} \cdot \frac{\partial x}{\partial z} = -1$$

رموز المشتقة الجزئية

$$w_{xx} = \frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}(w_x)$$

Theorem : If $f(x,y)$ is contineous with a contineous partial derivative then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \qquad (\text{or} \quad f_{xy} = f_{yx})$$

ex. 5 : If $w = x^2 + 3x^4y^3$ then ① Find w_{xxyx} ② show that $w_{xy} = w_{yx}$ ③ Prove that $w_{yyyy} = 0$

Sol: ①

$$w = x^2 + 3x^4y^3$$
$$w_x = 2x + 12x^3y^3$$
$$w_{xx} = 2 + 36x^2y^3$$
$$w_{xxy} = 0 + 108x^2y^2$$
$$w_{xxyx} = 216xy^2$$

②

$$\left.\begin{aligned} w_{xy} &= 36x^3y^2 \\ w_y &= 9x^4y^2 \\ w_{yx} &= 36x^3y^2 \end{aligned}\right\} \therefore \quad w_{yx} = w_{xy}$$

③

$$w = x^2 + 3x^4y^3$$
$$w_y = 9x^4y^2$$
$$w_{yy} = 18x^4y$$
$$w_{yyy} = 18x^4$$
$$w_{yyyy} = 0$$

ex. 6 : If $z = f(2u+3v, 3u-2v)$ then show that :

$$\frac{\partial^2 z}{\partial u \partial v} = 6\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2}$$

where $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ and $x = 2u+3v$ & $y = 3u-2v$

Soln let $x = 2u+3v$ & $y = 3u-2v$

$\therefore \quad z = f(x, y)$

$$\therefore \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$z_u = z_x(2) + z_y(3)$$

$$\frac{\partial^2 z}{\partial u \partial v} = \frac{\partial}{\partial v}(z_u) = \frac{\partial}{\partial v}(2z_x + 3z_y)$$

$$= \frac{\partial}{\partial x}(2z_x + 3z_y)\cdot\frac{\partial x}{\partial v} + \frac{\partial}{\partial y}(2z_x + 3z_y)\cdot\frac{\partial y}{\partial v}$$

$$= (2z_{xx} + 3z_{yx})\cdot(3) + (2z_{xy} + 3z_{yy})\cdot(-2)$$

$$= 6z_{xx} + 9z_{yx} - 4z_{xy} - 6z_{yy}$$

$$= 6z_{xx} + 5z_{xy} - 6z_{yy}$$

or

$$= 6\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2}$$

Total Differential :

التفاضل الكلي

$$\boxed{d\,(f(x,y)) = \frac{\partial f}{\partial x}\cdot dx + \frac{\partial f}{\partial y}\cdot dy}$$

where dx & dy tend to zero

ex. a $d(x^3y^5) = \frac{\partial}{\partial x}(x^3y^5)\,dx + \frac{\partial}{\partial y}(x^3y^5)\,dy$

$= 3x^2y^5\,dx + 5x^3y^4\,dy$

ex. b $d(u^2v^4) = \frac{\partial}{\partial u}(u^2v^4)\,du + \frac{\partial}{\partial v}(u^2v^4)\,dv$

$= 2uv^4\,du + 4u^2v^3\,dv$

ex. 7 : If $z = f(x,y)$ and $g(x,y) = c$ then show that :

$$\frac{dz}{dx} = \frac{g_y f_x - f_y g_x}{g_y}$$

Soln $z = f(x,y) \Rightarrow dz = d(f(x,y)) \Rightarrow$

$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \Rightarrow dz = f_x\,dx + f_y\,dy$

$\therefore \quad \frac{dz}{dx} = f_x + f_y\left(\frac{dy}{dx}\right)$ ——— (1)

$g(x,y) = c \Rightarrow d(g(x,y)) = d(c)$

$\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = 0 \Rightarrow g_x\,dx + g_y\,dy = 0$

$\therefore \quad \frac{dy}{dx} = -\frac{g_x}{g_y}$ ——— (2)

by substituting Eq.(2) into Eq.(1), gets :

$$\frac{dz}{dx} = f_x + f_y\left(-\frac{g_x}{g_y}\right) = \frac{g_y f_x - f_y g_x}{g_y}$$

Transformation :

ex. 8 : If $x = e^u \cos v$ & $y = e^u \sin v$, then find $\frac{\partial u}{\partial x}$.

Sol$\underline{n}$ Note : $\frac{\partial u}{\partial x} \neq \frac{1}{(\frac{\partial x}{\partial u})}$

method ① : Inverse Transformation طريقة التحويل العكسي

$\frac{\partial x}{\partial u} = e^u \cos v$; $\frac{\partial x}{\partial v} = -e^u \sin v$

$\frac{\partial y}{\partial u} = e^u \sin v$; $\frac{\partial y}{\partial v} = e^u \cos v$

hence ;

$x = e^u \cos v \Rightarrow x^2 = e^{2u} \cos^2 v$

$y = e^u \sin v \Rightarrow y^2 = e^{2u} \sin^2 v$

بالجمع

$$x^2 + y^2 = e^{2u} (\cos^2 v + \sin^2 v) = e^{2u}$$

$\therefore 2u = \ln(x^2 + y^2) \Rightarrow u = \frac{1}{2} \ln(x^2 + y^2)$

& $\frac{\sin v}{\cos v} = \frac{y}{x} \Rightarrow \tan v = \frac{y}{x} \Rightarrow v = \tan^{-1} \frac{y}{x}$

$$\therefore \boxed{\begin{aligned} u &= \frac{1}{2} \ln(x^2+y^2) \\ v &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}}$$ Inverse Transformation

$\therefore \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} \Rightarrow \frac{e^u \cos v}{e^{2u}}$ $\therefore \boxed{\frac{\partial u}{\partial x} = \frac{\cos v}{e^u}}$

Note : Polar Transformation

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Inverse Polar Transformation

$$r = \sqrt{x^2 + y^2} \quad , \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

method ② : General Method : It is the best but in which the solution is long " الطريقة العامة "

$$x = e^u \cos v \quad \Rightarrow \quad dx = d(e^u \cos v)$$

$$dx = \frac{\partial}{\partial u}(e^u \cos v)\,du + \frac{\partial}{\partial v}(e^u \cos v)\,dv$$

$$dx = e^u \cos v\,du - e^u \sin v\,dv \quad \text{——— (1)}$$

$$y = e^u \sin v \quad \Rightarrow \quad dy = d(e^u \sin v)$$

$$dy = \frac{\partial}{\partial u}(e^u \sin v)\,du + \frac{\partial}{\partial v}(e^u \sin v)\,dv$$

$$dy = e^u \sin v\,du + e^u \cos v\,dv \quad \text{——— (2)}$$

by using Gramer's Rule between Eqns.(1) & (2)

$$e^u \cos v\,du - e^u \sin v\,dv = dx \quad \text{——— (1)}$$
$$e^u \sin v\,du + e^u \cos v\,dv = dy \quad \text{——— (2)}$$

$$du = \frac{\begin{vmatrix} dx & -e^{u}\sin v \\ dy & e^{u}\cos v \end{vmatrix}}{\begin{vmatrix} e^{u}\cos v & -e^{u}\sin v \\ e^{u}\sin v & e^{u}\cos v \end{vmatrix}}$$

$$dv = \frac{\begin{vmatrix} e^{u}\cos v & dx \\ e^{u}\sin v & dy \end{vmatrix}}{\begin{vmatrix} e^{u}\cos v & -e^{u}\sin v \\ e^{u}\sin v & e^{u}\cos v \end{vmatrix}}$$

$$\therefore \quad du = \frac{e^{u}\cos v\,dx + e^{u}\sin v\,dy}{e^{2u}\cos^2 v + e^{2u}\sin^2 v} = \frac{e^{u}\cos v\,dx + e^{u}\sin v\,dy}{e^{2u}}$$

$$= \frac{\cos v}{e^{u}}\,dx + \frac{\sin v}{e^{u}}\,dy$$

$$= \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\cos v}{e^{u}} \quad \& \quad \frac{\partial u}{\partial y} = \frac{\sin v}{e^{u}}$$

