Subject :Engineering and numerical Analysis	**موضوع: تحليلات عددية و هندسية**
Weekly Hours : Theortical:2	**الساعات الأسبوعية: نظري: 2**
Tutorial :1	**مناقشة: 1**
Experimental :	**عملي:**
Units:4	**الوحدات: 4**

week	Contents	المحتويات	الأسبوع
1.	Laplace transformation (L.T)	تحويلات لابلاس	.1
2.	Inverse Laplace Transformation	تحويلات لابلاس العكسية	.2
3.	Solution of differential equations using Laplace transformation	حل المعادلات التفاضلية بواسطة تحويلات لابلاس	.3
4.	Applications	تطبيقات	.4
5.	Power series method	طريقة متواليات القوى	.5
6.	Partial differential equations (P.D.E)	المعادلات التفاضلية الجزئية	.6
7.	Separation of variables	فصل المتغيرات	.7
8.	Application of heat transfer	تطبيقات على انتقال الحرارة	.8
9.	Solution of non – linear equations	حل المعادلات اللاخطية	.9
10.	Alteration method	طريقة التكرار	.10
11.	Newton – Raphsen method	نيوتن رافسون	.11
12.	Applications	تطبيقات	.12
13.	Solution of simultaneously linear equations	حل المعادلات الآنية الخطية	.13
14.	Determinant method (Grammar Rule)	طريقة المصفوفات (قانون كريمر)	.14
15.	Matrix method	طريقة المحددات	.15
16.	Applications about topics (9-15)	تطبيقات على المواضيع (9- 15)	.16
17.	Applications about topics (9-15)	تطبيقات على المواضيع (9- 15)	.17
18.	Numerical interpolation	التحليلات العددية	.18
19.	Finite difference method	الفرو قات المحددة	.19
20.	Forward & Baehword & Center expression	صيغة الاشتقاق الأمامية ،الخلفية ، والمركزية	.20
21.	Newton s & Lagrange forms	صيغة نيوتن – لاكرانج	.21
22.	Numerical differentiation	التفاضل العددي	.22
23.	Numerical integration (three methods)	التكامل العددي (ثلاث طرق)	.23
24.	Applications	تطبيقات	.24
25.	Curve fitting	تطابق المنحنيات	.25
26.	Solution of D.E by finite difference method using Taylor series	حل المعادلات التفاضلية بطريقة الفرو قات المحددة	.26
27.	Applications	تطبيقات	.27
28.	Laplace equation in Engineering Fields	لمعادلة لابلاس في مواضيع هندسية	.28
29.	Applications Laplace equation in Engineering Fields	تطبيقات في مواضيع هندسية لمعادلة لابلاس	.29
30.	Applications Laplace equation in Engineering Fields	تطبيقات في مواضيع هندسية لمعادلة لابلاس	.30

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29.	=	=	.29
30.	=	=	.30

المصادر

1. التحليلات العددية والهندسية (المؤلف د. حسن الدلفي ، د. محمود عطاالله)

2. Numerical methods for engineering (steven chapra)
3. Numerical analysis (geraled)

(المصادر اعلاه متوفرة في المكتبة)

Laplace Transformation :-

1. Definition

The Laplace transform is one of mathematical tools by solving of ordinary linear differential equations. In comparsion with the classical method of solving L.D.E, the Laplace transform method has the following two altractive features :

ⓐ The homogeneous equation (C.F) and the particular Integral (P.I) are solved in one operator.

ⓑ The Laplace transform converts the diff. eq. into on algebric equation in s domain. It is then possible to manipulate the algebric eq. by simple algebric rules to obtain the solution in the s-domain. The final solution is obtained by taking the inverse Laplace transform.

② Mathematical definition for Laplace transform

Let $f(t)$ be a given function which defined for all $t \geqslant 0$. We multiply $f(t)$ by e^{-st} and integrat t from zero to infinity. Then, if the resulting integral exists, it is a function of s as $F(s)$:-

$$F(s) = \int_0^{\infty} e^{-st} \cdot f(t) \cdot dt$$

The fuction $F(s)$ is called the Laplace transform of the original function $f(t)$ and will be denoted by $\mathcal{L}(f(t))$, Thus:

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} \cdot f(t) \cdot dt \quad \text{---} (1)$$

* the operation just described, which yields $F(s)$ from a given $f(t)$, is called the Laplace transform

* the variable s is referred to as the Laplace operator, it may be real. Later it will be found useful to consider s complex.

