University of Technology
Mechanical Engineering Department
Advance Engineering Mathematics
Sheet No.(5): Double & Triple Integrals
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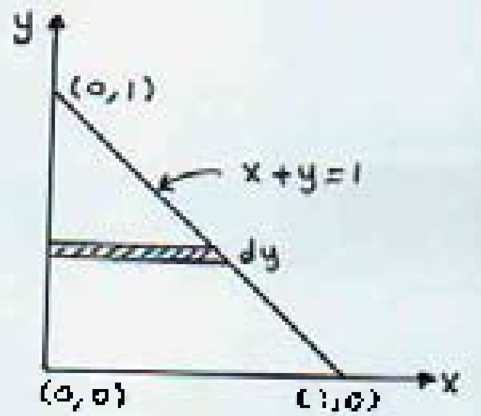
Prob. 1 : Evaluate the integral $\iint_R \sin(x+y)\cos(x-y)\,dx\,dy$ where R is the triangle whose vortices are (0,0), (1,0), (0,1).

Soln : The above vortices can be represented by the following figure.



$$\iint_R \sin(x+y)\cos(x-y)\,dx\,dy$$

$$\iint_R (\sin x\cos y + \sin y\cos x) * (\cos x\cos y + \sin x\sin y)\,dx\,dy$$

$$\iint_R \{\sin x\cos x(\sin^2 y + \cos^2 y) + \sin y\cos y(\sin^2 x + \cos^2 x)\,dx\,dy$$

$$\int_0^1\int_0^{1-y} ((\sin x\cos x + \sin y\cos y)\,dx)\,dy$$

$$\int_0^1 \frac{\sin^2 x}{2} + x\sin y\cos y \Big|_0^{1-y} dy$$

$$\int_0^1 \left(\frac{\sin^2(1-y)}{2} + (1-y)\sin y\cos y\right) dy$$

$$\int_0^1 \frac{1}{2}\cdot\frac{1}{2}(1-\cos 2(1-y)) + \frac{\sin 2y}{2} - \frac{y}{2}\sin 2y\ dy$$

$$= \int_0^1 \frac{1}{4} - \frac{1}{4}(\cos 2 \cos 2y + \sin 2 \sin 2y) + \frac{\sin 2y}{2} - \frac{y}{2}\sin 2y \, d$$

$$= \frac{1}{4}\int_0^1 dy - \frac{\cos 2}{4}\int_0^1 \cos 2y \, dy + \left(\frac{1}{2} - \frac{\sin 2}{4}\right)\int_0^1 \sin 2y \, dy$$

$$- \frac{1}{2}\int_0^1 \underbrace{y}_{u} \underbrace{\sin 2y \, dy}_{dv}$$

$$= \left[\frac{y}{4} - \frac{1}{4}\cos 2 \frac{\sin 2y}{2} - \left(\frac{1}{2} - \frac{\sin 2}{4}\right)\frac{\cos 2y}{2} \right]_0^1 - \frac{1}{2}\left[y\left(-\frac{\cos 2y}{2}\right) \right.$$

$$\left. + \frac{1}{2}\int_0^1 \cos 2y \, dy \right]$$

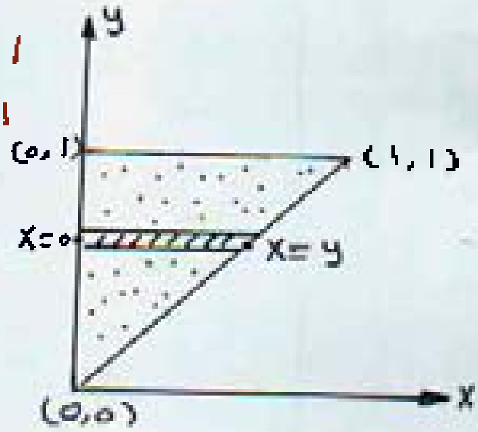
$$= \frac{1}{4} - \frac{1}{8}\cos 2 \sin 2 - \left(\frac{1}{2} - \frac{\sin 2}{4}\right)\left(\frac{\cos 2}{2} - \frac{1}{2}\right) + \frac{\cos 2}{4} + \frac{\sin 2}{8}$$

= (   ) unit ... volume.

Prob. 2 : Evaluate the following integral $\int_0^1 \int_x^1 \frac{1}{y} \sin\frac{x}{y} \cos\frac{x}{y} \, dy \, dx$

sol$^{\underline{n}}$: R : $y = x$ to $y = 1$

$x = 0$ $\quad x = 1$



$$\int_0^1 \int_0^y \frac{1}{y} \sin\frac{x}{y} \cos\frac{x}{y} \, dx \, dy$$

$$= \frac{1}{2}\int_0^1 \left(-\cos^2\frac{x}{y}\right)\Big|_0^y \, dy$$

$$= \frac{1}{2}\int_0^1 (1 - \cos^2 1) \, dy =$$

$$= \frac{1}{2}(1 - \cos^2 1) y$$

$$= \frac{1}{2}(1 - \cos^2 1)$$

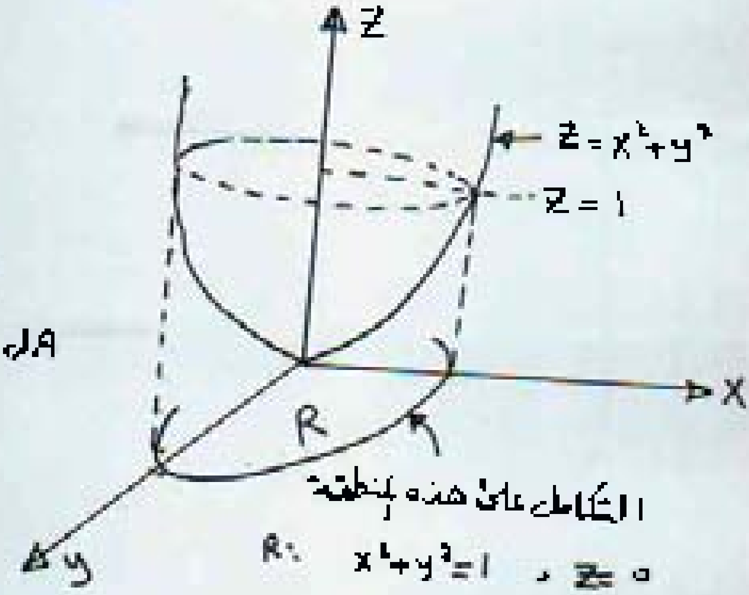
* حولنا الشريحة من التوازي مع y إلى التوازي مع x كي تكون عملية التكامل سهلة

Prob. 3 Find the surface area of the paraboloid $z=x^2+y^2$ below the plane $z=1$

Sol'n

$$\frac{\partial z}{\partial x} = 2x$$

$$\frac{\partial z}{\partial y} = 2y$$



$$S = \iint_R \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= \iint \sqrt{1+4(x^2+y^2)}\, dA$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\cdot r\, dr\, d\theta$$

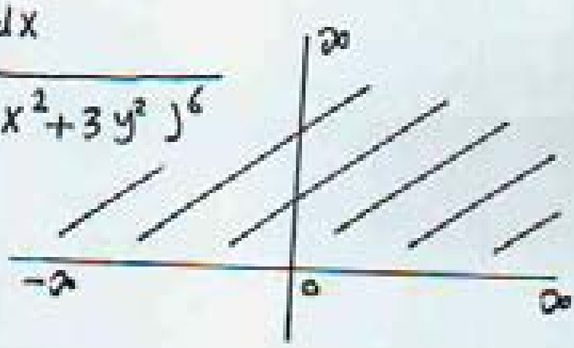
$$= \int_0^{2\pi} \frac{1}{8}\cdot \frac{(1+4r^2)^{3/2}}{3/2}\Big|_0^1\, d\theta$$

$$= \frac{\pi}{6}\left[(5)^{3/2}-1\right] = (\qquad) \text{ unit area.}$$

Prob. 4: Find $\displaystyle\int_{-\infty}^{\infty}\int_0^{\infty} \frac{dy\, dx}{(4+3x^2+3y^2)^6}$

Sol'n: R: $x=-\infty$ to $x=\infty$

$y=0$ $y=\infty$



$$I = \int_0^{\pi}\int_0^{\infty} \frac{r\, dr\, d\theta}{(4+3r^2)^6} = \int_0^{\pi}\int_0^{\infty} (4+3r^2)^{-6}\, \frac{1}{6}\cdot 6r\, dr\, d\theta$$

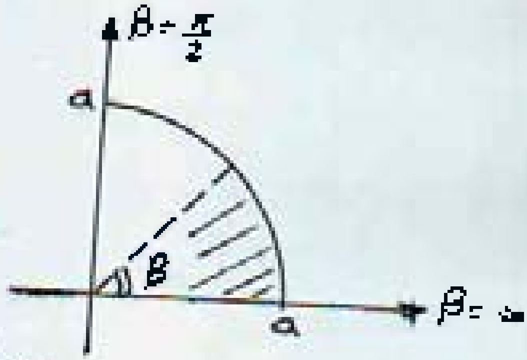
$$= -\frac{\pi}{30}\left[\left(\frac{1}{4+3(\infty)^2}\right)^5 - \frac{1}{(4+3(0)^2)^5}\right]$$

Prob. 5: show by transforming to polar coordinates that

$$\int_0^{a\sin\beta}\int_0^{\sqrt{a^2-y^2}} \ln(x^2+y^2)\,dx\,dy = a^2\beta\left(\ln a-\tfrac{1}{2}\right)$$

$$0<\beta<\frac{\pi}{2}$$

Sol^n: R:



$x=0$ to $x=\sqrt{a^2-y^2}$

$\therefore\ x^2+y^2=a^2$

$r=a$

$y=0$ to $y=a\sin\beta$

$$\int_0^{\beta}\int_0^{a} \ln r^2 \; r\,dr\,d\theta = 2\int_0^{\beta}\int_0^{a} \underbrace{\ln r}_{u}\;\underbrace{r\,dr}_{dv}\,d\theta$$

$$= 2\int_0^{\beta}\left\{ \ln(r)\cdot\frac{r^2}{2}\Big|_0^a - \int_0^a \frac{r^2}{2}\cdot\frac{1}{r}\,dr \right\} d\theta$$

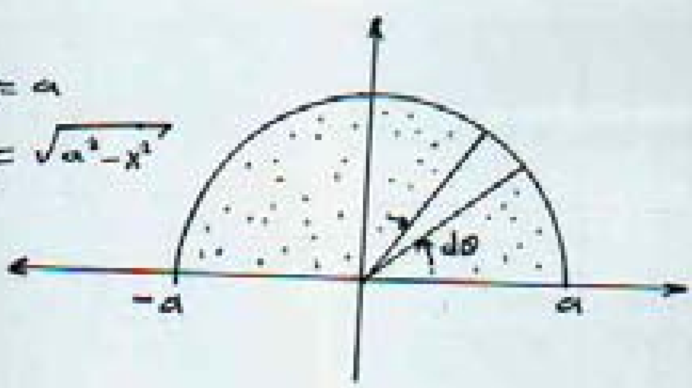
$$= 2\int_0^{\beta}\left(\frac{a^2}{2}\ln a - \frac{a^2}{4}\right)d\theta = a^2\theta\left(\ln a-\frac{1}{2}\right)\Big|_0^{\beta}$$

$$= a^2\beta\left(\ln a-\frac{1}{2}\right)$$

Prob. 6: Evaluate $\displaystyle\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} (x^2+y^2)^{3/2}\,dy\,dx$

Sol^n: R: $x=-a$ to $x=a$

$y=0$ to $y=\sqrt{a^2-x^2}$



$$I=\int_0^{\pi}\int_0^{a} r^3\; r\,dr\,d\theta$$

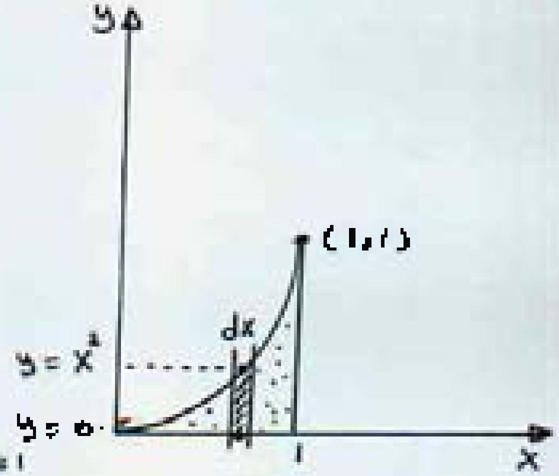
$$=\int_0^{\pi}\frac{r^5}{5}\Big|_0^a\,d\theta$$

$$=\frac{\pi a^5}{5}$$

Prob. 7 : Evaluate $\int_0^1 \int_{\sqrt{y}}^1 e^{5x^3}\, dx\, dy$

Soln : R: $x = \sqrt{y}$ to $x = 1$

$y = 0$ $\quad y = 1$



$$I = \int_0^1 \int_0^{x^2} e^{5x^3}\, dy\, dx$$

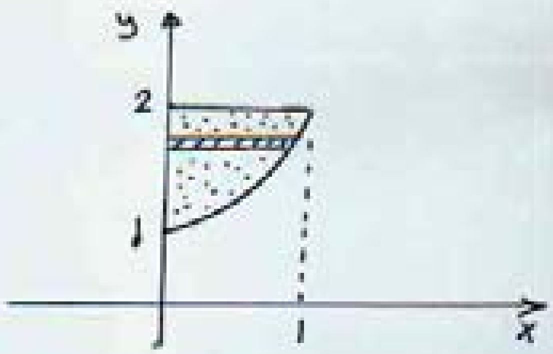
$$= \int_0^1 e^{5x^3} \cdot y \Big|_0^{y=x^2} dx$$

$$= \int_0^1 x^2 e^{5x^3}\, dx = \frac{e^{5x^3}}{15}\Bigg|_{x=0}^{x=1} = \frac{1}{15}(e^5 - 1)$$

Prob. 8 : Evaluate $\int_0^1 \int_{x^2+1}^2 \frac{x y e^y}{y-1}\, dy\, dx$

R: $y = x^2 + 1$ $\quad y = 2$

$x = 0$ $\quad x = 1$



$$\int_1^2 \int_0^{\sqrt{y-1}} x\, y\, \frac{e^y}{y-1}\, dx\, dy$$

$$= \int_1^2 \frac{x^2}{2} \frac{y e^y}{y-1} \Bigg|_0^{x=\sqrt{y-1}} dy$$

$$= \int_1^2 \frac{\cancel{(y-1)}}{2} \frac{y e^y}{\cancel{(y-1)}}\, dy = \int_1^2 \frac{y}{u} \frac{e^y\, dy}{dv}$$