<u>Home work</u> : If $x = f(u, v)$ & $y = g(u, v)$, then show that

$$\frac{\partial u}{\partial x} = \frac{g_v}{g_v\, f_u - f_v\, g_u}$$

Hint : use general method

Chain Rule :

قاعدة التفاضل المتسلسل

law (1) : If $w = f(x,y)$ and $x = g(t)$, $y = h(t)$

then $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial w}{\partial y} \cdot \dfrac{dy}{dt}$

law (2) : If $w = f(x,y)$ and $x = g(t,r)$ & $y = h(t,r)$

then $$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t}$$

ex. 9 : If $w = u^3 + v^5 + uv$, $u = \sin r$, $v = \cos r$

then find $\dfrac{dw}{dr}$

Sol$\underline{n}$ $$\frac{dw}{dr} = \frac{\partial w}{\partial u} \cdot \frac{du}{dr} + \frac{\partial w}{\partial v} \cdot \frac{dv}{dr}$$

$$= (3u^2 + v)\cos r + (5v^4 + u)(-\sin r)$$

ex. 10 : If $w = f(x-y, y-z, z-x)$ then show that

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Sol$\underline{n}$ put $u = x - y$, $v = y - z$. $t = z - x$

$\therefore$ $w = f(u, v, t)$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial w}{\partial t} \cdot \frac{\partial t}{\partial x}$$

$$= w_u \cdot (1) + w_v \cdot (0) + w_t \cdot (-1)$$

$$= w_u - w_t \qquad (1)$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= w_u(-1) + w_v(1) + w_t(0)$$

$$= w_v - w_u \qquad \text{———————— (2)}$$

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= w_u(0) + w_v(-1) + w_t(0)$$

$$= w_t - w_v \qquad \text{———————— (3)}$$

$$\therefore \quad \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = Eq.(1) + Eq.(2) + Eq.(3) = 0$$

* ~~ * ~.~. * ~.~. * ~.~. *

Gradient Vector :

$$\vec{\nabla f} = \overrightarrow{\text{grad} f} = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

$$\vec{\nabla}(\ \ ) = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$$

$\vec{\nabla}$ = Del_operator

Note : $D = \frac{d}{dx}$ = differential operator

ex. 11 : If $f(x,y,z) = x^3 y + z^5 x$ then find $\vec{\nabla f}$

Soln $\frac{\partial f}{\partial x} = 3x^2 y + z^5$; $\frac{\partial f}{\partial y} = x^3$; $\frac{\partial f}{\partial z} = 5z^4 x$

$$\vec{\nabla f} = (3x^2 y + z^5) i + x^3 j + 5z^4 x \ k$$

Divergence Vector

$\vec{\nabla} \cdot \vec{F}$ = Divergence of $\vec{F}$

$\vec{F}(x,y,z) = f_1(x,y,z)\,i + f_2(x,y,z)\,j + f_3(x,y,z)\,k$

= Vector Function

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(f_1) + \frac{\partial}{\partial y}(f_2) + \frac{\partial}{\partial z}(f_3)$$

Ex.12: Find div$(\vec{F})$ for:

① $\vec{F}(x,y,z) = x^3y\,i + yz^5\,j + xz^2\,K$

② $\vec{F}(x,y,z) = x\,i + y\,j + z\,K$

Soln ① $\text{div}\,\vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(x^3y) + \frac{\partial}{\partial y}(yz^5) + \frac{\partial}{\partial z}(xz^2)$

$= 3x^2y + z^5 + 2xz$

② $\text{div}\,\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$

$= 1 + 1 + 1$

$= 3$

Curl $\vec{F}$

$$\text{curl}\,\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= +\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)i - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)j + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)K$$

Directional Derivative : المشتقة الاتجاهية

$$\boxed{\frac{df}{ds} = D_{\vec{u}} = \vec{\nabla f} \cdot \vec{u}}$$

where $\vec{u}$ is a unit vector

Theorem :

1. $\text{Max.}\left(\frac{df}{ds}\right) = |\vec{\nabla f}|$

2. $\text{Min.}\left(\frac{df}{ds}\right) = -|\vec{\nabla f}|$

and the direction is $\vec{u} = \frac{\vec{\nabla f}}{|\vec{\nabla f}|}$

Proof :

$$\frac{df}{ds} = \vec{\nabla f} \cdot \vec{u}$$

$$= |\vec{\nabla f}|\ \overbrace{|\vec{u}|}^{1} \cos\theta$$

$$= |\vec{\nabla f}| \cos\theta$$

but

$$-1 \leq \cos\theta \leq 1$$

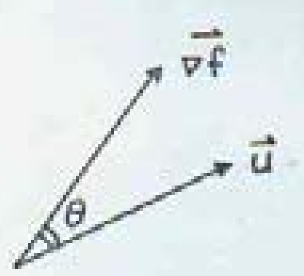
$$-|\vec{\nabla f}| \leq |\vec{\nabla f}| \cos\theta \leq |\vec{\nabla f}|$$

$$-|\vec{\nabla f}| \leq \frac{df}{ds} \leq |\vec{\nabla f}|$$

$\therefore$ $\text{Max.}\left(\frac{df}{ds}\right) = |\vec{\nabla f}|$ when $\theta = 0$

$\text{Min.}\left(\frac{df}{ds}\right) = -|\vec{\nabla f}|$

if $\theta = 0$ then $\vec{u} \parallel \vec{\nabla f}$



$\therefore$ $\vec{\nabla f} = t\ \vec{u}$

$|\vec{\nabla f}| = t\ |\vec{u}|$

$\therefore \quad t = |\vec{\nabla} f|$

$\therefore \quad \vec{\nabla} f = |\vec{\nabla} f| \ \vec{u}$

$$\therefore \quad \vec{u} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|}$$

ex. 13 : let $w = f(x, y, z) = x^2 + xy + z^3$ and let $P_1(2, 1, 1)$, Find

1. The maximum value of the directional derivative of f at P_1 (what is the direction)
2. The value of the directional derivative at P_1 towards $P_2(5, 4, 2)$.

Soln

$$\frac{\partial f}{\partial x} = 2x + y \qquad \left.\frac{\partial f}{\partial x}\right|_{P_1} = 5$$

$$\frac{\partial f}{\partial y} = x \qquad \left.\frac{\partial f}{\partial y}\right|_{P_1} = 2$$

$$\frac{\partial f}{\partial z} = 3z^2 \qquad \left.\frac{\partial f}{\partial z}\right|_{P_1} = 3$$

$\therefore \quad \vec{\nabla} f = 5i + 2j + 3K$

$$\text{Max.}\left(\frac{df}{ds}\right) = |\vec{\nabla} f| = \sqrt{5^2 + 2^2 + 3^2} = \sqrt{38}$$

The direction $\quad \vec{u} = \dfrac{\vec{\nabla} f}{|\vec{\nabla} f|} = \dfrac{1}{\sqrt{38}}(5i + 2j + 3K)$

2. $\overrightarrow{P_1 P_2} = 3i + 3j + K$

$$\vec{u} = \frac{\overrightarrow{P_1 P_2}}{|\overrightarrow{P_1 P_2}|} = \frac{3i + 3j + K}{\sqrt{3^2 + 3^2 + 1^2}} = \frac{1}{\sqrt{19}}(3i + 3j + K)$$

$$\frac{df}{ds} = \vec{\nabla} f \cdot \vec{u} = (5i + 2j + 3K) \cdot \frac{1}{\sqrt{19}}(3i + 3j + K)$$

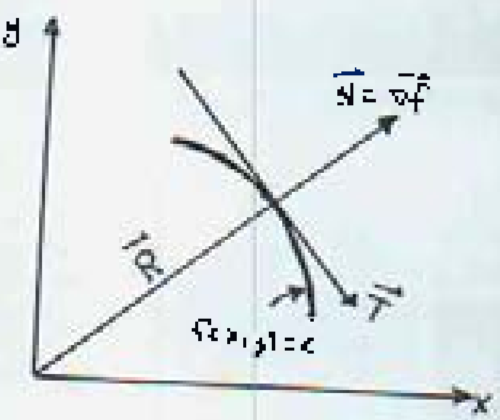
$$= \frac{24}{\sqrt{19}}$$

Equation of Tangent Plane and Normal line to the Surface $f(x,y,z) = c$

Theorem (1) : If $w = f(x,y)$ then the vector

$$\vec{\nabla f} = \frac{\partial f}{\partial x}\, i + \frac{\partial f}{\partial y}\, j \quad \text{is normal to the curve } f(x,y) = c$$