Example (1) :-

Let $f(t) = 1$ when $t \geq 0$, Find $F(s)$ or $\mathcal{L}(f(t))$

sol.

$$\mathcal{L}(f(t)) = \mathcal{L}(1) = \int_0^\infty e^{-st} \cdot 1 \cdot dt = \frac{-1}{s} e^{-st} \Big|_0^\infty \quad \text{For } s > 0$$

$$= \left(\frac{-1}{s} \underbrace{e^{-s \cdot \infty}}_{0}\right) - \left(\frac{-1}{s} e^{-s \cdot 0}\right) = \frac{1}{s}$$

$$\therefore \boxed{\mathcal{L}(1) = \frac{1}{s}}$$

Example (2): Let $f(t) = e^{at}$ when $t \geq 0$ and a = constant

Find $\mathcal{L}(e^{at})$:-

sol.: $$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} \cdot e^{at} \cdot dt = \int_0^\infty e^{(a-s)t} \cdot dt$$

$$= \frac{1}{a-s} e^{(a-s)t} \Big|_0^\infty = \frac{1}{a-s}$$

$$\therefore \boxed{\mathcal{L}(e^{at}) = \frac{1}{a-s}} \quad \text{for } a - s > 0$$

Example ③ :- Let $f(t) = \sin at$ when $t \geqslant 0$ and $a =$ constant

Find $\mathcal{L}(\sin at)$

Sol.

$$\mathcal{L}(\sin at) = \int_0^\infty e^{-st} . \sin at \, dt = \left[\lim_{T \to 0} \int_0^T e^{-st} . \sin at . dt \right]$$

by parts

Let $u = \sin at$ $\quad du = a \cos at$

$dv = e^{-st} dt$ $\quad V = \frac{-1}{s} e^{-st}$

$$\int u \, dv = uv - \int v \, du$$

$$\int e^{-st} . \sin at = \frac{-1}{s} e^{-st} . \sin at + \underbrace{\frac{a}{s} \int e^{-st} \cos at}$$

let $u = \cos at$

$du = -\sin at$, $dv = e^{-st}$, $V = \frac{-1}{s} e^{-st}$

$$\Rightarrow \int e^{-st} . \sin at \, dt = \frac{-1}{s} e^{-st} . \sin at + \frac{a}{s} \left[\frac{-1}{s} e^{-st} \cos at - \frac{a}{s} \int e^{-st} , \sin at \, dt \right]$$

$$\int e^{-st} . \sin at \, dt = \frac{-1}{s} e^{-st} \sin at - \frac{a}{s^2} e^{-st} \cos at - \frac{a^2}{s^2} \int e^{-st} \sin at \, dt$$

Let $I = \int e^{-st} . \sin at \, dt$

$$\left(1 + \frac{a^2}{s^2}\right) I = \frac{-1}{s} e^{-st} \sin at - \frac{a}{s^2} e^{-st} \cos at$$

$$\left(\frac{s^2 + a^2}{s^2}\right) I = \frac{-1}{s} e^{-st} \sin at - \frac{a}{s^2} e^{-st} \cos at$$

$$\Rightarrow I = \frac{s^2}{s^2 + a^2} \left(\frac{-1}{s} e^{-st} . \sin at \right) - \frac{s^2}{s^2 + a^2} \left(\frac{a}{s^2} e^{-st} . \cos at \right)$$

$$\therefore \int e^{-st} . \sin at . dt = \frac{1}{s^2 + a^2} \left[-e^{-st} (s \sin at + a \cos at) \right] \Big|_0^\infty$$

$$\therefore \mathcal{L}(\sin at) = \lim_{T \to \infty} \left[\frac{1}{s^2 + a^2} \left(-e^{-st} (s \sin at + a \cos at) \right) \right]_0^T$$

$$= \lim_{T \to \infty} \left[\frac{-e^{-sT} (s \sin aT + a \cos aT)}{s^2 + a^2} - \frac{-e^{-s(0)} (s \sin a0 + a \cos a0)}{s^2 + a^2} \right]$$

$$= \lim_{T\to\infty}\left[\frac{\cancelto{0}{e^{-sT}}(s\sin aT + a\cos aT)}{s^2+a^2} + \frac{a}{s^2+a^2}\right]$$

$$\therefore \boxed{\mathcal{L}(\sin at) = \frac{a}{s^2+a^2}}$$

Example (4): Let $f(t) = t$ when $t \geq 0$ find $\mathcal{L}(t)$

sol.:

$$\mathcal{L}(t) = \int_0^\infty e^{-st}\cdot t\cdot dt = \lim_{T\to\infty}\int_0^T t\cdot e^{-st}\cdot dt$$

let $u = t$, $du = dt$

$dv = e^{-st}$, $V = \frac{-1}{s}e^{-st}$

$$\therefore \mathcal{L}(t) = \lim_{T\to\infty}\left[-\frac{t}{s}e^{-st} + \frac{1}{s}\int e^{-st}\cdot dt\right]_0^T$$

$$= \lim_{T\to\infty}\left[\frac{-t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right|_0^T$$

$$= \lim_{T\to\infty}\left[\frac{-T}{s}\cancelto{0}{e^{-sT}} - \frac{1}{s^2}\cancelto{0}{e^{-sT}} + \frac{0}{s}e^{-s0} + \frac{1}{s^2}\cancelto{1}{e^{-s\cdot 0}}\right]$$