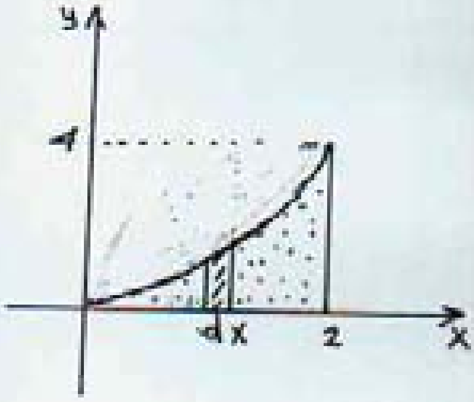
$$= \frac{1}{2}\left\{ e^y y - \int_1^2 e^y\, dy \right\} = \frac{1}{2} e^y (y-1) \Big]_1^2$$

$$= \frac{1}{2} e^2$$

Prob. 9 : Evaluate $\int_0^4 \int_{\sqrt{y}}^2 \cos(4x^3+5)\, dx\, dy$

Sol$\underline{n}$: R: $x=\sqrt{y}$ $\quad x=2$

$y=0 \quad y=4$



$$\int_0^2 \int_0^{x^2} \cos(4x^3+5)\, dy\, dx$$

$$= \int_0^2 y \cos(4x^3+5) \Big|_0^{x^2} dx$$

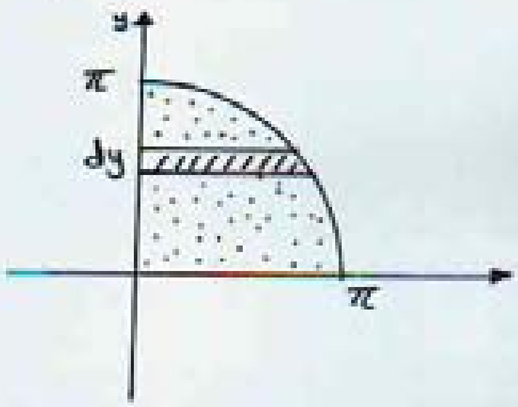
$$= \int_0^2 x^2 \cos(4x^3+5)\, dx = \frac{\sin(4x^3+5)}{12} \Bigg|_{x=0}^{x=2}$$

$$= \frac{1}{12}(\sin 37 - \sin 5) = (\qquad)$$

Prob. 10 : Evaluate $\int_0^\pi \int_0^{\sqrt{\pi^2-y^2}} \frac{x^2 y}{\sqrt{x^2+y^2}}\, dx\, dy$

Sol$\underline{n}$: R: $x=0 \quad x=\sqrt{\pi^2-y^2}$

$y=0 \quad y=\pi$



R: $r=0 \quad r=\pi$

$\theta=0 \quad \theta=\frac{\pi}{2}$

$$I = \int_0^{\pi/2}\int_0^\pi \frac{(r\cos\theta)^2(r\sin\theta)}{r} \cdot r\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^\pi r^3 \cos^2\theta \sin\theta\, dr\, d\theta$$

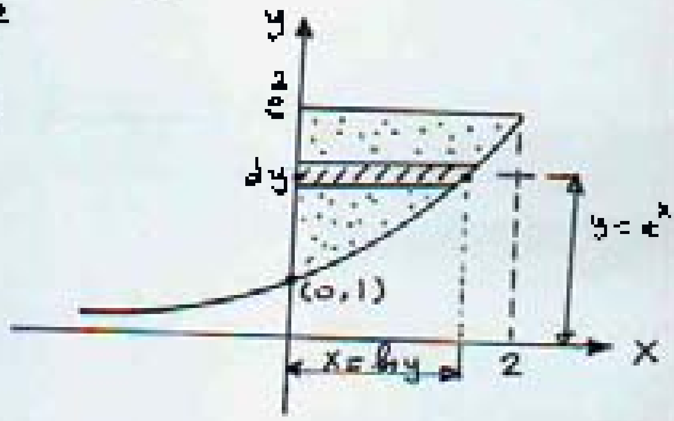
$$= \int_0^{\pi/2} \frac{r^4}{4}\Big|_0^\pi \sin\theta\cos^2\theta\, d\theta = \frac{\pi^4}{4}\left(\frac{-\cos^3\theta}{3}\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi^4}{12}$$

Prob. 11 : Evaluate $\int_0^2 \int_{e^x}^{e^2} \frac{y \sin y}{\ln y} \, dy \, dx$

Soln : R: $y = e^x$ $\quad$ $y = e^2$

$x = 0$ $\quad$ $x = 2$



$$\int_1^{e^2} \int_0^{\ln y} \frac{y \sin y}{\ln y} \, dx \, dy$$

$$= \int_1^{e^2} x \Big|_{x=0}^{x=\ln y} \frac{y \sin y}{\ln y} \, dy$$

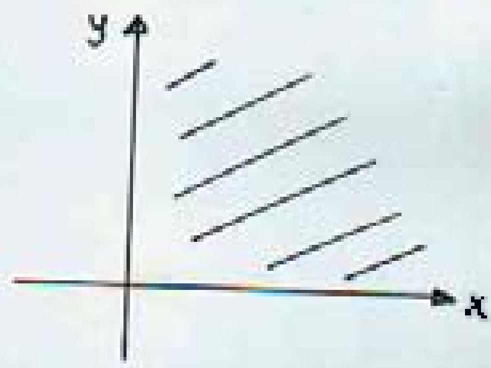
$$= \int_1^{e^2} \ln y \frac{y \sin y}{\ln y} \, dy = \int_1^{e^2} \underset{u}{y} \, \underset{dv}{\sin y \, dy}$$

$$= -y \cos y \Big|_1^{e^2} + \int_1^{e^2} \cos y \, dy = -y \cos y + \sin y \Big|_1^{e^2} = (\quad)$$

Prob. 12 : Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx \, dy$

soln : R: $x = 0$ $\quad$ $x = \infty$

$y = 0$ $\quad$ $y = \infty$



$$I = \int_0^{\pi/2} \int_0^\infty e^{-r^2} \, r \, dr \, d\theta$$

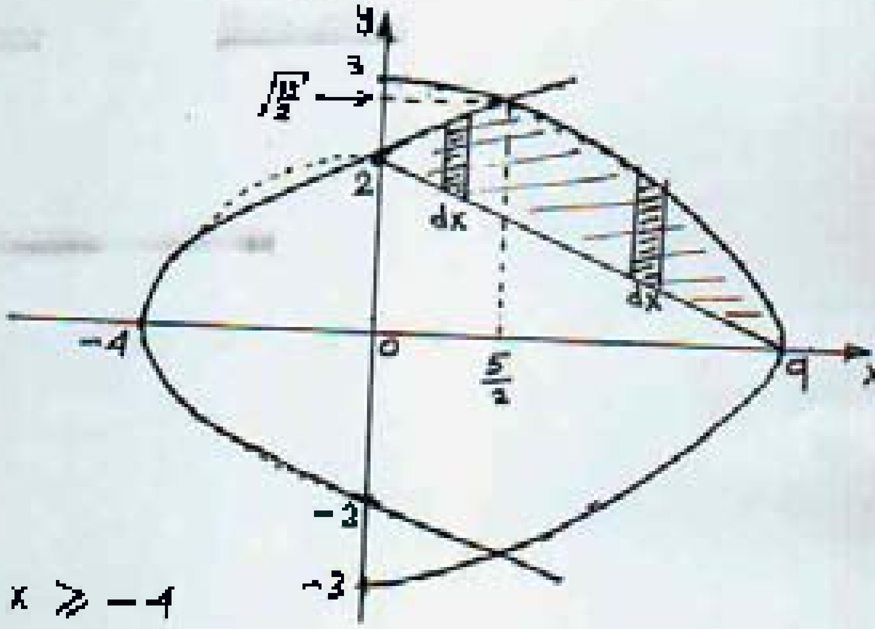
$$= -\frac{1}{2} \int_0^{\pi/2} e^{-r^2} \Big|_0^\infty d\theta$$

$$= -\frac{\pi}{4} \left[\frac{1}{\infty} - 1 \right] = \frac{\pi}{4}$$

Prob. 13 : Evaluate $\int_0^2 \int_0^x \frac{dy \, dx}{\sqrt{x^2+y^2}}$

Prob. 14: Using double integral to find the area of the region that is bounded by $x = 9 - y^2$ & $x = y^2 - 4$ & $\frac{x}{9} + \frac{y}{2} = 1$

Sol:



① $x = 9 - y^2$

$9 - x = y^2$

$9 - x \geq 0$

$\therefore x \leq 9$

② $x = y^2 - 4$

$x + 4 = y^2$

$x + 4 \geq 0 \qquad \therefore x \geq -4$

To find the intersection points of two curves

$9 - y^2 = y^2 - 4 \implies y = \sqrt{\frac{13}{2}}, \quad x = \frac{5}{2}$

$$\text{area} = \iint_R dy\,dx = \int_0^{5/2}\int_{2-\frac{2}{9}x}^{\sqrt{x+4}} dy\,dx + \int_{\frac{5}{2}}^{9}\int_{2-\frac{2}{9}x}^{\sqrt{9-x}} dy\,dx$$

$$= \int_0^{5/2} (\sqrt{x+4}) - (2 - \tfrac{2}{9}x)\,dx + \int_{5/2}^{9} (\sqrt{9-x}) - (2 - \tfrac{2}{9}x)\,dx$$

$$= \frac{2}{3}(x+4)^{3/2} - 2x + \frac{x^2}{9}\Bigg|_{x=0}^{x=\frac{5}{2}} - \frac{2}{3}(9-x)^{3/2} - 2x + \frac{x^2}{9}\Bigg|_{x=\frac{5}{2}}^{x=9}$$

$$= \left[\frac{2}{3}\left(\frac{13}{2}\right)^{3/2} - 5 + \frac{25}{36} - \frac{16}{3} - 18 + 9 + \frac{2}{3}\left(\frac{13}{2}\right)^{3/2} + 5 - \frac{25}{36}\right]$$

$$= \frac{4}{3}\left(\frac{13}{2}\right)^{3/2} - \frac{16}{3} - 9$$

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Advance Engineering Mathematics
Chapter ( ): Vectors
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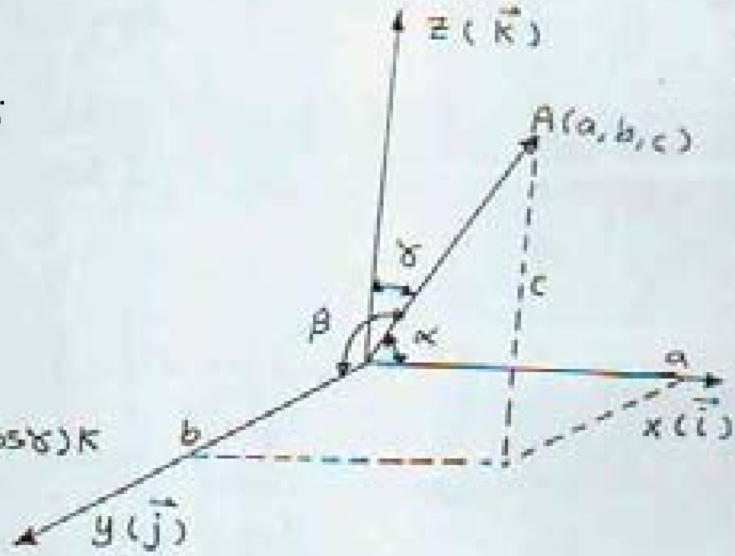
Vectors :



$\vec{A} = \vec{OA} = a\hat{i} + b\hat{j} + c\hat{k}$

$|\vec{A}| = \sqrt{a^2 + b^2 + c^2}$

unit vector $= \vec{u} = \dfrac{\vec{A}}{|\vec{A}|}$

$= (\cos\alpha)\hat{i} + (\cos\beta)\hat{j} + (\cos\gamma)\hat{k}$

where

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$(a, b, c) \Rightarrow$ direction numbers

$(\alpha, \beta, \gamma) \Rightarrow$ direction angles

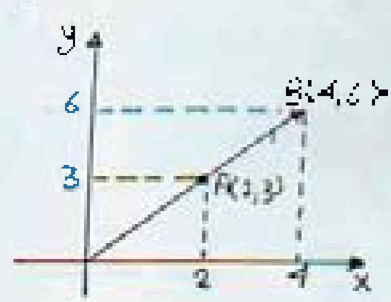
$(\cos\alpha, \cos\beta, \cos\gamma) \Rightarrow$ direction cosines

Paralel Vectors :

let $\vec{A}$ & $\vec{B}$ are two vector quantities, then

if $\vec{A} \parallel \vec{B} \Rightarrow \vec{B} = t\,\vec{A}$

where t is a scalar quantity



ex.1 : $\vec{A} = 2\hat{i} + 3\hat{j}$ & $\vec{B} = 4\hat{i} + 6\hat{j}$

$\therefore \vec{B} = 2(2\hat{i} + 3\hat{j})$

$= 2\vec{A}$ parallel

Product of two vectors :

let $\vec{A} = a_1 i + a_2 j + a_3 k$

$\vec{B} = b_1 i + b_2 j + b_3 k$

① Dot . Product (scalar product)

وهي طريقة مهمة لمعرفة
الزاوية بين متجهين

$$\vec{A} \cdot \vec{B} = |\vec{A}| \, |\vec{B}| \cos\theta$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3$$

Properties :

1. $\vec{A} \cdot \vec{A} = |\vec{A}|^2$
2. $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
3. $\vec{A} \perp \vec{B} \Rightarrow \vec{A} \cdot \vec{B} = 0$
4. $\cos\theta = \dfrac{\vec{A} \cdot \vec{B}}{|\vec{A}| \, |\vec{B}|}$
5. $i \cdot i = j \cdot j = k \cdot k = 1$

   $i \cdot j = j \cdot k = k \cdot i = 0$

② cross product (vector product)

طريقة مهمة لمعرفة المتجه
العمود على متجهين

$$\vec{A} \times \vec{B} = \vec{n} \, |\vec{A}| \, |\vec{B}| \sin\theta$$

where $\vec{n}$ is a unit vector normal to both $\vec{A}$ & $\vec{B}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = + i(a_2 b_3 - a_3 b_2) - j(a_1 b_3 - a_3 b_1) + k(a_1 b_2 - a_2 b_1)$$

Properties:

1. $\vec{A} \times \vec{A} = 0$
2. $\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$
3. $\vec{A} \parallel \vec{B} \Rightarrow \vec{A} \times \vec{B} = 0$

   $(\vec{A} \times \vec{B} \Leftrightarrow \vec{A} \cdot \vec{B} = 0)$
4. $i \times i = j \times j = k \times k = 0$

$i \times j = k$	$j \times i = -k$
$j \times k = i$	$k \times j = -i$
$k \times i = j$	$i \times k = -j$

Triple Product:

1. Vector triple product $= \vec{A} \times (\vec{B} \times \vec{C})$

$$= (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

2. scalar triple product $= \vec{A} \cdot (\vec{B} \times \vec{C})$

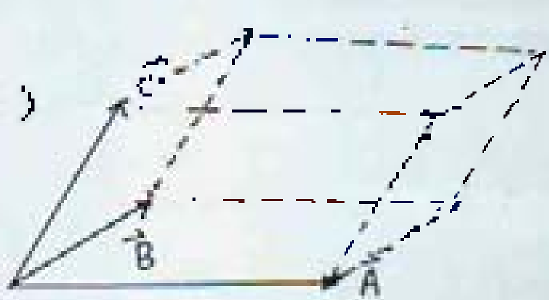
$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Volume of a box & pyramid حجم المكعب والهرم

volume of the box $= |\vec{A} \cdot (\vec{B} \times \vec{C})|$

volume of pyramid (tetrahedron) رباعي السطوح

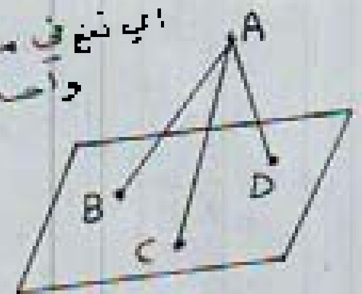
$$= \frac{1}{6} |\vec{A} \cdot (\vec{B} \times \vec{C})|$$



$\underline{ex\text{-}1}$: let $A(2,1,1)$, $B(3,2,5)$, $C(0,2,2)$, $D(4,5,6)$

show that A, B, C, D are non-coplanar points.

sol: if $\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = 0$ then A, B, C and D are coplanar points. ⇐ اي تقع في مستوى واحد



$\vec{AB} = i + j + 4k$

$\vec{AC} = -2i + j + k$

$\vec{AD} = 2i + 4j + 5k$

* اي ثلاث نقاط يوجد مستوى وحيد يحتويها

$\therefore \quad \vec{AB} \cdot (\vec{AC} \times \vec{AD}) = \begin{vmatrix} 1 & 1 & 4 \\ -2 & 1 & 1 \\ 2 & 4 & 5 \end{vmatrix}$

$= 1(5*1 - 4*1) - 1(5*(-2) - 2*1) + 4(4*(-2) - 2*1)$

$= 1 + 12 - 40 \neq 0$

$\therefore$ A, B, C and D are non-coplanar points.