Proof : $\vec{R} = x\, i + y\, j$



$$\vec{T} = \frac{\vec{dR}}{ds} = \frac{dx\, i + dy\, j}{ds}$$

$$\vec{T} = \left(\frac{1}{ds}\right) \vec{dR}$$

$\therefore \quad \vec{dR} \parallel \vec{T} \qquad (\text{i.e } \vec{dR} \perp \vec{N})$

$$f(x,y) = c \quad \Rightarrow \quad d(f(x,y)) = d(c)$$

$$\frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy = 0 \Rightarrow \left(\frac{\partial f}{\partial x}\, i + \frac{\partial f}{\partial y}\, j\right) \cdot (dx\, i + dy\, j) = 0$$

$\therefore \quad \vec{\nabla f} \cdot \vec{dR} = 0 \qquad (\text{i.e. } \vec{\nabla f} \perp \text{curve})\ (\vec{\nabla f} = \vec{N})$

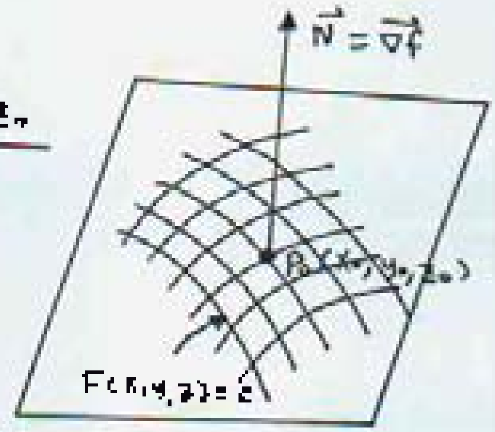
Theorem (2) : If $w = f(x,y,z)$ then the vector

$$\vec{\nabla f} = \frac{\partial f}{\partial x}\, i + \frac{\partial f}{\partial y}\, j + \frac{\partial f}{\partial z}\, k \quad \text{is normal to the surface}$$

$$f(x,y,z) = c$$

Eq. of Normal line :

$$\frac{x - x_0}{\frac{\partial f}{\partial x}} = \frac{y - y_0}{\frac{\partial f}{\partial y}} = \frac{z - z_0}{\frac{\partial f}{\partial z}}$$



Eq. of Tangent Plane :

$$\frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(z - z_0) + \frac{\partial f}{\partial z}(z - z_0) = 0$$

ex. 14 : Find the equation of the tangent plane and the normal line for surface $x^2 + 3xy + z^3 = 5$ at $P_0(1, 1, 1)$.

Soln

$$F(x, y, z) = x^2 + 3xy + z^3$$

$$F_x = 2x + 3y \quad \Rightarrow \quad F_x = 5$$

$$F_y = 3x \quad \Rightarrow \quad F_y = 3$$

$$F_z = 3z^2 \quad \Rightarrow \quad F_z = 3$$

N.L.

$$\frac{x - 1}{5} = \frac{y - 1}{3} = \frac{z - 1}{3}$$

T.P.

$$5(x - 1) + 3(y - 1) + 3(z - 1) = 0$$

$$5x + 3y + 3z - 11 = 0$$

Maximum and Minimum Points for the Surface $z = f(x,y)$

Definition:

$f(a+h, b+k) < f(a,b)$

max. point

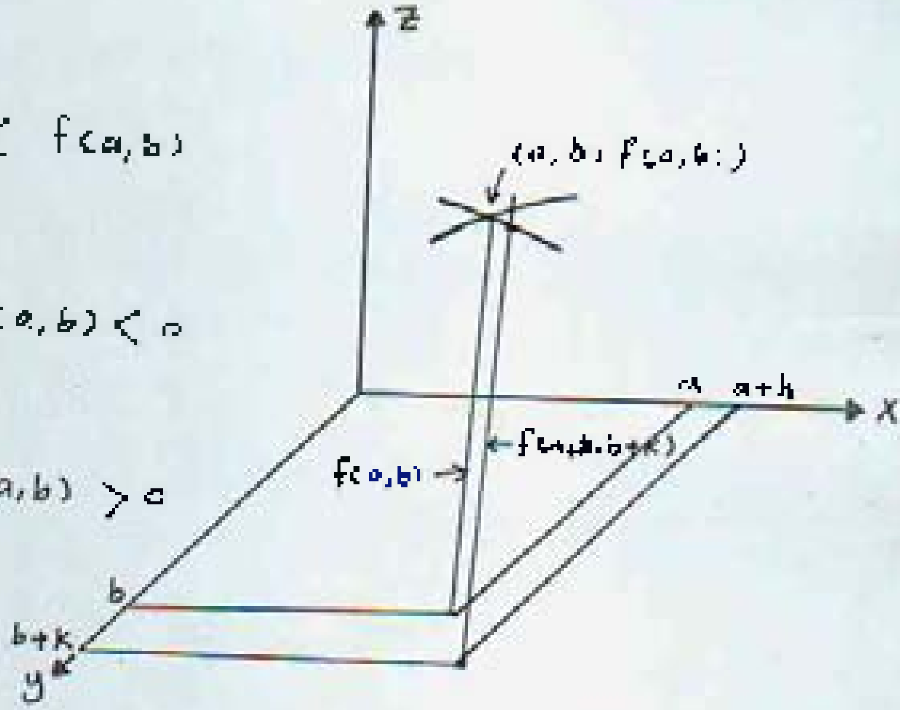
$f(a+h, b+k) - f(a,b) < 0$

min. point

$f(a+h, b+k) - f(a,b) > 0$

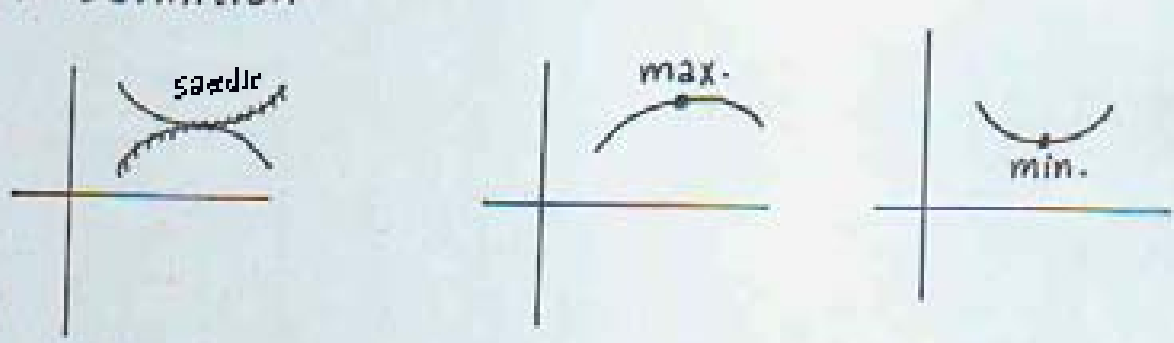
for all values of

h & K



M – Test (for $z = f(x,y,z)$)

1. Find $f_x = 0$, $f_y = 0$ and solve (say at $x = a$, $y = b$)
2. Find Point $(a, b, f(a,b))$ is a critical point
3. Find $M = f_{xx} f_{yy} - (f_{xy})^2$
4. If $M > 0$ and $f_{xx} < 0$ then $(a, b, f(a,b))$ is a max. point
5. If $M > 0$ and $f_{xx} > 0$ then $(a, b, f(a,b))$ is a min. point.
6. If $M < 0$ then $(a, b, f(a,b))$ is a saddle point.
7. If $M = 0$ or $f_{xy} \neq f_{yx}$ then the test fails, use the definition.



ex. 15 : Find the max., min., or a saddle points (if any) for $z = f(x,y) = x^2 + 2y^2 - 2xy - y + x$

Sol:

$f_x = 2x - 2y + 1 = 0 \Rightarrow 2x - 2y = -$

$f_y = 4y - 2x - 1 = 0 \Rightarrow -2x + 4y = 1$

add

$2y = 0$

$\therefore\ y = 0$

$2x = -1 + 2y \Rightarrow x = \frac{1}{2}(-1 + 2 * 0) = -\frac{1}{2}$

$\therefore\ z = f(-\frac{1}{2}, 0) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$