$$\Rightarrow \boxed{\mathcal{L}(t) = \frac{1}{s^2}}$$

Example (5): Let $f(t) = \sinh at$ where a = constant and $t \geq 0$ find $\mathcal{L}(\sinh at)$

sol.:

$$\mathcal{L}(\sinh at) = \mathcal{L}\left[\frac{e^{at} - e^{-at}}{2}\right] = \lim_{T\to\infty}\int_0^T e^{-st}\left[\frac{e^{at}-e^{-at}}{2}\right]dt$$

$$= \frac{1}{2}\underbrace{\lim_{T\to\infty}\int_0^T e^{-st}\cdot e^{at}\,dt}_{\text{equal } \mathcal{L}\,e^{at}} - \frac{1}{2}\underbrace{\lim_{T\to\infty}\int_0^T e^{-st}\cdot e^{-at}\,dt}_{\text{equal } \mathcal{L}\,e^{-at}}$$

$$= \frac{1}{2}\left[\mathcal{L}(e^{at}) - \mathcal{L}(e^{-at})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{\cancel{s}+a-\cancel{s}+a}{s^2-a^2}\right] = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}a}{s^2+a^2}$$

$$\boxed{\mathcal{L}\,\sinh at = \frac{a}{s^2-a^2}}$$

A short list of some important elementary functions and their Laplace transforms is given in the following table :-

No	$f(t)$	$\mathcal{L}(f(t))$	No.	$f(t)$	$\mathcal{L}(f(t))$
1.	1	$1/s$	6.	$\cos at$	$s/(s^2+a^2)$
2.	t	$1/s^2$	7.	$\sin at$	$a/(s^2+a^2)$
3.	t^2	$2!/s^3$	8.	$\sinh at$	$a/(s^2-a^2)$
4.	t^n	$n!/s^{(n+1)}$	9.	$\cosh at$	$s/(s^2-a^2)$
5.	e^{at}	$1/(s-a)$			

H.W ① prove that

a. $\mathcal{L}(\cos at) = \dfrac{s}{s^2+a^2}$ b. $\mathcal{L}(\cosh at) = \dfrac{s}{s^2-a^2}$

② Find $\mathcal{L}$ of a. e^{-3t} b. t^2

Some important properties of Laplace transforms:

① Linearity Property :-

IF C_1 and C_2 are constants while $f_1(t)$ and $f_2(t)$ are functions with Laplace transformation $F_1(s)$ and $F_2(s)$, Then

$$\mathcal{L}\left[c_1 f_1(t) + c_2 f_2(t)\right] = c_1 \mathcal{L}(f_1(t)) + c_2 \mathcal{L}(f_2(t)) + \cdots$$
$$= c_1 F_1(s) + c_2 F_2(s)$$

Example:

① $\mathcal{L}\left[4t^2 - 3\cos 2t + 5e^{-t}\right] = 4\mathcal{L}t^2 - 3\mathcal{L}\cos 2t + 5\mathcal{L}e^{-t}$

$$= \frac{8}{s^3} + \frac{3s}{s^2+4} + \frac{5}{s+1}$$

② $\mathcal{L}\left[4e^{-5t} + 6t^3 - 3\sin 4t + 2\cos 2t\right]$

$$= 4\mathcal{L}e^{-5t} + 6\mathcal{L}t^3 - 3\mathcal{L}\sin 4t + 2\mathcal{L}\cos 2t$$

$$= \frac{4}{s+5} + 6\cdot\frac{3!}{s^4} - 3\cdot\frac{4}{s^2+16} + 2\cdot\frac{s}{s^2+4}$$

② First shifting property :-

IF $\mathcal{L}(f(t)) = F(s)$, Then

$$\boxed{\mathcal{L}\left[e^{at} \cdot f(t)\right] = F(s-a)}$$

Example :-

ⓐ $\mathcal{L}(e^{-t} \cdot \cos 2t)$ $\Rightarrow \mathcal{L} \cos 2t = \frac{s}{s^2+4}$, $\mathcal{L}\, e^{-t} = \frac{1}{s+1}$

$\therefore \mathcal{L}(e^{-t} \cdot \cos 2t) = \frac{s+1}{(s+1)^2+4}$

ⓑ $\mathcal{L}(e^{3t} \cdot t^2) \Longrightarrow$ $\mathcal{L}\, t^2 = \frac{2!}{s^3}$, $\mathcal{L}\, e^{3t} = \frac{1}{s-3}$

$\therefore \mathcal{L}(e^{3t} \cdot t^2) = \frac{2}{(s-3)^3}$

③ Second shifting property :-

IF $\mathcal{L}(f(t)) = F(s)$ and $g(t) = \begin{Bmatrix} f(t-a) & , & t > a \\ 0 & , & t < a \end{Bmatrix}$