لذلك فانها تشغل حجم مكعب

$\underline{ex.2}$: let $\vec{A} = 2i + 3j + k$, $\vec{B} = i - j - k$ & $\vec{C} = 5i + j + 2k$. Find the vector of length 3 units that is normal to $\vec{C}$ and lies in the plane determined by $\vec{A}$ and $\vec{B}$.

sol:

المطلوب ايجاد متجه طوله ثلاث وحدات وعمودي على المتجه $\vec{C}$ ويقع في المستوى (AB) اي الذي يحوي $\vec{B}$, $\vec{A}$ in dr

let $\vec{L}$ is the required vector

and $\vec{N}$ is the vector normal on $\vec{A}$ & $\vec{B}$

and M is the plane determined by $\vec{A}$ & $\vec{B}$

$\therefore \quad \vec{N} = \vec{A} \times \vec{B}$

$\therefore \quad \vec{N} \perp M \quad \Rightarrow$ { المتجه العمود على متجهين يكون عمودياً على المستوى الذي يحويهما }

$\because \quad \vec{L}$ lies in M

$\therefore \quad \vec{N} \perp \vec{L} \quad \Rightarrow$ { المتجه العمودي على مستوٍ يكون عمودياً على جميع المتجهات المحتواة في ذلك المستوى }

$\because \quad \vec{L} \perp \vec{C} \quad \Rightarrow$ { من منطوق السؤال }

$\therefore \quad \vec{L} = \vec{C} \times \vec{N} = \vec{C} \times (\vec{A} \times \vec{B})$

$= (\vec{C} \cdot \vec{B})\vec{A} - (\vec{C} \cdot \vec{A})\vec{B}$

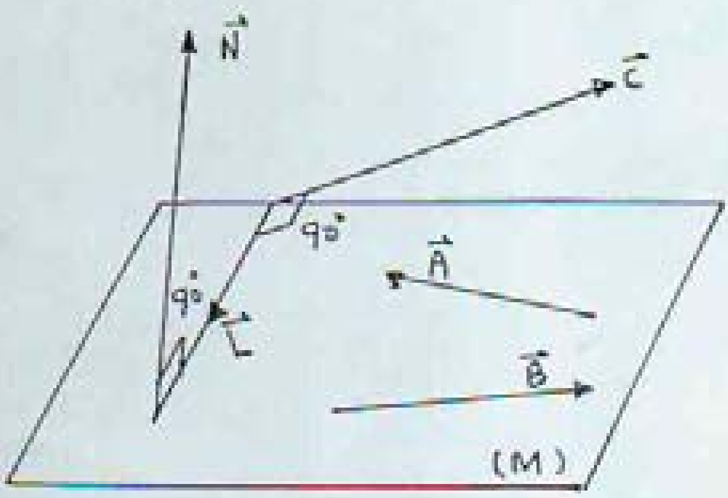
$\therefore \quad \vec{L} = (5-1-2)(2i+3j+k) - (10+3+2)(i-j-k)$

$= 2(2i+3j+k) - 15(i-j-k)$

$= 4i + 6j + 2k - 15i + 15j + 15k$

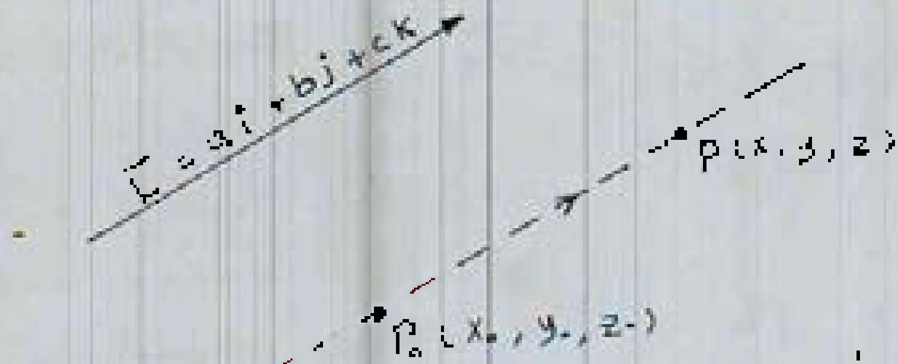
$= -11i + 21j + 17k$

$\therefore$ المتجه المطلوب $= 3 * \dfrac{\vec{L}}{|\vec{L}|} = 3 * \dfrac{-11i + 21j + 17k}{\sqrt{(-11)^2 + (21)^2 + (17)^2}}$



Equation of a Line in a Space :-

المطلوب : ايجاد معادلة المستقيم المار بالنقطة P_0 والموازي للمتجه $\vec{L}$



نفرض أن $p(x, y, z)$ أي نقطة على المستقيم

$$\overrightarrow{P_0P} = (x - x_0)i + (y - y_0)j + (z - z_0)k$$

$$\overrightarrow{P_0P} \parallel \vec{L} \Rightarrow \overrightarrow{P_0P} = t\,\vec{L}$$

$$(x - x_0)i + (y - y_0)j + (z - z_0)k = t\,(ai + bj + ck)$$

$$\therefore \left.\begin{aligned} x - x_0 &= t\,a \\ y - y_0 &= t\,b \\ z - z_0 &= t\,c \end{aligned}\right\}$$ parametric form of the eq. الشكل الوسيطي

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$ standard form الشكل القياسي

$a \neq 0 \quad b \neq 0 \,,\, c \neq 0$

* ملاحظة : لايجاد معادلة المستقيم يجب ان يتوفر شرطان :-

① نقطة معلومة ② المتجه الموازي معلوم

ex. 3 : Find the equation of the line that passes through

① $A(2, 1, 4)$, $B(3, 7, 5)$ ② $A(2, 1)$, $B(3, 1)$ ③ $A(2, 1, 4)$, $B(3, 7, 4)$

Sol$\underline{n}$:

① $\overrightarrow{AB} = (3-2)i + (7-1)j + (6-4)k$
$= i + 6j + 2k$ ⟹ المتجه الاتجاهي

∴ the eq. of Line is :

$$\frac{x-2}{1} = \frac{y-1}{6} = \frac{z-4}{2}$$

② $\overrightarrow{AB} = (3-2)i + (7-1)j$
$= i + 6j$

∴ the eq. of Line is $\dfrac{x-2}{1} = \dfrac{y-1}{6}$

③ $\overrightarrow{AB} = (3-2)i + (7-1)j + (4-4)k$
$= i + 6j$

$a = 1$, $b = 6$, $c = 0$

* ولا يمكن استعمال الصيغة التناظرية (لأن $c = 0$) لذلك نستعمل الصيغة الوسيطية

$$\left.\begin{array}{l} x - 2 = t \cdot 1 \\ y - 1 = t \cdot 6 \end{array}\right\} \Rightarrow \frac{x-2}{1} = \frac{y-1}{6}$$ مستقيم

$$z - 4 = t \cdot 0 \Rightarrow z = 4$$ مستوي

~ * ~ * ~ * ~

ex.4 : Find the vector that is parallel to the Line whose equation is $\dfrac{x-2}{3} = \dfrac{7-2y}{6} = \dfrac{3z+4}{6}$ also find at least two points on the Line.

Sol$\underline{n}$

* نكتب المعادلة بصورتها الصحيحة

$$\frac{x-2}{3} = -2\,\frac{y-\frac{7}{2}}{6} = 3\,\frac{z-(-\frac{4}{3})}{6}$$

$$\frac{x-2}{3} = \frac{y-\frac{7}{2}}{-3} = \frac{z-(-\frac{4}{3})}{2} = t \quad \text{——(1)}$$

∴ $P(2, \frac{7}{2}, -\frac{4}{3})$ & $\vec{L} = 3i - 3j + 2k$

from Eq(1), gets :

$x = 3t + 2$

$y = \frac{7-11t}{2}$

$z = \frac{6t-4}{3}$

$\left.\right\}$ لكل t

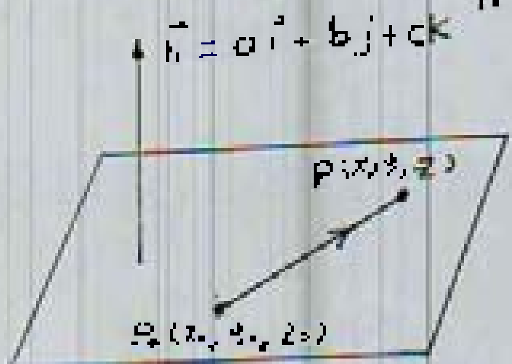
$t = 0 \Rightarrow P_0(1, \frac{7}{2}, -\frac{4}{3})$

$t = 3 \Rightarrow P_3(11, -\frac{11}{2}, \frac{20}{3})$

$\left.\right\}$ two points

Equation of a Plane : (معادلة المستوى)

المطلوب: ايجاد معادلة المستوى المار بالنقطة P_0 والعمودي على المتجه $\vec{N}$



نفرض أن $P(x, y, z)$ أية نقطة على المستوى

$\therefore \overrightarrow{P_0P} = (x - x_0)i + (y - y_0)j + (z - z_0)k$

$\overrightarrow{P_0P} \perp \vec{N} \Rightarrow \overrightarrow{P_0P} \cdot \vec{N} = 0$

$\Rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$ax + by + cz + \underbrace{(-ax_0 - by_0 - cz_0)}_{d} = 0$ كمية ثابتة نعرفها

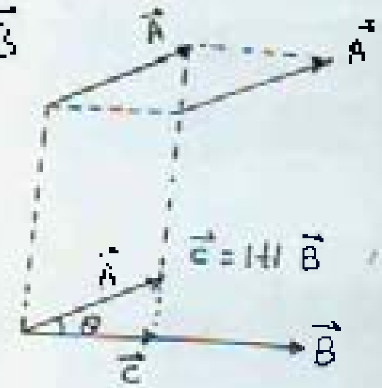
$$\boxed{ax + by + cz + d = 0}$$

الخلاصة: لايجاد معادلة المستوى يجب ان يتوفر لدينا شرطان

① نقطة معلومة ② المتجه العمودي على المستوى

Projection of two vectors :

$\vec{C}$ = vector projection of $\vec{A}$ onto $\vec{B}$



$$\vec{C} = \overrightarrow{\operatorname{proj}_{\vec{B}} \vec{A}}$$

$|\vec{C}|$ = scalar projection of $\vec{A}$ onto $\vec{B}$

$$|\vec{C}| = \operatorname{proj}_{\vec{B}} \vec{A}$$

$$|\vec{c}| = |\vec{A}| * |\cos\theta| = |\vec{A}| * \left| \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|} \right|$$

$$|\vec{c}| = \frac{|\vec{A}\cdot\vec{B}|}{|\vec{B}|}$$

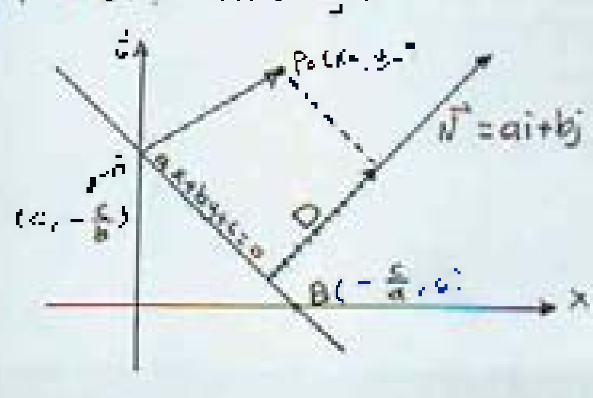
but $\vec{c} = |t|\,\vec{B}$ (parallel vectors)

$$\therefore |\vec{c}| = |t|\,|\vec{B}| \Rightarrow \frac{|\vec{A}\cdot\vec{B}|}{|\vec{B}|} = |t|\,|\vec{B}|$$

$$\therefore |t| = \frac{|\vec{A}\cdot\vec{B}|}{\vec{B}\cdot\vec{B}}$$

$$\therefore \vec{c} = \overrightarrow{\operatorname{proj}_{\vec{B}} \vec{A}} = \left(\frac{|\vec{A}\cdot\vec{B}|}{\vec{B}\cdot\vec{B}} \right) \vec{B}$$

ex : show that $D = \dfrac{|ax_0 + by_0 + c|}{\sqrt{a^2+b^2}}$ represents the short distance from $P_0(x_0, y_0)$ to the line $ax + by + c = 0$



Sol$_{\underline{n}}$ $\overrightarrow{AP_0} = x_0\, i + (y_0 + \frac{c}{b})\, j$

$\overrightarrow{AB} = -\frac{c}{a}\, i + \frac{c}{b}\, j$

$= (-\frac{c}{ab})(b\,i - a\,j)$

يؤثر على الطول فقط ولا يؤثر على الاتجاه

$\therefore \quad \vec{L} = b\hat{i} - a\hat{j}$ يمثل اتجاه المستقيم

يمثل اتجاه العمود

$\therefore \quad \vec{N} = a\hat{i} + b\hat{j}$

the vector $a\hat{i} + b\hat{j}$ is normal to $b\hat{i} - a\hat{j}$

$\therefore \quad D = proj_{\vec{N}} \overrightarrow{AP_0} = \left| \frac{\overrightarrow{AP_0} \cdot \vec{N}}{|\vec{N}|} \right| \Rightarrow$ وضعنا المطلق لإهمال الإشارة السالبة ان وجدت

$$= \frac{|ax_0 + b(y_0 + \frac{c}{b})|}{\sqrt{a^2 + b^2}}$$

$$= \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

Note : In three dimensional the extension of the above equation represents the distance between the point $P_0(x_0, y_0, z_0)$ & the plane $ax+by+cz+d=0$ as follows :

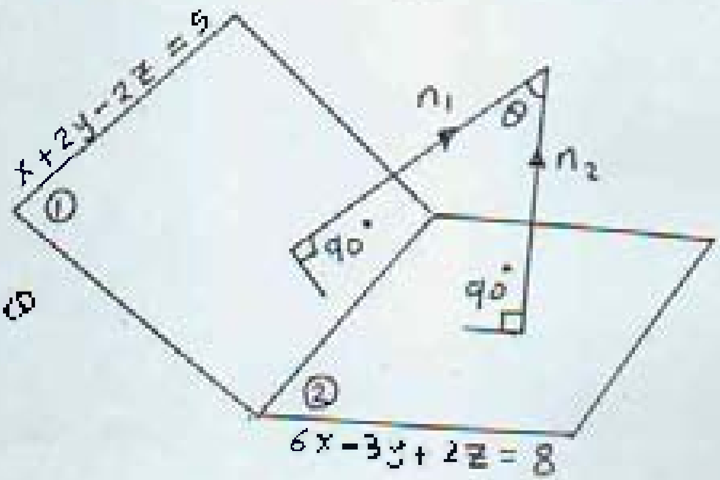
$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

University of Technology
Mechanical Engineering Department
Advance Engineering Mathematics
Sheet No.(4-6): Vector (Solved Problems)
Dr. Akeel Abdullah Mohammed

Prob. 1 : Find the acute angle of intersection of the planes (to the nearest degree)

$x + 2y - 2z = 5$ and $6x - 3y + 2z = 8$

Sol$\underline{n}$.