$\therefore\ (-\frac{1}{2}, 0, -\frac{1}{4})$ is a critical point

$$\left.\begin{array}{l} f_{xx} = 2 \\ f_{yy} = 4 \\ f_{xy} = f_{yx} = -2 \end{array}\right\} M = (2)(4) - (-2)^2 = 8 - 4 = 4$$

$\therefore\ M = 4 > 0$

$f_{xx} = 2 > 0$

$\therefore\ (-\frac{1}{2}, 0, -\frac{1}{4})$ is a minimum point

University of Technology
Mechanical Engineering Department
Advance Engineering Mathematics
Chapter ( ): Double & Triple Integrals
Dr. Akeel Abdullah Mohammed

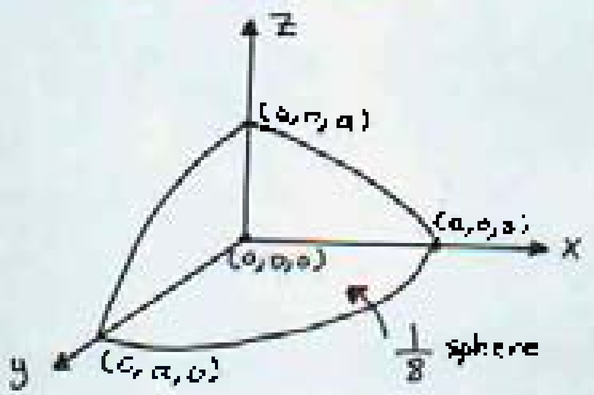
Double Integrals :

The equation of surface is $f(x,y,z)=0$ (or $z=f(x,y)$ which may be 1st order or 2nd order.

The Equations of Some Geometric Figures

1. Sphere

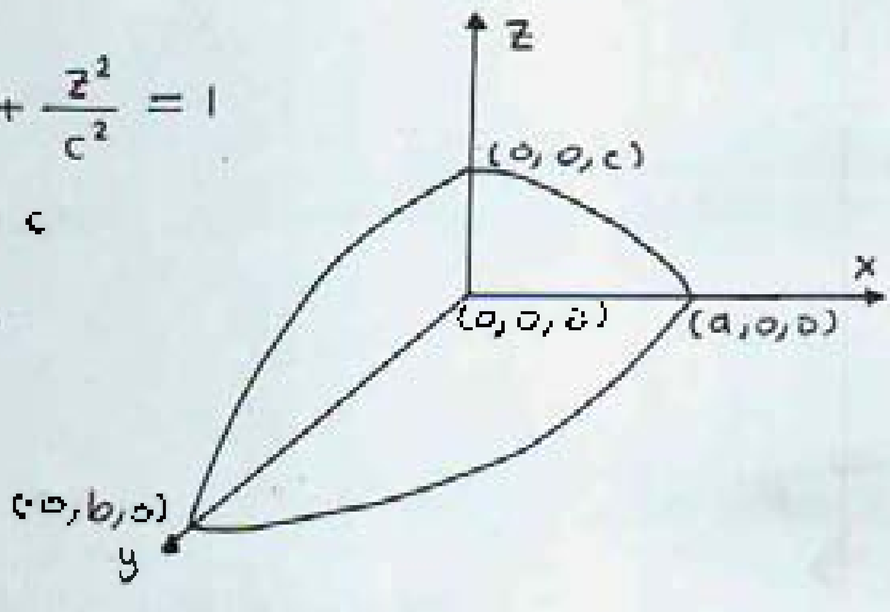
$$x^2+y^2+z^2=a^2$$



2. Ellipisoid

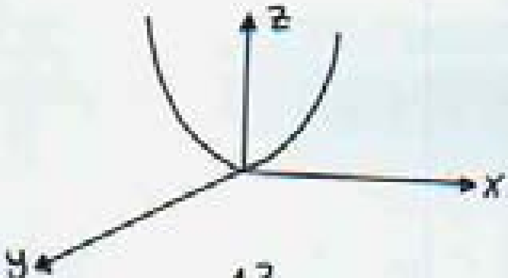
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$a \neq b \neq c$$

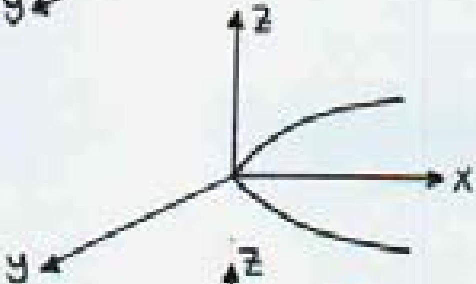


3. Paraboloid

$z = x^2 + y^2 \quad ; \quad z \geqslant 0$



$x = z^2 + y^2 \quad ; \quad x \geqslant 0$



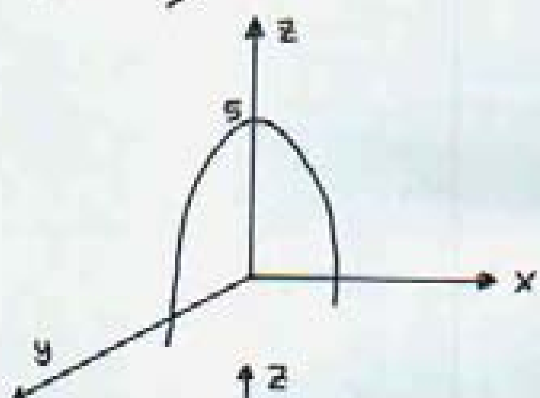
$y = z^2 + x^2 \quad ; \quad y \geqslant 0$

ex. sketch $z = 5 - x^2 - y^2$

$x^2 + y^2 = 5 - z$

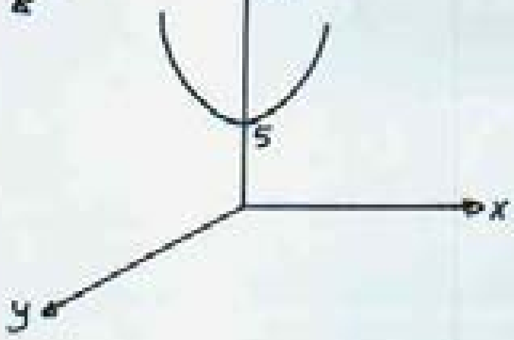
$5 - z \geqslant 0$

$z \leqslant 5$



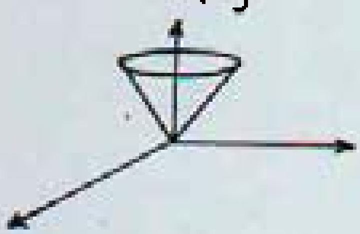
ex. sketch $z = 5 + x^2 + y^2$

$z - 5 \geqslant 0 \quad \therefore \quad z \geqslant 5$

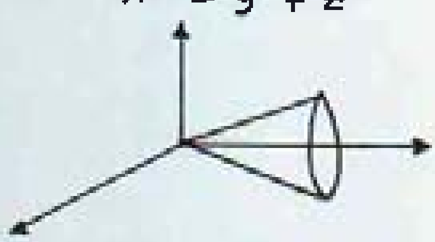


4. Cone

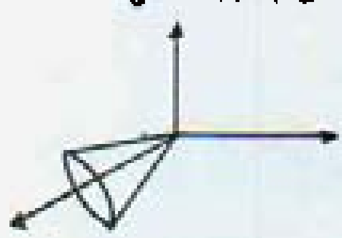
$z^2 = x^2 + y^2$



$x^2 = y^2 + z^2$



$y^2 = x^2 + z^2$

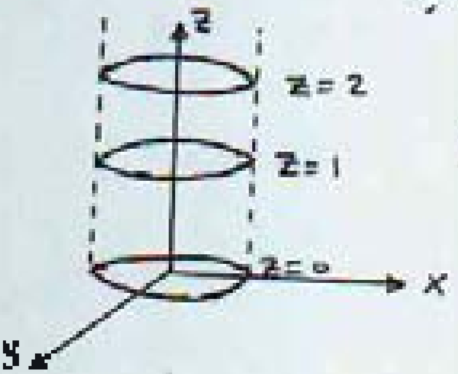


5. Cylinder

الأسطوانة : مجسم جميع مقاطعه المتوازية متساوية

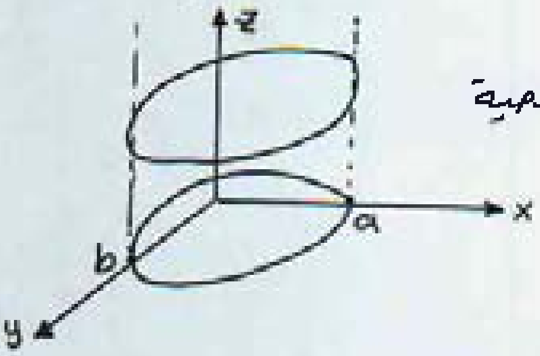
* اي معادلة تحتوي على متغيرين فقط هي أسطوانة
وتكون معادلة ذات درجة ثانية