Then

$$\mathcal{L}\, g(t) = e^{-as} \cdot F(s)$$

Example :

ⓐ $g(t) = \begin{Bmatrix} (t-2)^3 & t > 2 \\ 0 & t < 2 \end{Bmatrix}$ $\Rightarrow \mathcal{L}\, g(t) = G(s) = e^{-2s} \underbrace{\frac{3!}{s^4}}_{\text{equal } \mathcal{L} t^3}$

④ Change of Scale property :-

IF $\mathcal{L}(f(at)) = F(s)$, then $\boxed{\mathcal{L}(f(at)) = \frac{1}{a} F\left(\frac{s}{a}\right)}$

Examples :-

ⓐ $\mathcal{L}(\sin t) = \frac{1}{s^2+1}$, then

$\mathcal{L}(\sin \underset{a}{3} t) = \frac{1}{3}\left(\frac{1}{\left(\frac{s}{3}\right)^2+1}\right) = \frac{1}{3}\left(\frac{1}{\frac{s^2+9}{9}}\right) = \frac{1}{3} \cdot \frac{9}{s^2+9} = \frac{3}{s^2+9} = \frac{a}{s^2+a^2}$

ⓑ find $\mathcal{L}(3t)$
by change of scale property

$\mathcal{L}(3t) = \frac{1}{3}\left[\frac{1}{\left(\frac{s}{3}\right)^2}\right] = \frac{1}{3}\left(\frac{9}{s^2}\right) = 3 \cdot \frac{1}{s^2} = a \cdot \mathcal{L}\, f(t)$

(5) Laplace transformation of derivatives :-

IF $\mathcal{L}\, f(t) = F(s)$, then $\mathcal{L}\, f'(t) = s \cdot F(s) - f(0)$ and $\mathcal{L}\, f''(t) = s^2 \cdot F(s) - s\, f(0) - f'(0)$, therefore In general

$$\boxed{\mathcal{L}\, f^{(n)}(t) = s^n \cdot F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \cdots\cdots - s \cdot f^{(n-2)}(0) - f^{(n-1)}(0)}$$

Examples :-

(a) Find $\mathcal{L}\sin t$ by derivative property :-

we have

$$\frac{d}{dt} \underbrace{\cos t}_{f(t)} = \sin t = f'(t)$$

$$\therefore \mathcal{L}\, f'(t) = s \cdot F(s) - f(0)$$

$$= s \cdot \frac{s}{s^2+1} - 1 = \frac{s^2 - s^2 + 1}{s^2+1} = \frac{1}{s^2+1} = \mathcal{L}\sin t$$

H.W Find $\mathcal{L}\cos t$ by derivative property

Note that :- $-\cos t = \frac{d^2}{dt^2} \cos t = f''(t)$

(b) find $\mathcal{L}\, e^{at}$ by derivative property

Sol.

$$e^{at} = \frac{1}{a} \frac{d}{dt} e^{at} = \frac{1}{a} f'(t)$$

$$\Rightarrow \mathcal{L}\, \frac{1}{a} f'(t) = \frac{1}{a} \mathcal{L}\, f'(t) = \frac{1}{a}\left[s \cdot F(s) - f(0)\right]$$

$$= \frac{1}{a}\left[s \cdot \frac{1}{s-a} - 1\right] = \frac{1}{a}\left[\frac{s - s + a}{s-a}\right] = \frac{1}{a} \cdot \frac{a}{s-a} = \frac{1}{s-a}$$

(c) Prove that $\sin at = \frac{a}{s^2+a^2}$

Sol.

$$\sin at = \frac{-1}{a} \frac{d}{dt} \underbrace{\cos at}_{f(t)} = \frac{-1}{a} f'(t)$$

$$\Rightarrow \mathcal{L}\, f'(t) = \frac{-1}{a} \mathcal{L}\cos at = \frac{-1}{a}\left[s \cdot F(s) - f(0)\right] = \frac{-1}{a}\left[s \cdot \frac{s}{s^2+a^2} - 1\right]$$

$$= \frac{-1}{a}\left[\frac{\cancel{s^2} - \cancel{s^2} - a^2}{s^2+a^2}\right] = \frac{-1}{a}\left(\frac{-a^2}{s^2+a^2}\right)$$

$$= \frac{a}{s^2+a^2}$$

(6) Laplace transformation by integral:-

IF $\mathcal{L}\, f(t) = F(s)$, then $\boxed{\mathcal{L}\left[\int_0^t f(u).du\right] = \frac{1}{s} . F(s)}$

Examples:-

(a) $\mathcal{L} \int_0^t \underbrace{\sin 2u}_{f(u)}\, du = \frac{F(s)}{s}$

$\because \mathcal{L} \sin 2t = \frac{2}{s^2+4} \Rightarrow \mathcal{L} \int_0^t \sin 2u\, du = \frac{1}{s} . \frac{2}{s^2+4}$