$n_1 = i + 2j - 2k$

$n_2 = 6i - 3j + 2k$

$n_1 \cdot n_2 = |n_1||n_2| \cos\theta$

$$\theta = \cos^{-1} \frac{n_1 \cdot n_2}{|n_1|\,|n_2|}$$

$$= \frac{(i+2j-2k)\cdot(6i-3j+2k)}{\sqrt{1^2+2^2+2^2}\,\sqrt{6^2+3^2+2^2}}$$

$$= \cos^{-1} \frac{6-6-4}{(3)(7)} = \cos^{-1} \frac{-4}{21} = 101^\circ$$

$\therefore$ Acute angle $= 180^\circ - 101^\circ = 79^\circ$ Ans.

Prob. 2 : Find the point of intersection of the line and plane shown below

Line : $x = 1 + t$, $y = -1 + 3t$, $z = 2 + 4t$

Plane : $x - y + 4z = 7$

Sol$\underline{n}$

point of intersection satisfy the line and plane.

hence substitute line in the equation of plane to get :

$$(1+t) - (-1+3t) + 4(2+4t) = 7$$
$$1 + t + 1 - 3t + 8 + 16t = 7$$
$$14t = -3 \quad \text{to get} \quad t = -\frac{3}{14}$$

$\therefore \quad x = 1 + t \implies x_o = 1 - \frac{3}{14} = \frac{11}{14}$

$y = -1 + 3t \implies y_o = -1 + 3\left(-\frac{3}{14}\right) = -1 - \frac{9}{14} = -\frac{23}{14}$

$z = 2 + 4t \implies z_o = 2 + 4\left(-\frac{3}{14}\right) = \frac{16}{14}$

$\therefore$ point of intersection shall be ;

$$P_o\left(\frac{11}{14}, -\frac{23}{14}, \frac{16}{14}\right)$$

Prob. 3 : Find the coordinates of point of intersection between the line shown below and xy-plane ;

Line ; $x = 3 - t$; $y = 1 + 2t$; $z = 1 + 3t$

Soln :

from the equation of line shown above, intersection with xy-plane gives $z = 0$

$\therefore \quad 0 = 1 + 3t \implies t = -\frac{1}{3}$

$\therefore \quad x_o = 3 - \frac{1}{3} = \frac{8}{3}$

$\therefore \quad y_o = -1 + 2\left(-\frac{1}{3}\right) = -\frac{5}{3}$

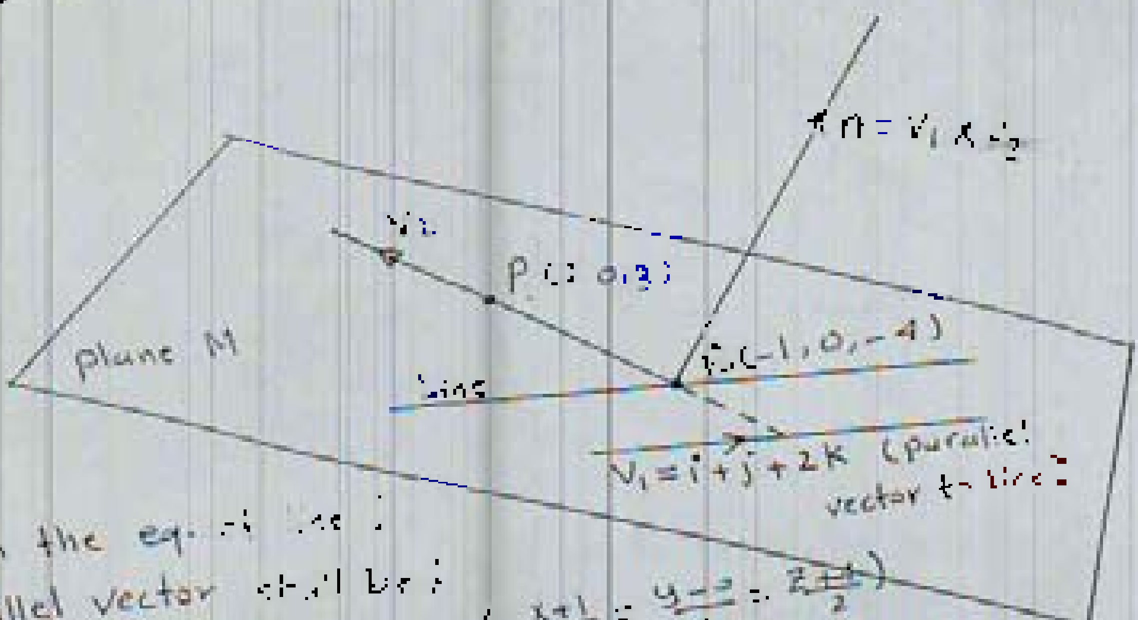
$\therefore$ the point of intersection shall be :

$$P\left(\frac{8}{3}, -\frac{5}{3}, 0\right)$$

Ans

Prob-1 : Find an equation of the plane that contains the point $(2,0,3)$ and the line $x = -1+t$; $y = t$; $z = -4+2t$

Sol$\underline{n}$



from the eq. of line ;
Parallel vector shall be ;
$v_1 = i + j + 2k$
and $P_0(-1, 0, -4)$
$v_2 = P_0P_1 = (3i + 0j + 7k) = 3i + 7k$

$\therefore n = v_1 \times v_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 3 & 0 & 7 \end{vmatrix} = i(7-0) - j(7-6) + k(0-3)$

$\therefore n = 7i - j - 3k$

$\therefore$ consider the normal vector $n = 7i - 9j + k$ and know point $P_0(-1, 0, -4)$ to find the eq of plane where;

$ax + by + cz = d$

$7x - y - 3z = d$

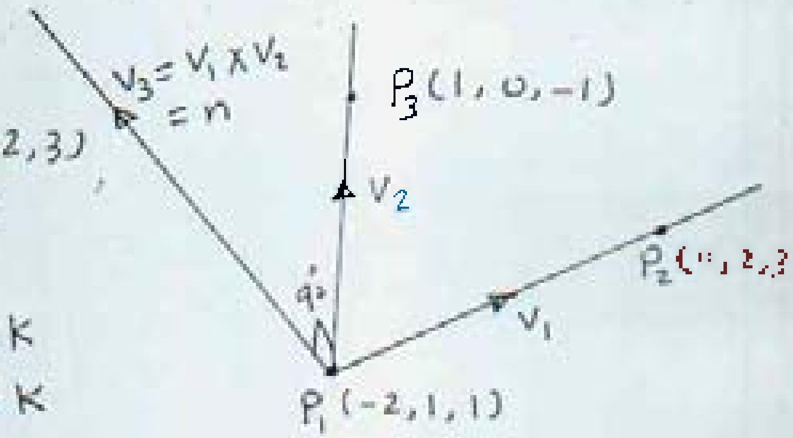
$7(-1) - 0 - 3(-4) = d$

$\therefore d = 5$

$\therefore 7x - 9y + z = 5$ $\underline{\text{Ans}}$

Prob. 5: Find the equation of plane passing through the points $(-2,1,1)$, $(0,2,3)$ and $(1,0,-1)$.

Soln:



let $P_1(-2,1,1)$, $P_2(0,2,3)$ and $P_3(1,0,-1)$

$$V_1 = \overrightarrow{P_1P_2} = 2i + j + 2k$$
$$V_2 = \overrightarrow{P_1P_3} = 3i - j - 2k$$

V_3 = normal vector $n = V_1 \times V_2$

$$V_3 = \begin{vmatrix} i & j & k \\ 2 & 1 & 2 \\ 3 & -1 & -2 \end{vmatrix} = i(-2+2) - j(-4-6) + k(-2-3)$$

$\therefore$ $V_3 = 10j - 5k$ From which $a=0$, $b=10$ and $c=-5$

from $P_1(-2,1,1)$ $x_0=-2$, $y_0=1$, $z_0=1$

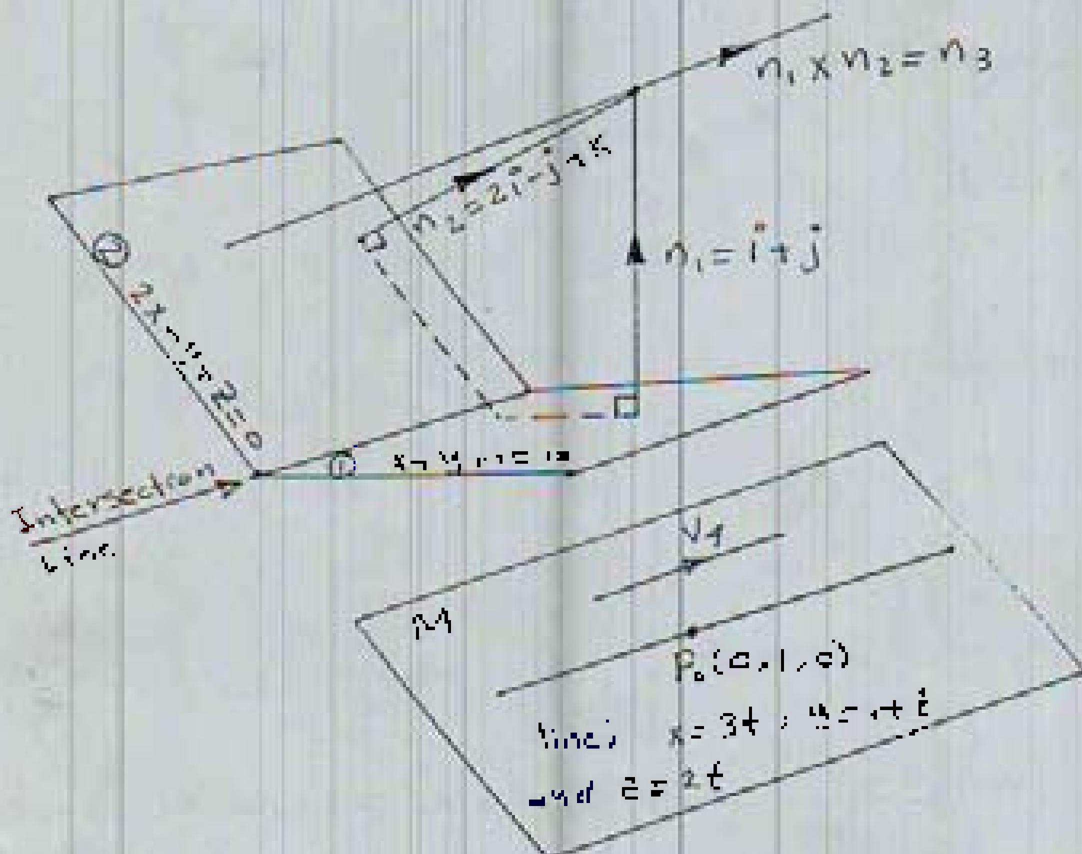
$\therefore$ Equation of plane shall be:

$$ax + by + cz + d = 0$$
$$0x + 10y - 5z + d = 0$$
$$0 + 10(1) - 5(1) + d = 0$$
$$\therefore d = -5$$

$\therefore$ $\boxed{10y - 5z = 5}$ Ans

Prob. 6 : Find an equation of the plane containing the line $x = 3t$, $y = 1 + t$, $z = 2t$ and parallel to the intersection of the plane $2x - y + z = 0$ and $x + y + 1 = 0$.

Sol:



let M is the required plane.

From Equation of Line: $\frac{x - 0}{3} = \frac{y - 1}{1} = \frac{z - 0}{2}$

$V_4 = 3\hat{i} + \hat{j} + 2K$

$P_0\ (0, 1, 0)$

From Equation of Planes:

$n_1 = \hat{i} + \hat{j}$; $n_2 = 2\hat{i} - \hat{j} + K$

find n_3

$$n_3 = \begin{vmatrix} \hat{i} & \hat{j} & K \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{vmatrix} = \hat{i}(1 - 0) - \hat{j}(1 - 0) + K(-1 - 2)$$

$\therefore\ n_3 = \hat{i} - \hat{j} - 3K$

then find $n_4 = n_3 \times v_4 =$

or
$$n_4 = \begin{vmatrix} i & j & k \\ 1 & -1 & -3 \\ 3 & 1 & 2 \end{vmatrix} = i(-2+3) - j(2+9) + k(1+3)$$

$\therefore \quad n_4 = i - 11j + 4k$

$\therefore$ There are normal vector $n_4 = i - 11j - 4k$ with $a = 1$, $b = -11$, $c = 4$ and Point $P_o(0, 1, 0)$ with $x_o = 0$, $y_o = 1$, and $z_o = 0$ to find equation of the plane as follows:

$$x - 1 \cdot y + 4z = d$$
$$0 - 1(1) + 4(0) = d \qquad \therefore d = -11$$

$\therefore$ The equation of required plane N is:

$$\boxed{x - 11y + 4z = -11}$$

Ans

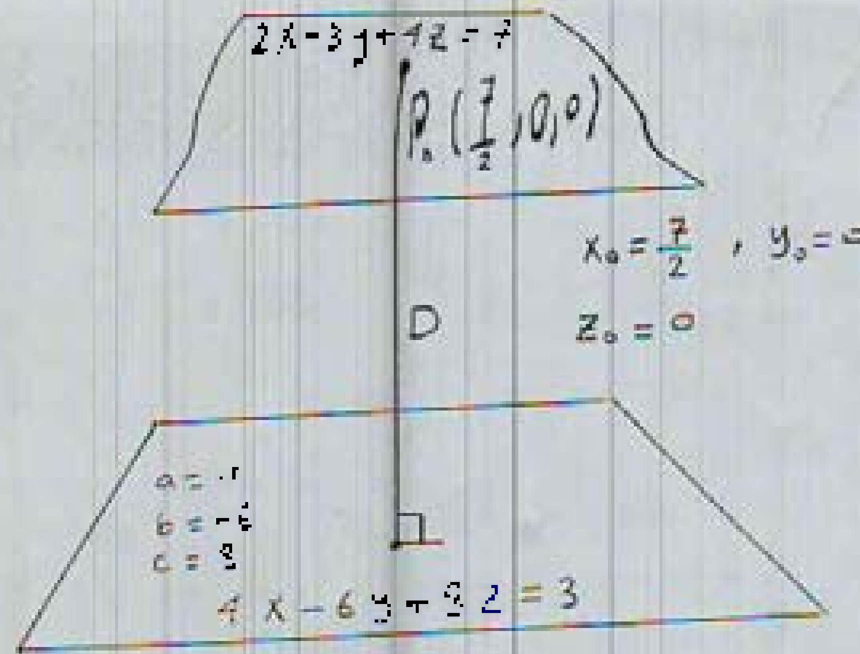
* ~ * ~ * ~ * ~ *

Prob. 7: Find distance between the given parallel planes $2x - 3y + 4z = 7$ and $4x - 6y + 8z = 3$

Sol'n: To find the distance between the planes, we may select an arbitrary point in one of the planes. hence, by selecting $y = z = 0$ in the equation $2x - 3y + 4z = 7$ we obtain $P_o(\frac{7}{2}, 0, 0)$ then;

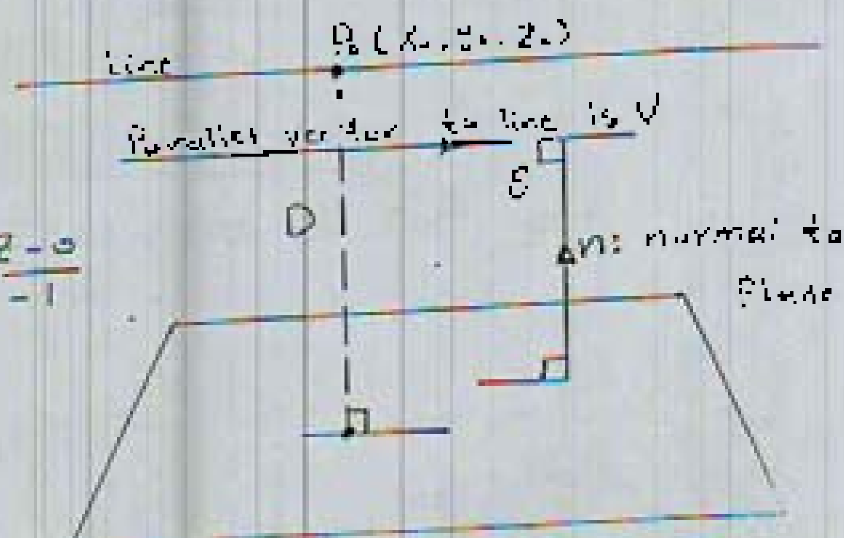
$$D = \left| \frac{ax_o + by_o + cz_o + d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

$$= \left| \frac{(4)(\frac{7}{2}) + (-6)(0) + 8(0) - 3}{\sqrt{4^2 + 6^2 + 8^2}} \right| = \frac{11}{\sqrt{116}} = \frac{11}{2\sqrt{29}}$$



Prob. 8: Show that the line $x=-1+t$; $y=3+2t$, $z=-t$ and the plane $2x-2y-2z+3=0$ are parallel then find the distance between them.