ex.1 : $x^2 + y^2 = a^2$ for all z



أسطوانة دائرية

ex.2 : sketch $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for all z



أسطوانة ناقصية

Triple Integrals :

$$V = \iiint dz\,dx\,dy$$

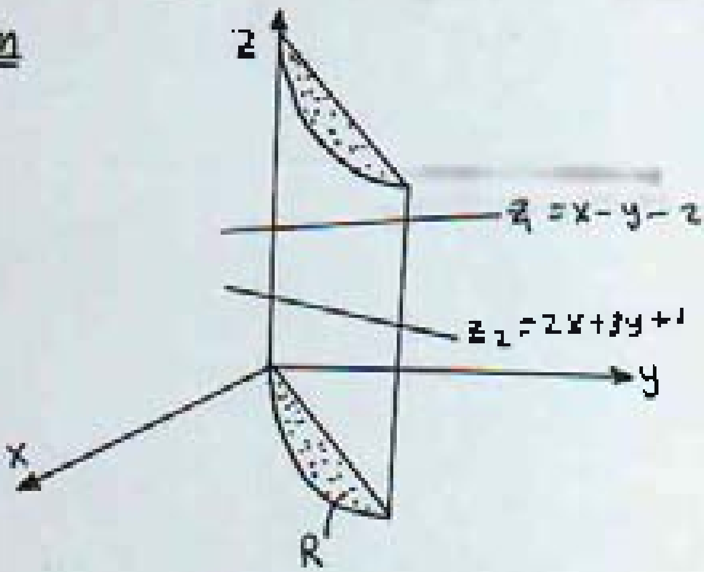
$$\text{Volume} = \iint_R z\,dA = \begin{cases} \iint_R f(x,y)\,dy\,dx \\ \iint_R f(x,y)\,dx\,dy \end{cases}$$

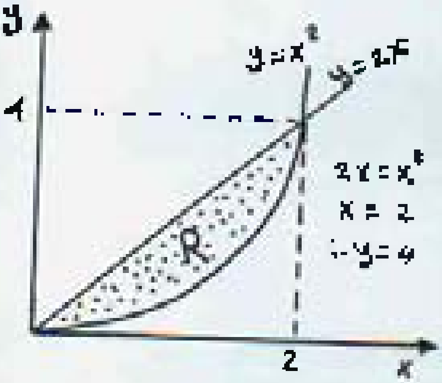
if $z = 1$

$$\therefore \text{ area} = A = \iint_R dA = \iint_R dx\,dy = \iint_R dy\,dx$$

Ex. 3: Find the volume of solid bounded by $z_1 = x - y - 2$, $z_2 = 2x + 3y + 1$, for the region $y = x^2$, $y = 2x$ for all z

Soln



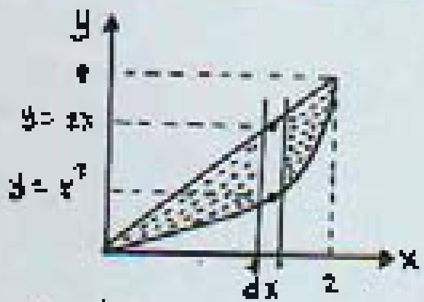


R in x-y plane

$$V = \left| \iint_R (z_2 - z_1)\, dA \right|$$

$$= \iint_R \left\{ (2x + 3y + 1) - (x - y - 2) \right\} dA = \iint_R (x + 4y + 3)\, dA$$

method ① تحريك dy مع ثبات dx

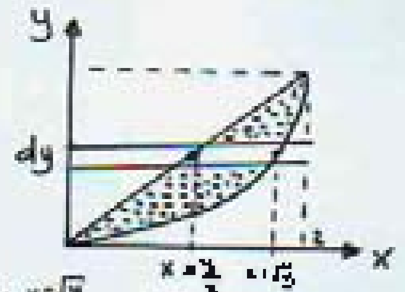


$$V = \int_0^2 \left(\int_{y=x^2}^{y=2x} (x + 4y + 3)\, dy \right) dx$$

$$= \int_0^2 xy + \frac{4y^2}{2} + 3y \Big]_{x^2}^{2x} dx$$

$$= \int_0^2 \left(x^3 + \frac{4x^4}{2} + 3x^2\right) - \left(2x^2 + 8x^2 + 6x\right) dx$$

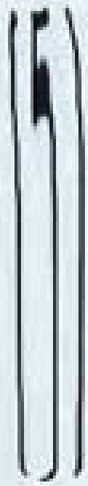
method ② تحريك (dx) مع ثبات dy



$$V = \int_0^4 \left(\int_{x=\frac{y}{2}}^{x=\sqrt{y}} (x + 4y + 3)\, dx \right) dy$$

$$= \int_0^4 \frac{x^2}{2} + 4yx + 3x \Big]_{\frac{y}{2}}^{\sqrt{y}} dy$$

$$= \int_0^4 \left\{ \frac{y}{2} + 4y^{3/2} + 3y^{1/2} - \left(\frac{y^2}{8} + 2y^2 + \frac{3}{2}y \right) \right\} dy$$



Ex. 4: Find the area of the region bounded by $y=\sqrt{x}$, $x+y=6$, $y=0$ (using double integral).

Sol: إيجاد نقطة تقاطع المستقيم مع المنحني

$x+y=6 \Rightarrow x+\sqrt{x}=6 \Rightarrow \sqrt{x}=6-x \Rightarrow$

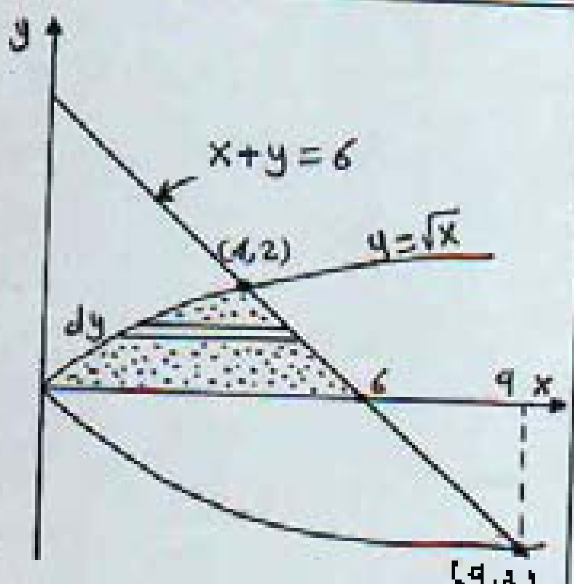
$x=36-12x+x^2 \Rightarrow x^2-13x+36=0 \Rightarrow$