(b) Find $\mathcal{L} \int_0^t (u^2 - u + e^{-u})\, du$

Sol.

where $\mathcal{L}\, t^2 = \frac{2!}{s^3} = \frac{2}{s^3}$, $\mathcal{L}\, t = \frac{1}{s^2}$, $\mathcal{L}\, e^{-t} = \frac{1}{s+1}$

$\therefore \mathcal{L} \int_0^t (u^2 - u + e^{-u})\, du = \frac{1}{s}\left[\frac{2}{s^3} - \frac{1}{s^2} + \frac{1}{s+1}\right] = \frac{2}{s^4} - \frac{1}{s^3} + \frac{1}{s(s+1)}$

(7) Multiplication by t^n:-

IF $\mathcal{L}\, f(t) = F(s)$, then $\boxed{\mathcal{L}\left[t^n . f(t)\right] = (-1)^n \frac{d^n}{ds^n} F(s)}$

Examples:-

Find $\mathcal{L}.T$ for the following functions:

(a) $t . e^{2t}$ (b) $t^2 . \cos at$

Sol.

(a) $\mathcal{L}\, t . e^{2t} \Rightarrow \mathcal{L}\, e^{2t} = \frac{1}{s-2}$, we have

$\mathcal{L}\left[t . e^{2t}\right] = -1^1 . \frac{d}{ds}\left(\frac{1}{s-2}\right) = -\frac{-1}{(s-2)^2} = \frac{1}{(s-2)^2}$

(b) $\mathcal{L}\, t^2 . \cos at \Rightarrow \mathcal{L} \cos at = \frac{s}{s^2+a^2}$, we have

$\mathcal{L}\left[t^2 . \cos at\right] = -1^2 . \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right)$

$= \frac{d}{ds}\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right] = \frac{d}{ds}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$

$= \frac{-2s(s^2+a^2)^2 - (a^2-s^2).4s(s^2+a^2)}{(s^2+a^2)^4} = \frac{-2s\left[3a^2-s^2\right]}{(s^2+a^2)^3}$

⑧ Division by t :-

If $\mathcal{L} f(t) = F(s)$, then $\boxed{\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(u).du}$

ex. ⓐ Find $\mathcal{L}\, \frac{\sin t}{t}$, sol. since $\mathcal{L} \sin t = \frac{1}{s^2+1}$

$$\therefore \mathcal{L}\, \frac{\sin t}{t} = \int_s^\infty \frac{1}{u^2+1}\, du = \tan^{-1} u \Big|_s^\infty = \tan^{-1}\infty - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

ⓑ show that $\mathcal{L}\left[\frac{e^{-at} - e^{-bt}}{t}\right] = \ln \frac{s+b}{s+a}$

sol. by linearity property $\Rightarrow \mathcal{L}\, \frac{e^{-at}}{t} - \mathcal{L}\, \frac{e^{-bt}}{t}$

$$= \int_s^\infty \frac{1}{u+a}\, du - \int_s^\infty \frac{1}{u+b}\, du = \ln(u+a)\Big|_s^\infty - \ln(u+b)\Big|_s^\infty$$

$$= \cancel{\ln(\infty+a)} - \ln(s+a) - \cancel{\ln(\infty+b)} + \ln(s+b) =$$

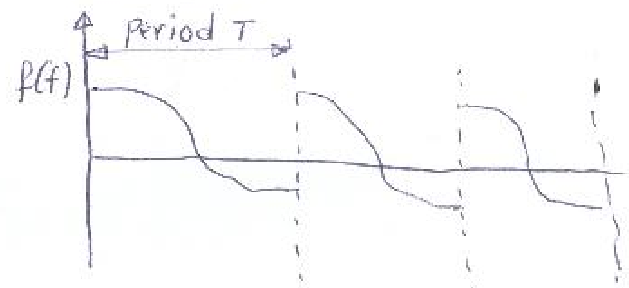
$$= \ln(s+b) - \ln(s+a) = \ln \frac{s+b}{s+a}$$

⑨ Periodic Function :-

Let $f(t)$ have period $T > 0$, So that

$f(t+T) = F(t)$, then

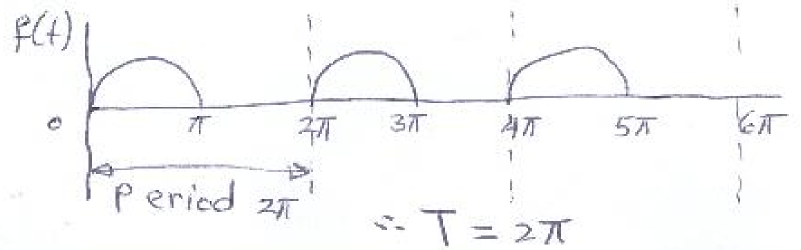
$$\boxed{\mathcal{L}\; f(t) = \frac{\int_0^T e^{-st}.f(t).dt}{1 - e^{-sT}}}$$



ex. Graph the function

$$f(t) = \begin{Bmatrix} \sin t & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{Bmatrix} \text{ then find } \mathcal{L}\, f(t)$$

sol.