Sol:



From Eq. of Line:

$$\frac{x+1}{1}=\frac{y-3}{2}=\frac{z-0}{-1}$$

$$v=i+2j-k$$

$$P_0=P_0(-1,3,0)$$

From Eq of Plane:

$$\vec{n}=2i-2j-2k$$

• لإثبات حالة التوازي بين المستقيم والمستوي يجب أن تكون الزاوية المحصورة بين المتجه العمود n على المستوي والمتجه الموازي للمستقيم V هي 90° .

$$n \cdot v = |n|\,|v| \cos\theta$$

$$\cos\theta = \frac{n \cdot v}{|n|\,|v|} = \frac{(2i-2j-2k)\cdot(i+2j-k)}{\sqrt{2^2+2^2+2^2}\,\sqrt{1^2+2^2+1^2}}$$

$$= \frac{2-4+2}{\sqrt{12}\,\sqrt{6}} = 0$$

$$\therefore \quad \theta = \cos^{-1} 0 = \frac{\pi}{2}$$

$\therefore$ The plane and Line are parallel

Hence, to find the distance between the line and the plane, we will find the distance between the point P_0 that lies on the line and the plane ;

$P_0(-1, 3, 0)$ and plane $2x-2y+2z+3=0$

where $x_0 = -1$, $y_0 = 3$ and $z_0 = 0$
and $a = 2$; $b = -2$; and $c = 2$; $d = 3$

and $$D = \left| \frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2+b^2+c^2}} \right|$$

$$= \left| \frac{(2)(-1)+(-2)(3)+(2)(0)+3}{\sqrt{2^2+2^2+2^2}} \right|$$

$$= \left| \frac{-2-6+0+3}{\sqrt{12}} \right| = \left| \frac{-5}{2\sqrt{3}} \right|$$

$$= \frac{5}{2\sqrt{3}}$$ Ans

Prob. 9 : Let L_1 and L_2 be the lines;

$$L_1: \quad \text{[illegible]} \qquad y = 5 - 4t \qquad z = -1 + 5t \quad ; \quad L_2: \quad x = 2 + 8t \qquad y = 4 - 3t \qquad z = 5 + t$$

a) Are the lines parallel.
b) Does the lines intersect or they are skew.
c) If they are skew find the distance between them.

Soln : From equation of L_1 the parallel vector shall be $\vec{V}_1 = 4i - 4j + 5K$ and from the L_2 the parallel vector $\vec{V}_2 = 8i - 3j + K$

hence L_1 shall be parallel to L_2 if $\vec{V}_1 \parallel \vec{V}_2$ when the angle between them is either (0) or (180°).

hence, $\vec{V}_1 \cdot \vec{V}_2 = |\vec{V}_1| |\vec{V}_2| \cos\theta$

$$\theta = \cos^{-1} \frac{\vec{V}_1 \cdot \vec{V}_2}{|\vec{V}_1| |\vec{V}_2|} = \cos^{-1} \frac{(4i - 4j + 5K)\cdot(8i - 3j + K)}{\sqrt{4^2 + 4^2 + 5^2}\sqrt{8^2 + 3^2 + 1^2}}$$

$$= \frac{32 + 12 + 5}{\sqrt{57}\sqrt{74}} = \cos^{-1} \frac{49}{\sqrt{57}\sqrt{74}} = 41^\circ$$

$\therefore$ the two lines are non-parallel hence, to determine whether they intersect or not.

ملاحظة : ان المستقيمين المتقاطعين يشتركان في نقطة التقاطع التي تحققهما ولذلك لنفرض ان نقطة التقاطع هي $P_0(x_0, y_0, z_0)$ والتي تحقق معادلات المستقيمين اعلاه نستخرجها كما يلي :-

$$x_0 = 1 + 4t_1 \qquad\qquad x_0 = 2 + 8t_2$$
$$y_0 = 5 - 4t_1 \quad \text{and} \quad y_0 = 4 - 3t_2$$
$$z_0 = -1 - 5t_1 \qquad\qquad z_0 = 5 + t_2$$

This leads to three conditions on t_1 & t_2

$$1+4t_1 = 2+8t_2 \quad \text{———} (1)$$
$$5-4t_1 = 4-3t_2 \quad \text{———} (2)$$
$$-1+5t_1 = 5+t_2 \quad \text{———} (3)$$

بحل المعادلتين (1) & (2) آنياً نحصل على t_1 , t_2 :-

$$1+4t_1 = 2+8t_2 \quad [\text{ from Eq. (1) }]$$
$$\therefore \quad 4t_1 = 1+8t_2$$

$$\therefore \quad t_1 = 0.25 + t_2 \quad \text{———} (4)$$

by substituting Eq.(4) into Eq.(2), will get:

$$5-4(0.25+t_2) = 4-3t_2$$
$$5-1-4t_2 = 4-3t_2 \implies t_2 = 0$$
$$\text{and} \quad t_1 = 0.25$$

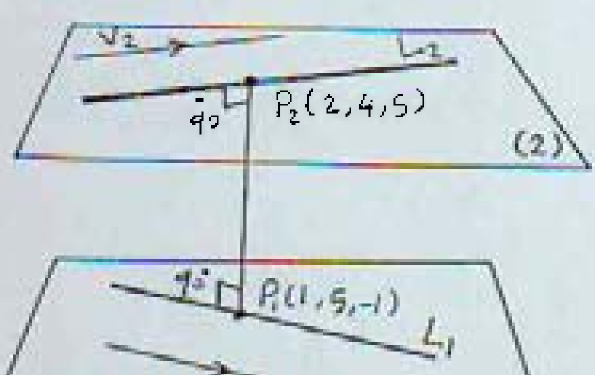
ولكي يكون المستقيمان متقاطعين يجب ان تتحقق المعادلة رقم (3) عند التعويض بقيم t_1 , t_2 فيها من خلال تساوي طرفيها الأيمن والأيسر .

$$\therefore \quad -1+5(0.25) \neq 5+0$$
$$\text{but} \quad 0.25 \neq 5$$

∴ المستقيمان غير متقاطعين وغير متوازيين

∴ يطلق على المستقيمان متخالفين (skew lines)

ولذلك سوف نجد البعد بينهما .



$$V_1 = 4\hat{i} - 4\hat{j} - 5\hat{k}$$
$$P_1 = P_1(1,5,-1)$$

$$V_2 = 8\hat{i} - 3\hat{j} + \hat{k}$$
$$P_2 = P_2(2,4,5)$$

المستوي رقم (1) سوف يحوي المستقيم L_1 والنقطة المعلومة P_1

وكذلك المستوي (2) سوف يحوي المستقيم L_2 والنقطة المعلومة P_2 ....

الاتجاهي للمتجهين الموازيين V_1 و V_2 لكل من المستقيمين L_1 و L_2 على التوالي ، والذي يكون عمودي على كل من المستويين .

سوف يكون n هو حاصل الضرب

$$\therefore \quad n = \vec{V_1} \times \vec{V_2} = \begin{vmatrix} i & j & k \\ 4 & -4 & 5 \\ 8 & -3 & 1 \end{vmatrix} = 11 i + 36 j + 20 k$$

hence, the equation of plane containing L_1 & P_1 is:

$$a x_0 + b y + c z + d = 0$$

$$11 x + 36 y + 20 z + d = 0$$

$$11 (2) + 36 (4) + 20 (5) + d = 0$$

$$\therefore \quad d = -266$$

$$\therefore \quad 11 x + 36 y + 20 z - 266 = 0$$

والآن يكون D البعد بين المستوي أعلاه والنقطة P_1 هو البعد بين المستقيمين المتخالفين .

$$D = \left| \frac{a x_0 + b y_0 + c z_0 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

where $P_1 (x_0, y_0, z_0) = P_1 (1, 5, -1)$

and $a = 11 \quad , b = 36 \quad , c = 20 \quad , d = -266$

$$\therefore \quad D = \left| \frac{(11)(1) + (36)(5) + (20)(-1) - 266}{\sqrt{11^2 + 36^2 + 20^2}} \right|$$

$$= \frac{95}{\sqrt{1817}} \quad \underline{Ans}$$

Prob 10 : Show that the lines :

$L_1 : \quad x = -1 + 4t \qquad , \qquad L_2 : \quad x = -13 + 12t$

$\qquad\quad y = 3 + t \qquad\qquad\qquad\qquad y = 1 + 6t$

$\qquad\quad z = 1 \qquad\qquad\qquad\qquad\quad z = 2 + 3t$

intersect and find the equation of plane they determines .

Sol$\underline{n}$. لكي يتقاطع المستقيمان يجب أن يشتركان في نقطة هي $P_\circ(x_\circ , y_\circ , z_\circ)$ والتي تحققهما .

$x_\circ = -1 + 4t_1 \qquad\qquad x_\circ = -13 + 12t_2$

$y_\circ = 3 + t_1 \qquad\qquad y_\circ = 1 + 6t_2$

$z_\circ = 1 \qquad\qquad z_\circ = 2 + 3t_2$

or ,

$$-1 + 4t_1 = -13 + 12t_2 \quad ----(1)$$

$$3 + t_1 = 1 + 6t_2 \quad ----(2)$$

$$1 = 2 + 3t_2 \quad ----(3)$$

From Eq.(1)

$$4t_1 = -12 + 12t_2$$

$$\therefore \quad t_1 = -3 + 3t_2 \quad ----(4)$$

by substituting Eq.(4) into Eq.(2) , will get :

$$3 + (-3 + 3t_2) = 1 + 6t_2$$

$$3 - 3 + 3t_2 = 1 + 6t_2$$

$$\therefore \quad t_2 = -\frac{1}{3} \quad \& \quad t_1 = -4$$

t_1 & t_2 must be satify Eq.(3)

$$1 = 2 + 3\left(-\frac{1}{3}\right)$$

$$1 = 1 \qquad \text{(i.e. satisfy)}$$

$\therefore \quad L_1$ & L_2 are intersect in point $P_\circ(-17, -1, 1)$

where :

$$x_0 = -1 + 4t_1 = -1 + 4(-4) = -17$$

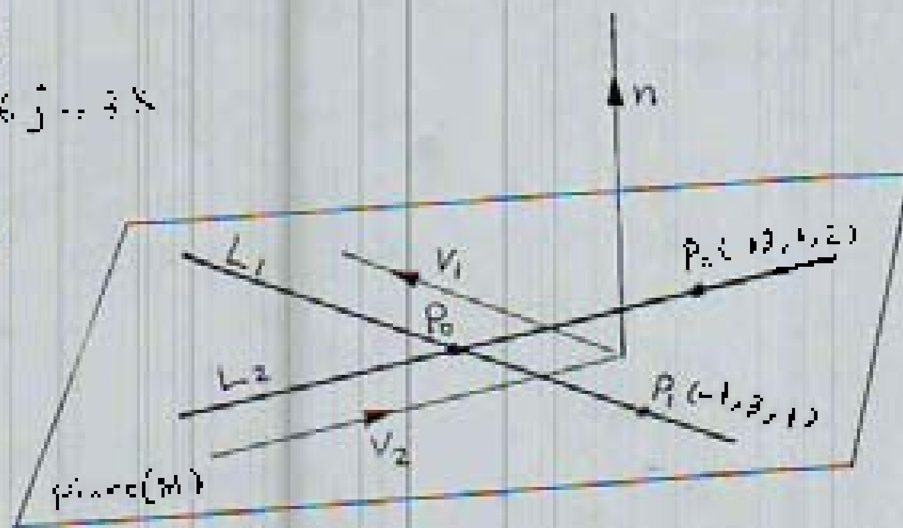
$$y_0 = 3 + t_1 = 3 - 4 = -1$$

$$z_0 = 1$$

والتي يمكن ان نحصل عليها أيضاً من معادلات المستقيم الثاني. وذلك بتعويض قيم t_2

$$\vec{V}_1 = 4i + j$$

$$\vec{V}_2 = 12i + 6j + 3k$$



$$\vec{n} = \vec{V}_1 \times \vec{V}_2$$

$$\vec{n} = \begin{vmatrix} i & j & k \\ 4 & 1 & 0 \\ 12 & 6 & 3 \end{vmatrix} = i(3-0) - j(12-0) + k(24-12)$$

$$\therefore \quad \vec{n} = 3i - 12j + 12k$$

The equation of plane (M) that contains L_1 & L_2 (i.e., it contains P_1, P_2 & P_0) is

$$3x - 12y + 12z + d = 0 \qquad \text{---- (5)}$$

by substituting point P_1, (or P_2 or P_0) into eq. (5) will get

$$3(-1) - 12(3) + 12(1) + d = 0$$

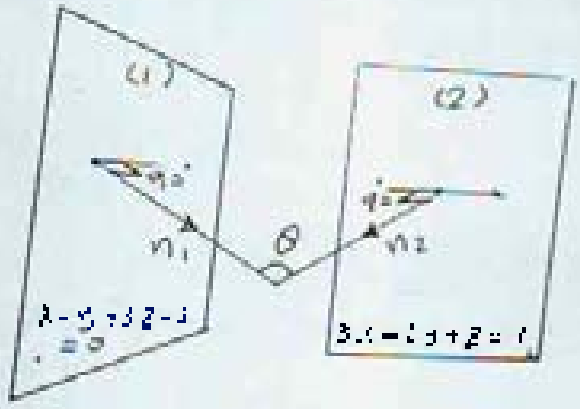
$$\therefore \quad d = 27$$

$$\therefore \quad 3x - 12y + 12z + 27 = 0$$

Ans

Prob. 11 : Determine weather the following planes are parallel or perpendicular or skew
$x - y + 3z - 2 = 0$ and $3x - 2y + z = 1$