$(x-9)(x-4)=0$

either $x=9$ is neglected

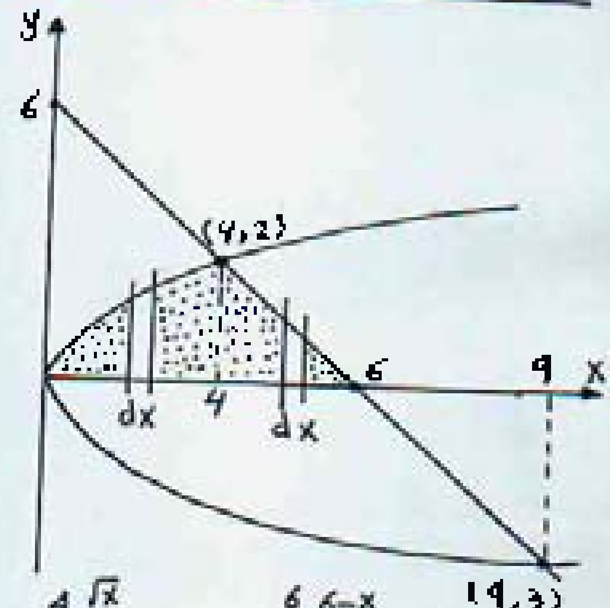
or $x=4$ $\therefore y=2$

method ①



$$A=\int_0^2\int_{y^2}^{6-y} dx\,dy$$

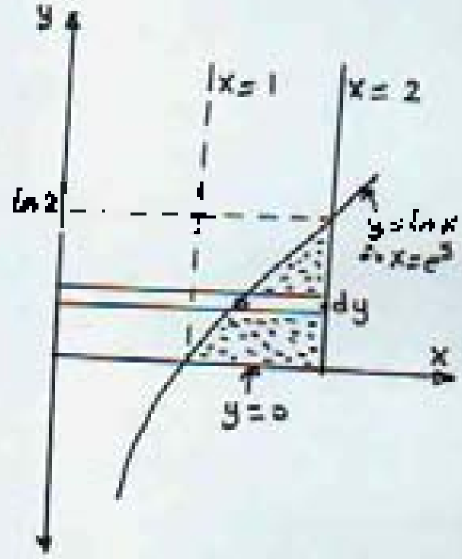
method ②



$$A=\int_0^4\int_0^{\sqrt{x}} dy\,dx+\int_4^6\int_0^{6-x} dy\,dx$$

ex. 5 : Reverse the order of the integral and evaluate.

① $\int_{1}^{2}\int_{0}^{\ln x} x \, dy \, dx$



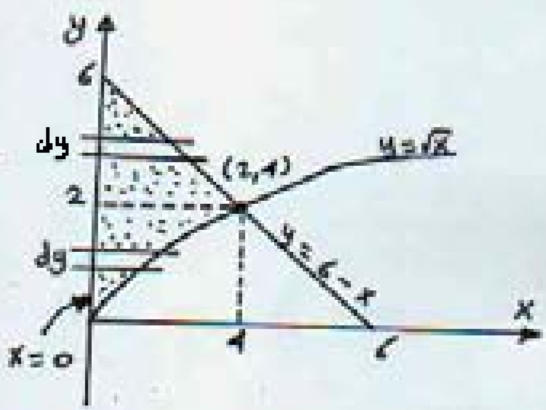
Sol$\underline{n}$ R: $x=1$ to $x=2$

$y=0$ to $y=\ln x$

$$\therefore \int_{1}^{2}\int_{0}^{\ln x} x \, dy \, dx = \int_{0}^{\ln 2}\int_{e^y}^{2} x \, dx \, dy$$

$$= \int_{0}^{\ln 2} \left. \frac{x^2}{2} \right|_{e^y}^{2} dy$$

② $\int_{0}^{4}\int_{\sqrt{x}}^{6-x} y \, dy \, dx$



Sol$\underline{n}$ R: $x=0$ to $x=4$

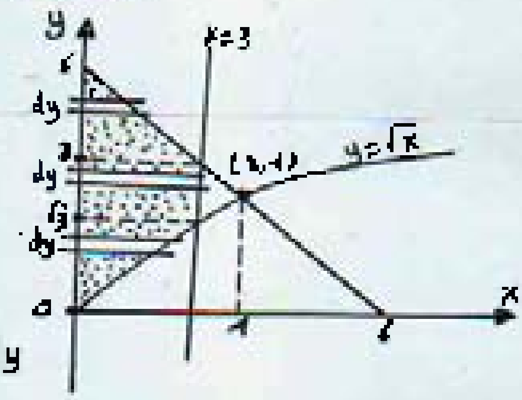
$y=\sqrt{x}$ to $y=6-x$

$\sqrt{x} = 6 - x \Rightarrow$ نجد نقاط التقاطع

$\therefore x=4$ & $y=2$ (from previos example)

$$\therefore \int_{0}^{4}\int_{\sqrt{x}}^{6-x} y \, dy \, dx = \int_{0}^{2}\int_{0}^{y^2} y \, dx \, dy + \int_{2}^{6}\int_{0}^{6-y} y \, dx \, dy$$

③ $\int_{0}^{3}\int_{\sqrt{x}}^{6-x} y \, dy \, dx$



Sol$\underline{n}$ R: $x=0$ to $x=3$

$y=\sqrt{x}$ to $y=6-x$

$$\int_{0}^{3}\int_{\sqrt{x}}^{6-x} y \, dy \, dx = \int_{0}^{\sqrt{3}}\int_{0}^{y^2} y \, dx \, dy + \int_{\sqrt{3}}^{3}\int_{0}^{3} y \, dx \, dy$$

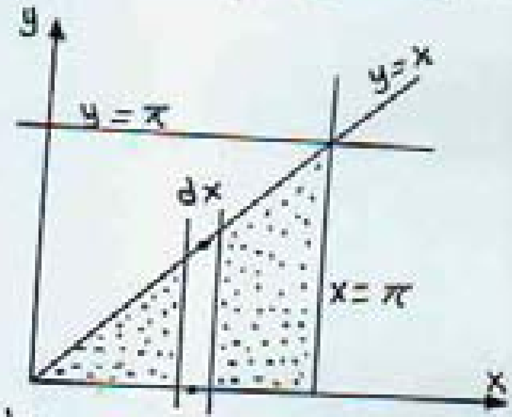
$$+ \int_{3}^{6}\int_{0}^{6-y} y \, dx \, dy$$

Ex. 6: Find $\int_0^{\pi}\int_y^{\pi} \frac{\sin x}{x}\,dx\,dy$

Sol$\underline{n}$ the order of integral should be reversed because of the difficult integral

R : $y=0$ to $y=\pi$

$x=y$ to $x=\pi$



$$\int_0^{\pi}\int_y^{\pi} \frac{\sin x}{x}\,dx\,dy = \int_0^{\pi}\int_0^{x} \frac{\sin x}{x}\,dy\,dx$$

$$= \int_0^{\pi} \frac{\sin x}{x}\, y \Big|_0^x dx = \int_0^{\pi} \frac{\sin x}{x}(x- \quad)\,dx$$

$$= \int_0^{\pi} \sin x\,dx = \cos x \Big|_0^{\pi} = -(\cos\pi - \cos 0) = 2$$

ملاحظة : هناك تكاملات صعبة جداً ولكي تكون سهلة نعكس رتبة التكامل ، مثل :

$$\iint e^{x^2}dx\,dy \qquad \iint \frac{\cos x}{\sqrt{4-x}} \qquad \iint \frac{dx}{1+x^4}dy \qquad \iint \frac{e^x}{x^3}dx\,dy$$

Area in Polar curve :

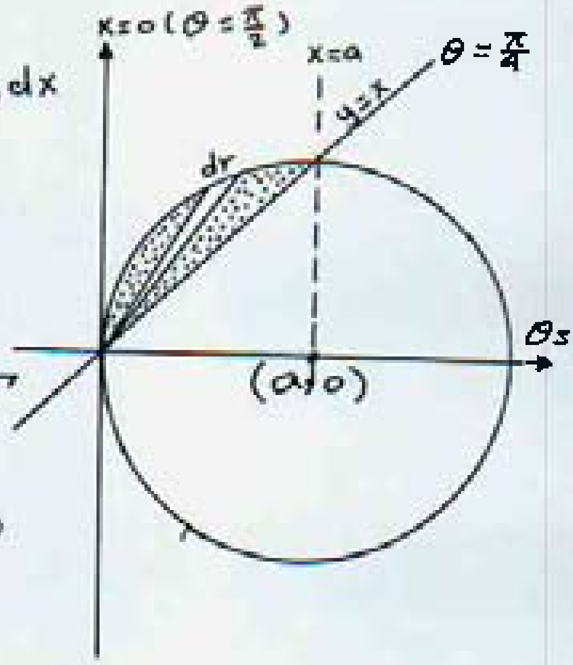
$$V = \iint_R f(x,y)\,dy\,dx = \iint_R f(r,\theta)\, r\,dr\,d\theta = \text{Volume}$$

$$A = \iint_R r\,dr\,d\theta = \frac{1}{2}\int r^2\,d\theta = \text{Area}$$

$x = r\cos\theta$, $y = r\sin\theta$, $\theta = \tan^{-1}\frac{y}{x}$

$r = \sqrt{x^2+y^2}$, $dx\,dy = r\,dr\,d\theta$

ex. 7 : Find $\displaystyle\int_0^a \int_x^{\sqrt{2ax-x^2}} (x^2+y^2)\,dy\,dx$



Soln   R :