$\therefore T = 2\pi$

$$\mathcal{L}\, f(t) = \frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi} \sin t \cdot e^{-st}\cdot dt + \int_{\pi}^{2\pi} 0 \cdot e^{-st}\cdot dt\right]$$

$$= \frac{1}{1-e^{-2\pi s}} \int_0^{\pi} \sin t \cdot e^{-st}\cdot dt \Rightarrow \text{by udv integral}$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{-st}(-s\sin t - \cos t)}{s^2+1}\right|_0^{\pi}$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{-s\pi}(0+1)}{s^2+1} - \frac{e^{-s\cdot 0}(0-1)}{s^2+1}\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{1+e^{-s\pi}}{s^2+1}\right] = \frac{1}{(1-e^{-s\pi})\cancel{(1+e^{-s\pi})}} * \frac{\cancel{1+e^{-s\pi}}}{s^2+1}$$

$$\Rightarrow \mathcal{L}\, f(t) = \frac{1}{(1-e^{-s\pi})(s^2+1)}$$

H.W

ⓐ Graph the function

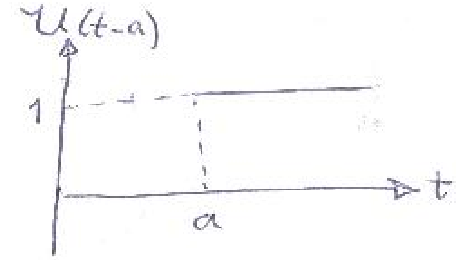
$$f(t) = \begin{cases} \cos t & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases}$$

ⓑ Find $\mathcal{L}\, f(t)$

(10) The unit step function :-

The unit step function, also called Heaviside's unit step function, as defined as :

$$\mathcal{U}(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$$



$$\therefore \quad \mathcal{L}\,\mathcal{U}(t-a) = \frac{e^{-as}}{s} \qquad s > 0$$

Examples:- (a) Prove that $\mathcal{L}\,\mathcal{U}(t-a) = \frac{e^{-as}}{s}$ if $s > 0$

Sol.

$$(\mathcal{U}-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases} \quad \text{So that}$$

$$\mathcal{L}(\mathcal{U}-a) = \int_0^a e^{-st}\cdot(0)\cdot dt + \int_a^\infty e^{-st}\cdot(1)\cdot dt$$

$$= 0 + \left.\frac{e^{-st}}{-s}\right|_a^\infty = \frac{e^{-as}}{s} \quad \text{if } s > 0$$

ex. (b) Express the function $f(t) = \begin{cases} 8 & t < 2 \\ 6 & t > 2 \end{cases}$ in term unit step function and thus obtain its Laplace transform

Sol

$$f(t) = 8 + \begin{cases} 0 & t < 2 \\ -2 & t > 2 \end{cases} = 8 - 2\begin{cases} 0 & t < 2 \\ 1 & t > 2 \end{cases}$$

$$f(t) = 8 - 2\,\mathcal{U}(t-2)$$

$$\therefore \mathcal{L}\,f(t) = \mathcal{L}\left[8 - 2\,\mathcal{U}(t-2)\right]$$

$$= \frac{8}{s} - 2\cdot\frac{e^{-2s}}{s} = \frac{8 - 2e^{-2s}}{s}$$

H.W Find

① $\mathcal{L}\left[2\mathcal{U}(t-1)+3\,\mathcal{U}(t-2)\right]$

② $\mathcal{L}\left[t\cdot\mathcal{U}(t-3)\right]$

③ Express the function

$$f(t)=\left\{\begin{array}{ll}\cos t & t>\pi/2\\ 3 & t<\pi/2\end{array}\right\}$$ in terms of the unit step function and find its Laplace transformation

Example :- Find $\mathcal{L}^{-1} \dfrac{s+1}{s^2+3s+2}$

Sol.

$$\frac{s+1}{s^2-3s+2} = \frac{s+1}{(s-1)(s-2)} = \frac{A}{(s-1)} + \frac{B}{(s-2)} = \frac{A(s-2)+B(s-1)}{(s-1)(s-2)}$$

$$\Rightarrow \quad s+1 = A(s-2) + B(s-1) \quad \text{———} \ast$$

Then $\quad 1 = A + B \quad$ —① $\qquad \Rightarrow$ we can be solved, then

$\qquad 1 = -2A + B \quad$ —② $\qquad A = -2 \;,\; B = 3$

$$\Rightarrow \mathcal{L}^{-1} \frac{s+1}{(s-1)(s-2)} = \mathcal{L}^{-1} \frac{-2}{s-1} + \mathcal{L}^{-1} \frac{3}{s-2}$$

$$= -2\, e^{t} + 3\, e^{2t}$$

$\ast$ Factors $(as+b)^2$ give P.F. $\dfrac{A}{as+b} + \dfrac{B}{(as+b)^2}$

ex. Find $\mathcal{L}^{-1} \dfrac{s^2}{(s+1)(s-1)^2}$

Sol.