Sol$\underline{n}$: from the equation of planes normal vectors shall be
$\vec{n}_1 = \hat{i} - \hat{j} + 3k$
$\vec{n}_2 = 3\hat{i} - 2\hat{j} + k$



$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

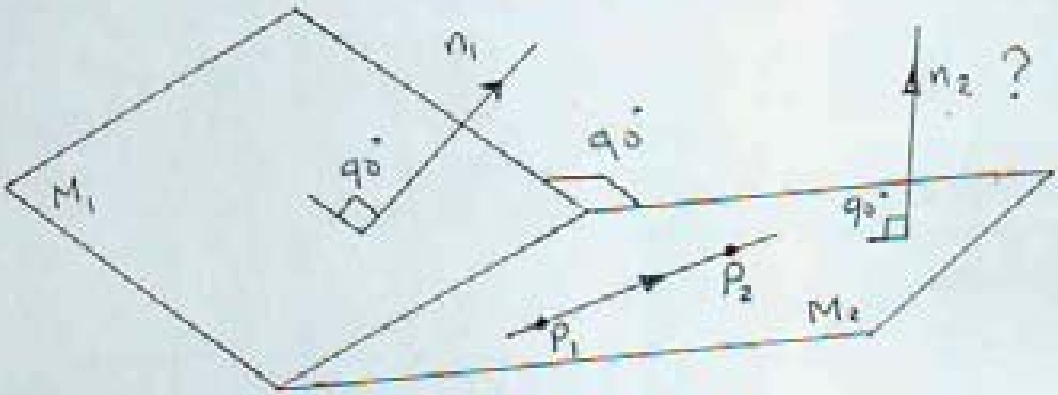
$$\theta = \cos^{-1}\frac{(\hat{i} - \hat{j} + 3k)\cdot(3\hat{i} - 2\hat{j} + k)}{\sqrt{1^2+1^2+3^2}\,\sqrt{3^2+2^2+1^2}} = \frac{3+2+3}{\sqrt{11}\,\sqrt{14}}$$

$= 49.35°$ not $90°$ (not perpendicular)
nor 0 (parallel)

$\therefore$ The two planes are skew.

Prob. 12 : Find the equation of plane through point $P_1(-2, 1, 4)$, $P_2(1, 0, 3)$ and perpendicular to the plane $4x - y + 3z = 2$.

Sol$\underline{n}$:



let M_2 is the required plane that contains P_1 & P_2

M_1 is the given plane

$$\therefore \overrightarrow{P_1P_2} = 3i - j - k$$

$$\overrightarrow{n_2} \perp M_2$$

$$\overrightarrow{n_2} = \overrightarrow{n_1} \times \overrightarrow{P_1P_2} = \begin{vmatrix} i & j & k \\ 4 & -1 & 3 \\ 3 & -1 & -1 \end{vmatrix}$$

$$= i(1+3) - j(-4-9) + k(-4+3)$$

$$= 4i + 13j - k$$

The equation of M_2 is

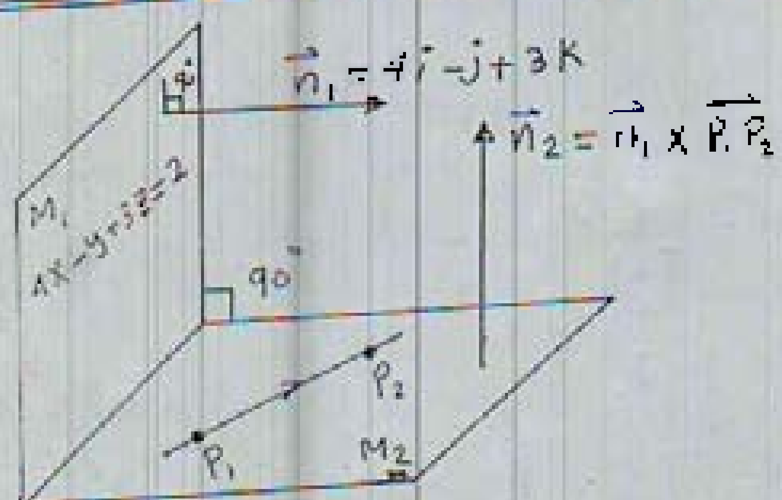
$$4x + 13y - z + d = 0$$

$4(-2) + 13(1) - 4 + d = 0$ (by substituting P_1)

$$\therefore d = -1$$

$$\therefore 4x + 13y - z - 1 = 0 \quad \underline{\text{Ans}}$$

• ملاحظة : ممكن رسم معالم هذا السؤال بالصورة التالية :-



1

University of Technology
Mechanical Engineering Department
Advance Engineering Mathematics
Chapter (3): Series
Dr. Akeel Abdullah Mohammed

Sequence : المتوالية

$[a_n]_{n=1}^{\infty} = a_1 , a_2 , a_3 , \ldots$

$[\frac{n+1}{n}]_{n=1}^{\infty} = 2 , \frac{3}{2} , \frac{4}{3} , \frac{5}{4} , \frac{6}{5} , \frac{7}{6} \rightarrow 1$

$a_n = \frac{n+1}{n}$ = general term

$\lim_{n \to \infty} a_n = \lim_{n \to \infty} (\frac{n+1}{n}) = \lim_{n \to \infty} (1 + \frac{1}{n}) = 1$

the sequence converges to 1

Convergence of Sequence : تقارب المتوالية

$[a_n]$ converges to L means that

$\lim_{n \to \infty} a_n = L$ where L is a single finite number

otherwise it diverges

Important Rules قوانين للحفظ

1. $\lim_{n \to \infty} x^n = 0 \qquad -1 < x < 1$

2. $\lim_{n \to \infty} x^n = \infty \qquad x > 1 , \quad x < -1$

3. $\lim\limits_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$ for all x

4. $\lim\limits_{n\to\infty} \left(\frac{a_n^p + \cdots}{b_n^k}\right) = \begin{cases} 0 & p < k \\ \frac{a}{b} & p = k \\ \infty & p > k \end{cases}$

5. $\lim\limits_{n\to\infty} \frac{x^n}{n!} = 0$ (any x)

6. $\lim\limits_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) = \lim\limits_{n\to\infty} \frac{f'(n)}{g'(n)}$ (L'Hospital Rule)

تستخدم قاعدة لوبيتال للحالات المبهمة 1^∞ , ∞^0 , 0^0 , $\infty-\infty$, $0\cdot\infty$, $\frac{\infty}{\infty}$, $\frac{0}{0}$ على ان تكون مكتوبة بالصيغة $\frac{0}{0}$ او $\frac{\infty}{\infty}$.

ex. 1: Test for convergence

1. $[2^n] \Rightarrow \lim\limits_{n\to\infty} 2^n = \infty$

$\therefore$ the sequence is diverge

$[(0.2)^n] \Rightarrow \lim\limits_{n\to\infty} (0.2)^n = 0$

$\therefore$ the sequence converges to 0

2. $[(-1)^n] = -1, 1, -1, 1, \ldots = \begin{cases} 1 & \text{if } n \text{ even} \\ -1 & \text{if } n \text{ odd} \end{cases}$

$\therefore [(-1)^n]$ diverges

3. $\left[\frac{2n^2+1}{3n^2+n+2}\right] \Rightarrow \lim\limits_{n\to\infty} \frac{2n^2+1}{3n^2+n+2} = \lim \frac{2+\frac{1}{n^2}}{3+\frac{1}{n}+\frac{2}{n^2}}$

$= \frac{2}{3}$ $\therefore$ the sequence converges to $\frac{2}{3}$

4. $\left[\left(1+\frac{3}{n}\right)^n\right] \Rightarrow \lim_{n\to\infty}\left(1+\frac{3}{n}\right)^n = e^3$

the sequence converges to e^3

5. $\left[\left(1-\frac{2}{n}\right)^{5n}\right] \Rightarrow \lim_{n\to\infty}\left[\left(1-\frac{2}{n}\right)^{5n}\right]$

$= \lim_{n\to\infty}\left[\left(1+\frac{(-2)}{n}\right)^n\right]^5 = (e^{-2})^5 = e^{-10}$

6. $\left[\frac{\ln n}{n}\right] \Rightarrow \lim_{n\to\infty}\left(\frac{\ln n}{n}\right) = \lim_{n\to\infty}\frac{\frac{1}{n}}{1} = 0$

the sequence converges to 0

7. $\left[n^{1/n}\right] \Rightarrow \lim_{n\to\infty}\left[n^{1/n}\right] = \lim_{n\to\infty} e^{\ln(n^{1/n})}$

$= \lim_{n\to\infty} e^{\frac{\ln n}{n}} = \lim_{n\to\infty} e^{1/n} = e^0 = 1$

نحول الصيغة الى بسط ومقام لكي نستطيع استخدام قاعدة لوبيتال

8. $\left[\frac{n+1}{5n^2+2}\right] \Rightarrow \lim_{n\to\infty}\frac{n\left(1+\frac{1}{n}\right)}{n^2\left(5+\frac{2}{n^2}\right)} = 0$

the sequence converges to 0

9. $\left[\frac{n^3+1}{n^2+3}\right] \Rightarrow \lim_{n\to\infty}\frac{3n^2}{2n} = \lim_{n\to\infty}\frac{6n}{2} = \infty$ diverges

10. $\left[\left(2+\frac{3}{n}\right)^n\right] \Rightarrow \lim_{n\to\infty}\left(2+\frac{3}{n}\right)^n = \lim_{n\to\infty} 2^n \cdot \lim_{n\to\infty}\left(1+\frac{3/2}{n}\right)$

$= \infty \cdot e^{3/2} = \infty$ diverges

11. $\sqrt{n^2+n} - n \Rightarrow \lim_{n\to\infty}\sqrt{n^2+n} - n = \infty$ diverges

12. $\left[\frac{e^n - e^{-n}}{e^n + e^{-n}}\right] \Rightarrow \lim_{n\to\infty}\frac{\left(1-\frac{1}{e^{2n}}\right)}{\left(1+\frac{1}{e^{2n}}\right)} = 1$ converges to 1

Series :

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

$\therefore$ The series converges to 1

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots = \infty$$

The series diverges

Definition : **Convergence of a series**

$\sum a_n$ converges to S means that $\lim_{n \to \infty} S_n = S$ where S is a finite single number, otherwise it diverges.

$S \equiv$ total sum

$S_n \equiv$ partial sum $= a_1 + a_2 + \cdots + a_n$

ex. 2 : Use the definition to test whether the series converges or not

1. $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$

Soln $\quad a_n = \frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2} = \frac{A(n+2) + B(n+1)}{(n+1)(n+2)}$

$$1 = A(n+2) + B(n+1)$$

if $n=-1 \Rightarrow A=1$

if $n=-2 \Rightarrow B=-1$

$\therefore \quad a_n = \frac{1}{n+1} - \frac{1}{n+2}$

$a_1 = \frac{1}{2} - \cancel{\frac{1}{3}}$

$a_2 = \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}$

$a_3 = \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}$

$\vdots$

$a_{n-2} = \cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}$

$a_{n-1} = \cancel{\frac{1}{n}} - \cancel{\frac{1}{n+1}}$

$a_n = \cancel{\frac{1}{n+1}} - \frac{1}{n+2}$

بالجمع

$$S_n = \frac{1}{2} - \frac{1}{n+2}$$

$\therefore \quad S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{n+2}\right) = \frac{1}{2}$

$\therefore$ the series converges to $\frac{1}{2}$

2. $\sum_{n=1}^{\infty} \ln \frac{n+1}{n+3} \Rightarrow a_n = \ln\left(\frac{n+1}{n+3}\right) = \ln(n+1) - \ln(n+3)$

$\therefore \quad a_1 = \ln 2 - \ln 4$

$a_2 = \ln 3 - \ln 5$

$a_3 = \ln 4 - \ln 6$

$a_4 = \ln 5 - \ln 7$

$a_{n-3} = \ln(n-2) - \ln(n)$

$a_{n-2} = \ln(n-1) - \ln(n+1)$

$a_{n-1} = \ln(n) - \ln(n+2)$

$a_n = \ln(n+1) - \ln(n+3)$

بالجمع

$$S_n = \ln 2 + \ln 3 - \ln(n+2) - \ln(n+3)$$

$\therefore \quad S = \lim_{n\to\infty} S_n = -\infty \quad \therefore$ the series diverges

Geometric Series :

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

عندما نأخذ متغير ونقسمه على الذي قبله فالمقدار يبقى نفسه

مثلاً : x^3 على x^2 = x

Theorem :

The series $\sum_{n=0}^{\infty} x^n$ converges to $S = \frac{1}{1-x}$ when $-1 < x < 1$, otherwise it diverges

$$1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x} , \quad -1 < x < 1$$

ex. 3: Test for convergence $\sum_{n=1}^{\infty} \frac{1}{2^n}$

Soln

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

$$= \frac{1}{2} \left\{ 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots \right\} \leftarrow \text{G.S. } x = \tfrac{1}{2} < 1$$

$$\therefore \quad S = \frac{1}{2} \left\{ \frac{1}{1 - \left(\frac{1}{2}\right)} \right\} = \frac{1}{2} * \frac{1}{\frac{1}{2}} = 1$$

$\therefore$ the series converges to S

ex. 4: Find the partial sum of $\sum_{n=1}^{\infty} \frac{2}{5^{n-1}}$

Soln

$$S_n = \frac{2}{5^0} + \frac{2}{5^1} + \frac{2}{5^2} + \frac{2}{5^3} + \cdots\cdots + \frac{2}{5^{n-1}} \quad \text{———} (1)$$

Eq. (1) is multiplied by $\frac{1}{5}$

$$\frac{1}{5} S_n = \frac{2}{5} + \frac{2}{5^2} + \frac{2}{5^3} + \frac{2}{5^4} + \cdots\cdots + \frac{2}{5^n} \quad \text{———} (2)$$

by subtracting Eq. 2 from Eq. (1) (i.e., Eq. 1 − Eq. 2):

$$\frac{4}{5} S_n = 2 + \frac{2}{5^{n-1}} - \frac{2}{5^n}$$

$\therefore \; S_n = \frac{5}{2}\left\{1 + \frac{4}{5^n}\right\}$

the total sum $\; S = \lim_{n\to\infty} \frac{5}{2}\left\{1 + \frac{4}{5^n}\right\} = \frac{5}{2}$

ex.5: Test the following series, Find the sum of the convergent series

1. $\sum_{n=1}^{\infty} \left(-\frac{3}{4}\right)^{n-1}$ 2. $\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^{n-1}$ 3. $\sum_{n=1}^{\infty} \left(-\frac{3}{2}\right)^{n+1}$

Soln

1. $\sum_{n=1}^{\infty} \left(-\frac{3}{4}\right)^{n-1} = 1 + \left(-\frac{3}{4}\right) + \left(-\frac{3}{4}\right)^2 + \cdots\cdots$

It is a geometric series $\; |x| = \frac{3}{4} < 1$

$\therefore$ G.S. converges to $\; S = \frac{1}{1-\frac{3}{4}} = 4$

2. $\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^{n-1} = 1 + \frac{e}{\pi} + \left(\frac{e}{\pi}\right)^2 + \cdots\cdots$

It is a geometric series $\; |x| = \frac{e}{\pi} = 0.865 < 1$

$\therefore$ G.S. converges to $\; S = \frac{1}{1-0.865} = 7.407$

3. $\sum_{n=1}^{\infty} \left(-\frac{3}{2}\right)^{n+1} = \left(-\frac{3}{2}\right)^2 + \left(-\frac{3}{2}\right)^3 + \left(-\frac{3}{2}\right)^4 + \cdots$

$$= \left(-\frac{3}{2}\right)^2 \underbrace{\left\{1 + \left(-\frac{3}{2}\right) + \left(-\frac{3}{2}\right)^2 + \cdots\right\}}_{\text{G.S. } |x| = \frac{3}{2} > 1}$$

$\therefore$ G.S diverges

Homework : Test the following series, Find the sum of the convergent series.

$$\sum_{n=1}^{\infty} \frac{1}{9n^2+3n-2} \qquad \sum_{n=2}^{\infty} \frac{1}{n^2-1} \qquad \sum_{n=3}^{\infty} \frac{5}{n-2}$$

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} \qquad \sum_{n=1}^{\infty} \frac{4^{n+2}}{7^{n-1}} \qquad \sum_{n=1}^{\infty} \frac{7}{6^{n-1}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n+3} - \frac{1}{n+4} \qquad \sum_{n=1}^{\infty} \frac{2^{n-1}}{4} \qquad \sum_{n=1}^{\infty} \frac{1}{5^n}$$

Convergence Tests : اختبارات التقارب

1. The general term test اختبار الحد العام

If $\lim\limits_{n\to\infty} a_n \neq 0$ then $\sum a_n$ diverges

If $\lim\limits_{n\to\infty} a_n = 0$ then the test is failure

ex. 6 : Test for convergence $\sum_{n=1}^{\infty} \frac{2n+1}{3n+4}$

Soln $\quad a_n = \frac{2n+1}{3n+4}$

$\therefore \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(\frac{2n+1}{3n+4}\right) = \frac{2}{3} \neq 0$

$\therefore$ the series diverges.