$x=0$ to $x=a$

$y=x$ to $y=\sqrt{2ax-x^2}$

hence, change into polar

$y = x \Rightarrow r\sin\theta = r\cos\theta$

$\therefore \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

& $y=\sqrt{2ax-x^2} \Rightarrow y^2 = 2ax - x^2 \Rightarrow x^2+y^2 = 2ax$

$\therefore r^2 = 2ar\cos\theta$ or $r = 2a\cos\theta$

which is eq. of circle and can be written as

$y^2 = 2ax - x^2 \Rightarrow x^2 - 2ax + a^2 - a^2 + y^2 = 0$

$\therefore (x-a)^2 + y^2 = a^2$ (eq. of circle)

$$I = \int_{\frac{\pi}{4}}^{\pi/2} \int_0^{2a\cos\theta} r^2 \cdot r\,dr\,d\theta = \int_{\frac{\pi}{4}}^{\pi/2} \frac{r^4}{4}\Big|_0^{2a\cos\theta} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\pi/2} 4a^4 (\cos^2\theta)^2\,d\theta = 4a^4 \int_{\frac{\pi}{4}}^{\pi/2} \left(\frac{1}{2}(1+\cos 2\theta)\right)^2 d\theta$$

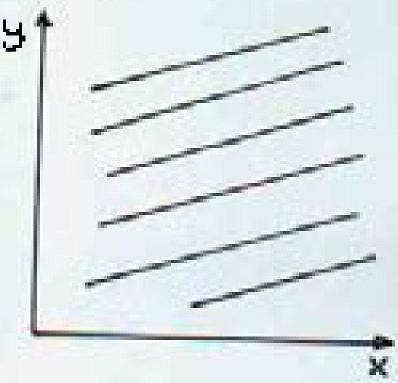
$$= a^4 \int_{\frac{\pi}{4}}^{\pi/2} \left\{1 + \frac{1}{2}(1+\cos 4\theta) + 2\cos 2\theta\right\} d\theta$$

$$= a^4 \left[\frac{3}{2}\theta + \frac{1}{8}\sin 4\theta + \sin 2\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= a^4 \left\{\frac{3}{2}\frac{\pi}{2} + 0 + 0 - \frac{3}{2}\frac{\pi}{4} - 0 - 1\right\} = \left(\frac{3}{8}\pi - 1\right)a^4$$

ex. 8 : Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} \, dy \, dx$

Sol$\underline{n}$ R: $x = 0$ to $x = \infty$
$y = 0$ to $y = \infty$



$$I = \int_0^{\pi/2}\int_0^{\infty} e^{-r^2} \, r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \left(-\tfrac{1}{2}\right) e^{-r^2} \Big|_0^{\infty} d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2} \frac{1}{e^{r^2}} \Big|_0^{\infty} d\theta = -\frac{1}{2}\int_0^{\pi/2} (0-1)\, d\theta = \frac{\pi}{4}$$

* ~ ~ ~ ~ ~ ~ *

ex. 9 : Find the volume of a solid bounded above by $x^2 + y^2 + z^2 = 2a^2$ and bounded below by $az = x^2 + y^2$

Sol$\underline{n}$

الشيء الناتج من تقاطع مجسمين هو منحنى (curve)

$(x^2+y^2) + z^2 = 2a^2$

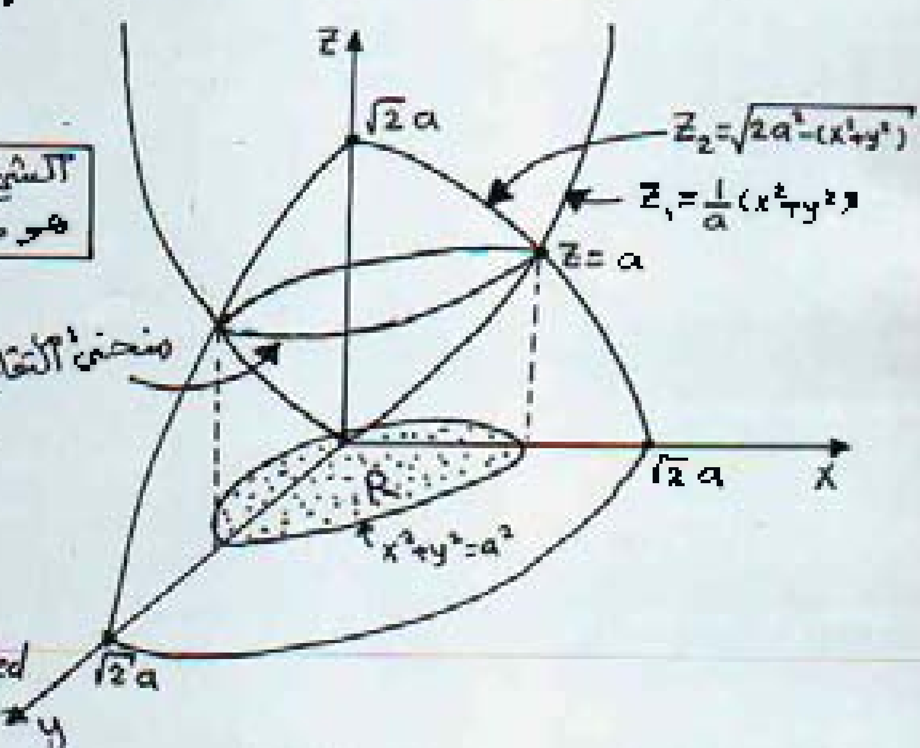
$az + z^2 = 2a^2$

$z^2 + az - 2a^2 = 0$

$(z-a)(z+2a) = 0$

$\boxed{z = a}$ مستوى التقاطع

or $z = -2a$ neglected



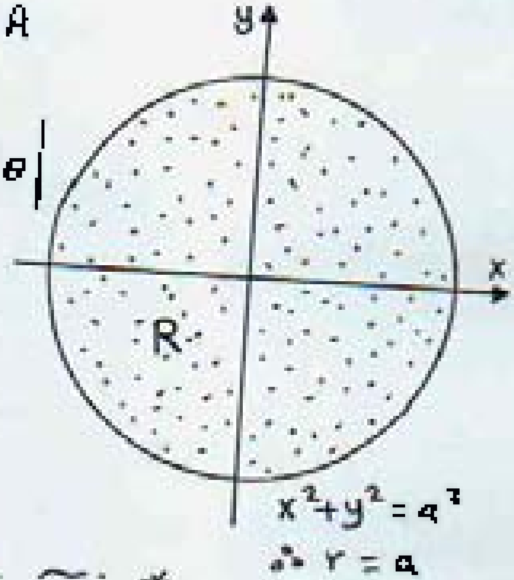
by substituting $z = a$ in one of the two equations gets

hence, $V = \iint_R (z_2 - z_1)\, dA$

$$V = \left| \int_0^{2\pi} \int_0^a \left\{ \sqrt{2a^2 - r^2} - \frac{1}{a} r^2 \right\} r\, dr\, d\theta \right|$$

$$= \int_0^{2\pi} -\frac{1}{2} \frac{(2a^2 - r^2)^{3/2}}{3/2} - \frac{r^4}{4a} \Big|_0^a d\theta$$