$$\frac{s^2}{(s+1)(s-1)^2} = \frac{A}{(s+1)} + \frac{B}{(s-1)} + \frac{C}{(s-1)^2}$$

$$= \frac{A(s-1)^2 + B(s-1)(s+1) + C(s+1)}{(s+1)(s-1)^2}$$

$$\therefore \quad s^2 = A(s-1)^2 + B(s-1)(s+1) + C(s+1)$$

$$s^2 = A(s^2 - 2s + 1) + B(s^2 - 1) + C(s+1)$$

$\therefore$ For $s^2 \Rightarrow 1 = A + B \quad \Rightarrow B = 1 - A \quad$ —①

$\qquad s \Rightarrow 0 = -2A + C \quad \Rightarrow C = 2A \quad$ —② } put in ③

ح. مطلق $\Rightarrow 0 = A - B + C \quad \overset{③}{\Rightarrow} \quad 0 = A - 1 + A + 2A \quad \Rightarrow A = \frac{1}{4}$

And $C = \frac{1}{2} \;,\; B = \frac{3}{4}$

$$\therefore \mathcal{L}^{-1} \frac{s^2}{(s+1)(s-1)^2} = \mathcal{L}^{-1} \frac{1}{4} \cdot \frac{1}{s+1} + \mathcal{L}^{-1} \frac{3}{4} \frac{1}{s-1} + \mathcal{L}^{-1} \frac{1}{2} \frac{1}{(s-1)^2}$$

$$= \frac{1}{4} e^{-t} + \frac{3}{4} e^{t} + \frac{1}{2} t . e^{t}$$

$*$ Factors $(as+b)^3$ gives P.F. $\dfrac{A}{as+b} + \dfrac{B}{(as+b)^2} + \dfrac{C}{(as+b)^3}$

ex.

Find $\mathcal{L}^{-1} \dfrac{s^2+1}{(s+2)^3}$

$$\frac{s^2+1}{(s+2)^3} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3}$$

$$\therefore s^2+1 = A(s+2)^2 + B(s+2) + C$$

let $s+2=0 \Rightarrow s=-2$

$5 = A(0) + B(0) + C \Rightarrow C = 5$

from

$s^2 : \quad 1 = A$

$s : \quad \Rightarrow C = 5$

الثابت $: \quad 1 = 4A + 2B + C \Rightarrow B = -4$

$$\therefore \mathcal{L}^{-1} \frac{s^2+1}{(s+2)^3} = \mathcal{L}^{-1} \frac{1}{s+2} + \mathcal{L}^{-1} \frac{-4}{(s+2)^2} + \mathcal{L}^{-1} \frac{5}{(s+2)^3}$$

$$= e^{-2t} - 4 e^{-2t} \cdot t + \frac{5}{2} e^{-2t} \cdot t^2$$

In general $\dfrac{P(s)}{(Q(s))^n} = \displaystyle\sum_{i=1}^{n} \dfrac{A_i}{(as+b)^i}$

$*$ A quadratic factor (as^2+bs+c) gives P.F $\dfrac{As+B}{as^2+bs+c}$

ex. Find $\mathcal{L}^{-1} \dfrac{s^2}{(s-2)(s^2+1)}$

$$\frac{s^2}{(s-2)(s^2+1)} = \frac{A}{(s-2)} + \frac{Bs+C}{(s^2+1)} = \frac{A(s^2+1)+(Bs+C)(s-2)}{(s-2)(s^2+1)}$$

$$\therefore s^2 = A(s^2+1) + (Bs+C)(s-2)$$

let $s-2=0 \Rightarrow s=2$

Then $4 = 5A + (Bs+C) \times 0 \Rightarrow A = 4/5$

by equating Coeff.

$s^2 \ : \ 1 = A + B \Rightarrow B = 1/5$

C.T $: \ 0 = A - 2C \Rightarrow C = 2/5$

$$\therefore \mathcal{L}^{-1} \frac{s^2}{(s-2)(s^2+1)} = \mathcal{L}^{-1} \frac{4/5}{s-2} + \mathcal{L}^{-1} \frac{\frac{1}{5}s + \frac{2}{5}}{s^2+1}$$

$$= \frac{4}{5} \mathcal{L}^{-1} \frac{1}{s-2} + \mathcal{L}^{-1} \frac{1}{5} \frac{s}{s^2+1} + \mathcal{L}^{-1} \frac{2}{5} \frac{1}{s^2+1}$$

$$= \frac{4}{5} e^{2t} + \frac{1}{5} \cos t + \frac{2}{5} \sin t$$

* Factors of quadratic function $(as^2+bs+c)^2$ has P.F

$$\frac{As+B}{as^2+bs+c} + \frac{Cs+D}{(as^2+bs+c)^2} \quad \text{In general}$$

$$\frac{P(s)}{(\varphi(s))^n} = \sum_{i=1}^{n} \frac{(As+B)_i}{(as^2+bs+c)^i}$$ where $\varphi(s)$ is a quadratic function

ex. find $\mathcal{L}^{-1} \dfrac{s^4+2s+4}{(s-1)(s^2+1)^2}$

sol.