2. The Integral Test "الاختبار التكاملي"

If $a_n = f(n)$ then the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_1^{\infty} f(x)\,dx$ are togather converge or diverge.

أي يتباعدان او يتقاربان سوية

ex. 7: Test for convergence $\sum_{n=1}^{\infty} \frac{1}{n^2}$; $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}$

1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Sol: let $f(x) = \frac{1}{x^2}$

$$\therefore \int_1^{\infty} f(x)\,dx = \lim_{n\to\infty} \int_1^{n} f(x)\,dx = \lim_{n\to\infty} \int_1^{n} \frac{1}{x^2}\,dx$$

$$= \lim_{n\to\infty} \frac{x^{-1}}{-1} \Big|_1^n = \lim_{n\to\infty} -\left(\frac{1}{n} - 1\right) = 1$$

$\therefore \int_1^{\infty} f(x)\,dx$ converges

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

2. $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}$, let $f(x) = \frac{1}{\sqrt{x}}$

$$\therefore \lim_{n\to\infty} \int_3^{n} f(x)\,dx = \lim_{n\to\infty} \int_3^{n} \frac{1}{\sqrt{x}}\,dx = \lim_{n\to\infty} 2\sqrt{x} \Big|_3^n$$

$$= \lim_{n\to\infty} 2(\sqrt{n} - \sqrt{3}) = \infty$$

$\therefore \int_3^{\infty} f(x)\,dx$ diverges

3. The Comparison Test : اختبار المقارنة

أ. إذا كان $a_n < b_n$ وكانت $\sum b_n$ متقاربة فأن $\sum a_n$ متقاربة وإلا يفشل الاختبار.

ب. إذا كان $a_n > b_n$ وكانت $\sum b_n$ متباعدة فأن $\sum a_n$ متباعدة وإلا يفشل الاختبار.

Theorem (P-Series)

تحفظ

$$\boxed{\sum \frac{1}{n^p} \begin{cases} \text{conveges} & \text{for } p > 1 \\ \text{diverges} & \text{for } p \leq 1 \end{cases}}$$

كيف نجد b_n ؟

هناك نمطان من الاسئلة نتعرف عليهما من خلال المثالين أدناه

ex. 8 : Test for convergence $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3}$

Soln :

$$\because \quad \sin^2 n \leq 1 \qquad \div\, n^3$$

$$\therefore \quad \frac{\sin^2 n}{n^3} \leq \frac{1}{n^3}$$

$$a_n \leq b_n$$

but $\sum \frac{1}{n^3}$ (converges , p-series theorem, $p = 3 > 1$)

$\therefore \sum \frac{\sin^2 n}{n^3}$ converges

ex. 9 : $\sum_{n=1}^{\infty} \frac{n^3}{e^{n^4}+17}$

Sol$\underline{n}$ by using comparison test

$$17 > 0 \Rightarrow 17 + e^{n^4} > e^{n^4} \Rightarrow$$

$$\frac{1}{17+e^{n^4}} < \frac{1}{e^{n^4}} \Rightarrow \frac{n^3}{17+e^{n^4}} < \frac{n^3}{e^{n^4}}$$

$$\Rightarrow a_n < b_n$$

$\sum b_n = \sum \frac{n^3}{e^{n^4}}$ by using integral test

$$\int_1^{\infty} f(x)\,dx = \lim_{m\to\infty} \int_1^m \frac{x^3}{e^{x^4}}\,dx$$

$$= \lim_{m\to\infty} \int_1^m \frac{-4}{-4} x^3 e^{-x^4}\,dx$$

$$= -\frac{1}{4} \lim_{m\to\infty} e^{-x^4} \Big|_1^m$$

$$= \frac{1}{4} \lim_{m\to\infty} \left(\frac{1}{e^1} - \frac{1}{e^{m^4}} \right)$$

$$= \frac{1}{4e^1}$$

$\therefore \sum b_n = \sum \frac{n^3}{e^{n^4}}$ converges

$\therefore \sum a_n = \sum \frac{n^3}{17+e^{n^4}}$ converges

by comparison test

4. The Limit Comparison Test : اختبار غاية المقارنة

نستخدم هذه الطريقة عندما تفشل الطريقة الثالثة حيث يجب ان يكون :

$$\lim_{n \to \infty} \frac{a_n}{b_n} \neq 0$$

عندئذ فأن $\sum a_n$ & $\sum b_n$ يتقاربان او يتباعدان سوية

و الا تفشل هذه الطريقة.

ex.10: Test for convergence $\sum_{n=1}^{\infty} \frac{n^3}{n^{1/2}+3}$

Soln

$$3 > 0$$

$$3 + n^{1/2} > n^{1/2}$$

$$\frac{1}{3+n^{1/2}} < \frac{1}{n^{1/2}} \qquad * \; n^3$$

$$\frac{n^3}{3+n^{1/2}} < \frac{n^3}{n^{1/2}}$$

$$a_n < b_n$$

$$\sum b_n = \sum \frac{n^3}{n^{1/2}} = \sum \frac{1}{n^{-2.5}}$$ (diverges, p-series theorem, $p = -2.5 < 1$)

فلا يمكن استخدام الـ (comparison test) لذلك هناك نقص للقاعدة ؟ لذا نستعيض عنها بـ (limit comparison test)

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^3}{n^{1/2}+3} \cdot \frac{n^{1/2}}{n^3}$$

$$= \lim_{n \to \infty} \frac{1}{\frac{n^{1/2}+3}{n^{1/2}}} = \lim_{n \to \infty} \frac{1}{1+\frac{3}{n^{1/2}}}$$

$$= 1 \neq 0$$

$\therefore$ the choosing is right and $\sum a_n$ diverges

ex. 11: Test for convergence $\sum_{n=1}^{\infty} \frac{2n^3+n-2}{5n^4+n^2+3}$

Soln choose $\sum b_n = \sum \frac{n^3}{n^4} = \sum \frac{1}{n}$

(diverges, p-series theorem, $p=1$)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2n^3+n-2}{5n^4+n^2+3}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{2n^4+n^2-2n}{5n^4+n^2+3} = \frac{2}{5} \neq 0$$

$\therefore$ the choosing is right

$\therefore$ $\sum a_n$ diverges

H.W. Test for convergence $\sum \frac{5n^4+n^2+3}{2n^3+n-2}$

5. The Ratio Test الاختبار النسبي

a. If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ then $\sum a_n$ converges

b. If $\lim \frac{a_{n+1}}{a_n} > 1$ then $\sum a_n$ diverges

c. If $\lim \frac{a_{n+1}}{a_n} = 1$ the test is failure

Note: $n! = n(n-1)(n-2)$

$0! = 1$

$(n+1)! = (n+1)n(n-1)(n-2)$

$= (n+1)n!$

* نستخدم الطريقة الخاصة بنا أ حتوت المتسلسلة على فاكتوريل

ex. 12: Test for convergence $\sum \frac{n!}{n^n}$

Soln $a_n = \frac{n!}{n^n}$; $a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} * \frac{n^n}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)\,n!}{(n+1)^n \cdot (n+1)} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{(n+1)^n}{n^n}} = \lim_{n\to\infty} \frac{1}{(\frac{n+1}{n})^n}$$

$$= \lim_{n\to\infty} \{(1+\frac{1}{n})^n\}^{-1} = \{e^1\}^{-1} = \frac{1}{2.7} < 1$$

$\therefore$ $\sum \frac{n!}{n^n}$ is convergence

6. The Root Test الاختبار الجذري

a. If $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$ then $\sum a_n$ converges

b. If $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$ then $\sum a_n$ diverges

c. If $\lim_{n\to\infty} \sqrt{a_n} = 1$ then the test is failure

ex. 13: Test for convergence $\sum_{n=1}^{\infty} (\frac{3n+1}{2n+3})^n$

Soln $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{(\frac{3n+1}{2n+3})^n} = \lim_{n\to\infty} \frac{3n+1}{2n+3}$

$$= \frac{3}{2} > 1$$

$\therefore$ the series diverges

Alternating Series : المتسلسلات المتناوبة

$$\sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 - \cdots$$

Definition 1 : The series $\sum (-1)^n a_n$ converges absolutely if $\sum |(-1)^n a_n|$ converges.

Definition 2 : The series $\sum (-1)^n a_n$ converges conditionally if $\sum |(-1)^n a_n|$ diverges & $\lim_{n\to\infty} a_n = 0$

Definition 3 : The series $\sum (-1)^n a_n$ diverges if $\sum |(-1)^n a_n|$ diverges & $\lim_{n\to\infty} a_n \neq 0$

ex. 14 : Test for convergence $\sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{k+2}{3k-1}\right)^k$

Soln $|(-1)^{k+1} a_k| = \sum \left(\frac{k+2}{3k-1}\right)^k$ using root test

$\lim_{k\to\infty} \frac{k+2}{3k-1} = \frac{1}{3} < 1$ it converges

$\therefore \sum |(-1)^{k+1} a_k|$ converges

$\therefore$ The alternating series converges absolutely

ex. 15 : Test for convergence $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+2}{k(k+3)}$

Soln $|(-1)^{k+1} a_k| = \frac{k+2}{k^2+3k}$ using limit comparison test

choose $b_k = \frac{k}{k^2} = \frac{1}{k}$ (p-series, diverges, p = 1)

$$\lim_{k \to \infty} \left| \frac{a_k}{b_k} \right| = \lim_{k \to \infty} \left(\frac{k+2}{k^2+3k} \right) * k$$

$$= \lim_{k \to \infty} \frac{k^2+2k}{k^2+3k} = 1 \neq 0$$

$\therefore$ The choosing is right & the series $|a_k|$ diverges

$$\therefore \lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{k+2}{k(k+3)} = 0$$

$\therefore \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+2}{k(k+3)}$ converges conditionally

Power Series (Series of Function) متسلسلة دوال

$$\sum a_n(x) = a_1(x) + a_2(x) + a_3(x) + \cdots$$

ex. $\sum_{n=1}^{\infty} \frac{x^n}{n} = \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$ هذه الدالة متقاربة لقيم معينة من x ومتباعدة لقيم أخرى

Interval of Convergence :

هي قيم x التي عندها تكون المتسلسلة متقاربة ولمعرفتها عادة نستعمل الاختبار النسبي . أحياناً قد نستخدم الاختبار الجذري او الاختبار بالمقارنة

$$\boxed{\lim_{n \to \infty} \left| \frac{a_{n+1}(x)}{a_n(x)} \right| < 1}$$

or

$$\boxed{\lim_{n \to \infty} \sqrt[n]{\left| [a_n(x)]^n \right|} < 1}$$

ex. 16 : Test for convergence $\sum \frac{\cos^n x}{n!}$

Soln : by using comparison test

$$|\cos^n x| \leqslant 1 \quad \Rightarrow \left| \frac{\cos^n x}{n!} \right| \leqslant \frac{1}{n!}$$

$$a_n(x) < b_n \quad \text{where } \sum b_n = \sum \frac{1}{n!}$$

$$b_{n+1} = \frac{1}{(n+1)!} = \frac{1}{(n+1)n!}$$

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{1}{(n+1)n!} \cdot \frac{n!}{1} = 0 < 1$$

$\therefore \sum b_n$ converges

$\therefore \sum a_n(x) = \sum \frac{\cos^n x}{n!}$ converges for all values of x

ex. 17 : Find the interval of convergence of $\sum_{n=0}^{\infty} \frac{(x-2)^n}{5^n n^3}$

Soln

$$a_n(x) = \frac{(x-2)^n}{5^n n^3}$$

$$a_{n+1}(x) = \frac{(x-2)^{n+1}}{5^{n+1}(n+1)^3}$$

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}(x)}{a_n(x)} \right| = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{5^{n+1}(n+1)^3} \cdot \frac{5^n n^3}{(x-2)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-2)}{5} \cdot \frac{n^3}{(n+1)^3} \right| = \lim_{n\to\infty} \left| \frac{(x-2)}{5(1+\frac{1}{n})^3} \right|$$

$$= \left| \frac{x-2}{5} \right|$$

The series converges when $\lim_{n\to\infty} \left|\frac{a_{n+1}(x)}{a_n(x)}\right| < 1$

$\left|\frac{x-2}{5}\right| < 1 \quad \Rightarrow \quad -1 < \frac{x-2}{5} < 1$

$-5 < x-2 < 5 \quad \Rightarrow \quad -3 < x < 7$

$\therefore$ the series converges when $-3 < x < 7$ and diverges when $x > 7$ & $x < -3$

hence,

at $x = 7 \Rightarrow \frac{(7-2)^n}{5^n\, n^3} = \sum \frac{1}{n^3}$ (P-series, $p=3 > 1$, it converges

at $x = -3 \Rightarrow \frac{(-3-2)^n}{5^n\, n^3} = \sum \frac{(-5)^n}{5^n n^3} = \sum \frac{(-1)^n}{n^3}$

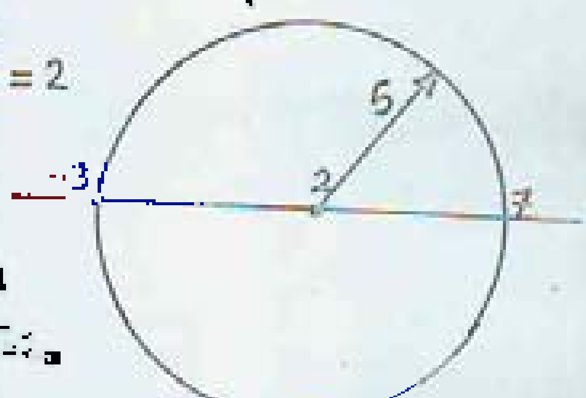
$\left|\sum \frac{(-1)^n}{n^3}\right| = \sum \frac{1}{n^3}$ (p-series, it converges)