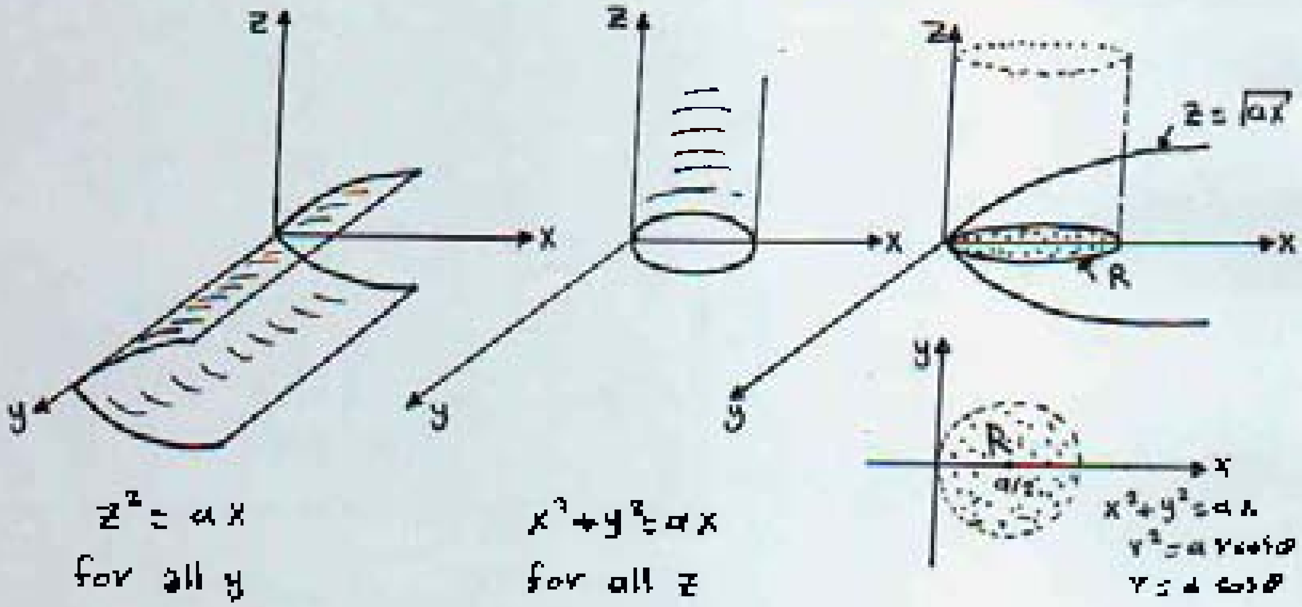
$= (\qquad)$ unit volume



* ~. ~. ~. ~. ~. ~. *

Ex. 10 : Find the volume common to the surfaces $x^2+y^2=ax$ & $z^2=ax$

Soln the two surfaces are cylinder because their equations have two variables and they are 2nd degree



$z^2 = ax$ for all y

$x^2+y^2=ax$ for all z

$$V = \left(\iint_R z\, dA \right) * 2 = 2 \iint_R \sqrt{a}\sqrt{x}\, dA$$

$$= 2 \int_0^{\pi/2} \int_0^{a\cos\theta} \sqrt{a} \cdot \sqrt{r\cos\theta} \cdot r\, dr\, d\theta \quad * 2$$

$$= 4\sqrt{a} \int_0^{\pi/2} \left(\int_0^{a\cos\theta} r^{3/2} (\cos\theta)^{1/2} dr \right) d\theta$$

$$V = 4\sqrt{a}\int_0^{\pi/2}(\cos\theta)^{1/2}\ \frac{r^{5/2}}{5/2}\Big|_0^{a\cos\theta}\ d\theta$$

$$= \frac{8}{5}\sqrt{a}\int_0^{\pi/2}(\cos\theta)^{1/2}\cdot\left\{a^{5/2}\ (\cos\theta)^{5/2}\right\}d\theta$$

$$= \frac{8}{5}a^3\int_0^{\pi/2}\cos^3\theta\ d\theta = \frac{8}{5}a^3\int_0^{\pi/2}(\cos\theta - \cos\theta\sin^2\theta)\,d\theta$$

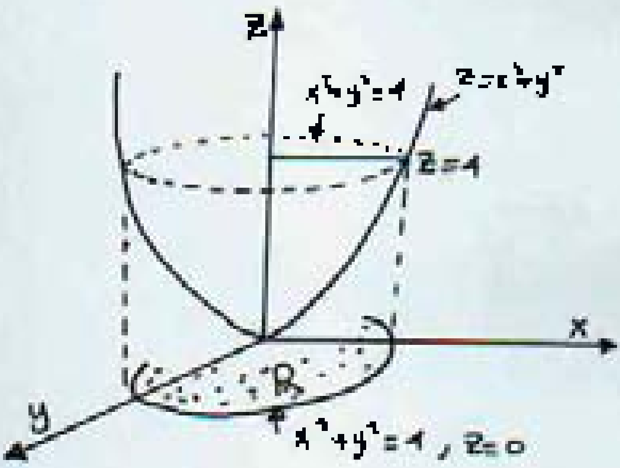
$$= \frac{8}{5}a^3\sin\theta - \frac{8}{5}\frac{a^3}{3}\sin^3\theta\Big|_0^{\pi/2} = \frac{16}{15}a^3$$

~ ~ ~ ~ ~ ~ ~

Surface area :

$$S = \iint_R \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\ dA$$

ex. 11 : Find the area of the surface $z = x^2+y^2$ cut by the plane $z=4$



Soln $\quad z = x^2+y^2$

$$\therefore\ \frac{\partial z}{\partial x} = 2x \quad , \quad \frac{\partial z}{\partial y} = 2y$$

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1+4x^2+4y^2}$$

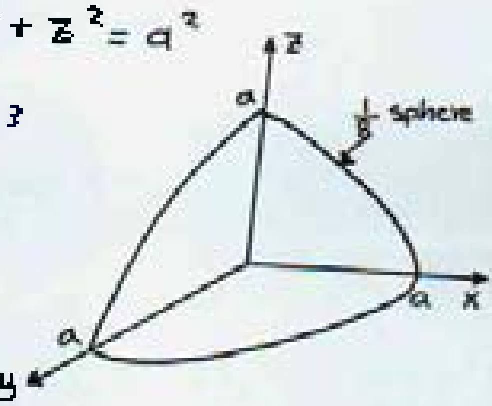
$$\therefore\ S = \iint_R \sqrt{1+4(x^2+y^2)}\ dA$$

$$= \int_0^{2\pi}\int_0^{2}\sqrt{1+4r^2}\cdot r\,dr\,d\theta$$

$$= \frac{\pi}{4}\ \frac{(1+4r^2)^{3/2}}{3/2}\Big|_0^2 = \frac{\pi}{6}\left((\sqrt{17})^3 - 1\right)$$

ex. 12 : For the sphere $x^2+y^2+z^2=a^2$

1. show that $V = \frac{4}{3}\pi a^3$
2. Show that $S = 4\pi a^2$



Soln $V = \iint z \, dA$

$$= \left\{ \int_0^{\pi/2} \int_0^a a^2 - r^2 \quad . \, r \, dr \, d\theta \right\} * 8$$

$$= -\frac{8}{3} \int_0^{\pi/2} (a^2 - r^2)^{3/2} \Big|_0^a d\theta$$

$$= \frac{8}{3} \int_0^{\pi/2} a^3 \, d\theta = \frac{8}{3} a^3 \frac{\pi}{2} = \frac{4}{3}\pi a^3$$

2. $z^2 = a^2 - x^2 - y^2$

$$2 z \frac{\partial z}{\partial x} = -2x \qquad \therefore \left(\frac{\partial z}{\partial x}\right)^2 = \frac{x^2}{a^2 - x^2 - y^2}$$

$$2 z \frac{\partial z}{\partial y} = -2y \qquad \therefore \left(\frac{\partial z}{\partial y}\right)^2 = \frac{y^2}{a^2 - x^2 - y^2}$$

$$\therefore \quad 1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1 + \frac{x^2+y^2}{a^2 - (x^2+y^2)}$$

$$= \frac{a^2 - x^2 - y^2 + x^2 + y^2}{a^2 - (x^2+y^2)} = \frac{a^2}{a^2 - r^2}$$

$$\therefore \quad S = \left[\int_0^{\pi/2} \int_0^a \frac{a}{\sqrt{a^2 - r^2}} \, r \, dr \, d\theta \right] * 8$$

$$= \frac{8a}{-2} \cdot \frac{\pi}{2} \cdot \frac{(a^2 - r^2)^{1/2}}{1/2} \Bigg|_0^a = 4\pi a^2$$