$\therefore \sum \frac{(-1)^n}{n^3}$ converges absolutely

$\therefore$ the series converges when $-3 \leq x \leq 7$

center of convergence $= \frac{-3+7}{2} = 2$

radius $= 7 - 2 = 5$



(اذا كان الحد الموجب (تعويض 7 في المتسلسلة)

بتساوي أخذ الحد السالب (تعويض -3 في المتسلسلة)

ex.18: Find the interval of convergence $\sum \frac{(2x-5)^n}{n^2}$

Sol: by using root test

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(2x-5)^n}{n^2}\right|} < 1$$

$$\lim_{n\to\infty} \left|\frac{2x-5}{n^{2/n}}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{2x-5}{e^{\ln n^{2/n}}}\right| < 1$$

$$\lim_{n\to\infty} \frac{|2x-5|}{e^{2\frac{\ln n}{n}}} < 1 \quad \text{using Lopital Rule}$$

$$\lim_{n\to\infty} \frac{|2x-5|}{e^{\frac{2}{n}/1}} < 1 \Rightarrow \frac{|2x-5|}{e^0} < 1$$

$$-1 < 2x-5 < 1 \Rightarrow 4 < 2x < 6$$

$\therefore$ the power series converges for $2 < x < 3$
and diverges for $2 > x$ & $x > 3$

if $x = 3 \Rightarrow \sum \frac{(3*2-5)^n}{n^2} = \sum \frac{1}{n^2}$ converges

if $x = 2 \Rightarrow \sum \frac{(2*2-5)^n}{n^2} = \sum \frac{(-1)^n}{n^2}$ converges absolutely

ex.19: Test for convergence $\sum_{n=1}^{\infty} (-2)^n (n+1)(x-1)^n$

Sol: $|(-1)^n a_n| = 2^n (n+1)(x-1)^n$

$$\rho = \lim_{n\to\infty} \sqrt[n]{|a_n(x)|} < 1 \quad \text{for convergence}$$

$$= \lim_{n\to\infty} [2^n (n+1)(x-1)^n]^{1/n} < 1$$

$$= \lim_{n\to\infty} 2(x-1)(n+1)^{1/n} < 1$$

$$= \lim_{n\to\infty} 2(x-1)\, e^{\ln(n+1)^{1/n}} < 1$$

$$= \lim_{n\to\infty} 2(x-1)\, e^{\frac{\ln(n+1)}{n}} < 1 \quad \text{using L'opital Rule}$$

$$= \lim_{n\to\infty} 2(x-1)\, e^{\frac{1}{n+1}} < 1$$

$x - 1 < \frac{1}{2}$ ∴ the series converges when $x < \frac{3}{2}$ and diverges when $x > \frac{3}{2}$

hence, if $x = \frac{3}{2} \Rightarrow \sum_{n=1}^{\infty} (-2)^n (n+1)(\frac{3}{2}-1)^n$

$$\Rightarrow \sum_{n=1}^{\infty} (-1)^n 2^n (n+1)(\tfrac{1}{2})^n = \sum_{n=1}^{\infty} (-1)^n (n+1)$$

$$\Rightarrow \sum_{n=1}^{\infty} |(-1)^n (n+1)| = \sum_{n=1}^{\infty} (n+1)$$

$\lim_{n\to\infty} (n+1) = \infty \neq 0$ ∴ it diverges

∴ $\sum (-1)^n (n+1)$ diverges also

∴ the alternating power series diverges at $x \geq \frac{3}{2}$

Tayler Series !

if f is defined at $x=a$ and it is differential of order n at $x=a$ then the series

$$f(a)+\frac{f'(a)}{1!}(x-a)^1+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots$$

converges to $f(x)$

$$f(x)=f(a)+f'(a)\frac{(x-a)}{1!}+f''(a)\frac{(x-a)^2}{2!}+f'''(a)\frac{(x-a)^3}{3!}+\cdots$$

Tayler Series

$$f(\ )=f(0)+f'(0)\frac{x}{1!}+f''(0)\frac{x^2}{2!}+f'''(0)\frac{x^3}{3!}+\cdots$$

Maclourin Series (special case)

ex. 20: Find Maclourin series ($a=0$) for $\sin x$

Soln

$f(x)=\sin x \qquad f(0)=\sin 0=0$

$f'(x)=\cos x \qquad f'(0)=\cos 0=1$

$f''(x)=-\sin x \qquad f''(0)=-\sin 0=0$

$f'''(x)=-\cos x \qquad f'''(0)=-1$

$$\therefore f(x)=f(0)+f'(0)x+f''(0)\frac{x^2}{2!}+\cdots$$
$$=0+1\,x+0\,\frac{x^2}{2!}+\frac{(-1)}{3}x^3+\cdots$$
$$=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$

(للحفظ) قوانين مهمة جداً

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

2. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

3. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

4. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

5. $e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

6. $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$

7. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$

8. $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$

9. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \cdots$

10. $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!} x^2 - \frac{n(n-1)(n-2)}{3!} x^3 + \cdots$

ex. 21: Find the Taylor series expansion of $\cos x$ about the point $a = 2\pi$

Sol/n the values of $\cos x$ and its derivatives at $a = 2\pi$ are the same as their values at $a = 0$, therefore;

$$f^{2k}(2\pi) = f^{2k}(0) = (-1)^k$$

$$f^{(2k+1)} \quad (2k+1)$$

$f(x) = \cos x \qquad f'(x) = -\sin x \qquad f'(0) = 0 \qquad f'(2\pi) = 0$

$f(0) = 1 \qquad f''(x) = -\cos x \qquad f''(0) = -1 \qquad f''(2\pi) = -1$

$f'''(x) = \sin x \qquad f'''(0) = 0 \qquad f'''(2\pi) = 0$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$= 1 + 0 - \frac{1}{2!}(x-2\pi)^2 + 0 + \frac{1}{4!}(x-2\pi)^4 - \cdots$$

$$= 1 - \frac{(x-2\pi)^2}{2!} + \frac{(x-2\pi)^4}{4!} - \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(x-2\pi)^{2k}}{(2k)!}$$

ex. 22: Find the series that converges to $\tan^{-1} x$ by the use of the series that converges to $\frac{1}{1-x}$

Sol: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

put $x = -t^2$

$\therefore \frac{1}{1+t^2} = 1 + (-t^2) + (-t^2)^2 + (-t^2)^3 + \cdots$

$= 1 - t^2 + t^4 - t^6 + t^8$ نجري التكامل للطرفين

$$\int_0^x \frac{dt}{1+t^2} = \int_0^x (1 - t^2 + t^4 - \cdots)\, dt$$

$$\tan^{-1} t \Big|_0^x = t - \frac{t^3}{3} + \frac{t^5}{5} - \cdots \Big|_0^x$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{\ldots}$$

Ex. 23: Find the series that converges to $\frac{x^4}{(1+x)^2}$ by differenatiating the series that converges to $\frac{1}{1+x}$

Sol$_{n}$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \dots$$

بأخذ المشتقة للطرفين

$$(-1)(1+x)^{-2} = 0 - 1 + 2x - 3x^2 + 4x^3$$

نضرب الطرفين ب $(-x^4)$

$$\frac{x^4}{(1+x)^2} = x^4 - 2x^5 + 3x^6 - 4x^7 + \dots$$

Ex. 24: Find the series that converges to $e^x \cos x$ by multiplication

Sol$_{n}$

$$e^x \cdot \cos x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots\right) \cdot \left(1 - \frac{x^2}{2} + \frac{x^4}{24} \dots\right)$$

$$= 1\,x^0 + 1\,x^1 + \left(-\frac{1}{2} + \frac{1}{2}\right)x^2 + \left(-\frac{1}{2} + \frac{1}{6}\right)x^3 + \left(\frac{1}{24} + \frac{1}{24} - \frac{1}{4}\right)x^4 + \dots$$

$$= 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 + \dots$$

Ex. 25: Expand using Maclaurine series ($\frac{x}{1-x^2}$, $\ln(1+x^4)$)

Sol$_{n}$ $f(x) = \frac{x}{1-x^2}$ & $g(x) = \ln(1+x^4)$

① $\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots \implies$ put $t = x^2$

$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \dots$ بالضرب ب x

② $\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$ بالضرب ب dt ثم التكامل

$$\int \frac{dt}{1+t} = \int (1 - t + t^2 - t^3 + \cdots) dt$$

$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} - \cdots$$

put $t = x^4$

$$\ln(1+x^4) = x^4 - \frac{x^8}{2} + \frac{x^{12}}{3} - \frac{x^{16}}{4} + \cdots = g(x)$$

$$\therefore \frac{x}{1-x^2} + \ln(1+x^4) = x + x^3 + x^4 + x^5 + x^7 - \frac{x^8}{2} + x^9 + x^{11} + \frac{x^{12}}{3} + x^{13} + x^{15} - \frac{x^{16}}{4} + \cdots$$

ex.26: Find $\int_{0.1}^{0.2} \frac{1-e^x}{x^3} \cdot dx$ using Maclaurine series

Soln $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$1 - e^x = -\left\{ x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right\}$$

$$\frac{1-e^x}{x^3} = -\left\{ \frac{1}{x^2} + \frac{1}{2!}\frac{1}{x} + \frac{1}{3!} + \frac{x}{4!} + \cdots \right\}$$

$$\therefore \int_{0.1}^{0.2} -\left\{ \frac{1}{x^2} + \frac{1}{2!}\frac{1}{x} + \frac{1}{3!} + \frac{x}{4!} + \cdots \right\} dx$$

$$= \frac{1}{x} - \frac{1}{2}\ln x - \frac{1}{6}x - \frac{x^2}{48} + \cdots \Big|_{0.1}^{0.2}$$

$$\cong (\quad)$$

ex. 27 : Evaluate $\lim_{x \to 0} \{ (\sin x - \tan x) / x^3 \}$ using Maclaurine series

Soln

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$$

$$\sin x - \tan x = -\frac{x^3}{2} - \frac{x^5}{8} - \dots$$

$$= x^3 \left(-\frac{1}{2} - \frac{x^2}{8} - \dots \right)$$

$$\therefore \lim_{x \to 0} \frac{\sin x - \tan x}{x^3} = \lim_{x \to 0} \left(-\frac{1}{2} - \frac{x^2}{8} - \dots \right)$$

$$= -\frac{1}{2}$$

ex. 28 : Evaluate $\lim_{x \to 1} \{ (\ln x / (x-1)) \}$

Soln let $f(x) = \ln(x)$ & $g(x) = x - 1$

Note :

If	$\lim_{x \to a}$	$\frac{f(x)}{g(x)}$	use	T-series
but	$\lim_{x \to 0}$	$\frac{f(x)}{g(x)}$	use	M-series

$f(x) = \ln(x)$ $\qquad f(1) = 0$

$\bar{f}(x) = 1/x$ $\qquad \bar{f}(1) = 1$

$\bar{\bar{f}}(x) = -1/x^2$ $\qquad \bar{\bar{f}}(1) = -1$

$$f(x) = f(a) + \frac{\bar{f}(a)}{1!}(x-a) + \frac{\bar{\bar{f}}(a)}{2!}(x-a)^2 + \dots$$

$$\ln(x) = 0 + \frac{1}{1!}(x-1) - \frac{1}{2!}(x-1)^2 + \dots$$

$$\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \cdots$$

$$\therefore \quad \frac{\ln(x)}{x-1} = 1 - \frac{1}{2}(x-1) + \cdots$$

$$\therefore \quad \lim_{x \to 1} \frac{\ln(x)}{x-1} = \lim_{x \to 1} \left\{ 1 - \frac{1}{2}(x-1) + \cdots \right\}$$

$$= 1$$

ex. 29 : Express $\int \sin x^2 \, dx$ as a power series

Soln from the series for $\sin x$ we obtain

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots \qquad -\infty < x < \infty$$

$$\therefore \int \sin x^2 \, dx = \frac{x^3}{3} - \frac{x^7}{7 \times 3!} + \frac{x^{11}}{11 \times 5!} - \cdots + C$$

ex. 30 : Estimate $\int_0^1 \sin(x^2) \, dx$ with an error of less than 0.001

Soln $\int_0^1 \sin(x^2) \, dx = \frac{1}{3} - \frac{1}{7 \times 3!} + \frac{1}{11 \times 5!} - \frac{1}{15 \times 7!} + \cdots$

but $\frac{1}{11 \times 5!} \approx 0.00076 < 0.001$ ∴ نكتفي بأول حدين

$$\therefore \int_0^1 \sin(x^2) \, dx \approx \frac{1}{3} - \frac{1}{42} \approx 0.31$$

ex. 31 : Estimate $\int_0^{0.5} \sqrt{1+x^4}\, dx$ with an error $< 10^{-4}$

Soln

$$(1+x^4)^{1/2} = 1 + \frac{1}{2} x^4 - \frac{1}{8} x^8 + \cdots$$

$$\int_0^{0.5} \sqrt{1+x^4}\, dx = x + \frac{1}{2*5} x^5 - \frac{1}{8*9} x^9 + \cdots \Big|_0^{0.5}$$

$$= 1 + 0.0031 - 0.00003 + \cdots$$

$$\simeq 1.0031$$

ex. 32 : Find the interval of the convergence for the series represent $\tan^{-1} x$?

Soln from Maclaurine series

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)}$$

$$|a_n(x)| = \frac{x^{2n-1}}{2n-1} \quad ; \quad |a_{n+1}(x)| = \frac{x^{2(n+1)-1}}{2(n+1)-1}$$

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}(x)}{a_n(x)} \right| < 1 \quad \text{for convergence}$$

$$= \lim_{n\to\infty} \frac{x^{2(n+1)-1}}{2(n+1)-1} \cdot \frac{2n-1}{x^{2n-1}} < 1$$

$$= \lim_{n\to\infty} \frac{x^2}{1 + \frac{2}{2n-1}} < 1 \quad \Rightarrow \quad x^2 < 1$$

$\therefore \quad x < 1$

Ex. 33 ! Evaluate $\lim\limits_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$

Soln $\lim\limits_{x \to 0} \frac{x - \sin x}{x \sin x}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$x \sin x = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \cdots$$

$$\therefore \lim_{x \to 0} \frac{x - \sin x}{x \sin x} = \lim_{x \to 0} \frac{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}{x^2 - \frac{x^4}{3!} + \frac{x^6}{5!}}$$

$$= \lim_{x \to 0} \frac{\frac{x^3}{3!} - \frac{x^5}{5!} + \cdots}{x^2 - \frac{x^4}{3!}}$$

$$= \lim_{x \to 0} \frac{\frac{x}{3!} - \frac{x^3}{5!}}{1 - \frac{x^2}{3!}} = 0$$

Homework : Use series to evalute the limits in

1. $\lim\limits_{x \to 0} \frac{x - \tan^{-1} x}{x^3}$

2. $\lim\limits_{x \to 0} \frac{\tan x - \sin x}{x^3 \cos x}$

3. $\lim\limits_{x \to 0} \frac{\ln(1 + x^2)}{1 - \cos x}$

4. $\lim\limits_{x \to 1} \frac{\ln(x^2)}{x - 1}$

5. $\lim$ $\sin x$