$$\frac{s^4+2s+4}{(s-1)(s^2+1)^2} = \frac{A}{(s-1)} + \frac{Bs+C}{s^2+1} + \frac{Ds+E}{(s^2+1)^2}$$

$$= \frac{A(s^2+1)^2 + (Bs+C)(s-1)(s^2+1) + (Ds+E)(s-1)}{(s-1)(s^2+1)^2}$$

$$\therefore s^4+2s+4 = A(s^2+1)^2 + (Bs+C)(s-1)(s^2+1) + (Ds+E)(s-1)$$

let $s-1=0 \Rightarrow s=1$, then

$$7 = 4A + (Bs+C)(0) + (Ds+E)(0) \Rightarrow A = 7/4$$

by equating Coeff.

$s^4 \ : \ 1 = A + B \Rightarrow B = -3/4$ $\qquad s^2 : 0 = 2A + B + C + D \Rightarrow D = \frac{-7}{2}$

$s^3 \ : \ 0 = C - B \Rightarrow C = -3/4$ $\qquad$ C.T $: 4 = A - C - E \Rightarrow E = -3/2$

$$\therefore \mathcal{L}^{-1} \frac{s^4+2s+4}{(s-1)(s^2+1)^2} = \frac{7}{4} \mathcal{L}^{-1} \frac{1}{s-1} - \frac{3}{4} \mathcal{L}^{-1} \frac{s+1}{s^2+1} - \frac{1}{2} \mathcal{L}^{-1} \frac{7s+3}{(s^2+1)^2}$$

where $\mathcal{L}^{-1} \frac{1}{s-1} = e^{t}$

$$\mathcal{L}^{-1} \frac{s+1}{s^2+1} = \mathcal{L}^{-1} \frac{s}{s^2+1} + \mathcal{L}^{-1} \frac{1}{s^2+1} = \cos t + \sin t$$

$$\mathcal{L}^{-1} \frac{7s+3}{(s^2+1)^2} = \mathcal{L}^{-1} \frac{7s}{(s^2+1)^2} + \mathcal{L}^{-1} \frac{3}{(s^2+1)^2} = \frac{7}{2} \cdot \mathcal{L}^{-1} \frac{2s}{(s^2+1)^2} + \frac{3}{2} \mathcal{L}^{-1} \frac{1 \times 2}{(s^2+1)^2}$$

$$= \frac{7}{2} \cdot t \cdot \cos t + \frac{3}{2} t \cdot \sin t$$

Then

$$\mathcal{L}^{-1} \frac{s^4+2s+4}{(s-1)(s^2+1)^2} = \frac{7}{2} e^{t} - \frac{3}{4} (\cos t + \sin t) - \frac{1}{2} \left[\frac{7}{2} \cdot t \cdot \cos t + \frac{3}{2} t \sin t \right]$$

(b) Long division

IF degree of $P(s)$ equal or more than that of $Q(s)$, can be written as the sum of rational functions by Long division

ex. Find $\mathcal{L}^{-1} \frac{s^2+s+1}{s^2-5s+6}$ $\Rightarrow$ degree of $P(s)$ = degree of $Q(s)$

sol. by long division

$$\begin{array}{r|l} & 1 \\ s^2-5s+6 & s^2+s+1 \\ & \mp s^2 \pm 5s \mp 6 \\ \hline & 6s-5 \end{array}$$

$$\therefore \frac{s^2+s+1}{s^2-5s+6} = 1 + \frac{6s-5}{s^2-5s+6}$$

where

$$\frac{6s-5}{s^2-5s+6} = \frac{6s-5}{(s-2)(s-3)} = \frac{A}{(s-2)} + \frac{B}{(s-3)}$$ by Partial fraction

$$\therefore 6s-5 = A(s-3) + B(s-2)$$

let $s-2=0 \Rightarrow s=2$, then $\Rightarrow 7 = -A + B(0) \Rightarrow A = -7$

let $s-3=0 \Rightarrow s=3$, then $\Rightarrow 13 = A(0) + B \Rightarrow B = 13$

$$\therefore \mathcal{L}^{-1} \frac{6s-5}{s^2-5s+6} = \mathcal{L}^{-1} \frac{7}{(s-2)} + \mathcal{L}^{-1} \frac{13}{(s-3)}$$

Then,

$$= -7e^{2t} + 13e^{3t}$$

$$\therefore \mathcal{L}^{-1} \frac{s^2+s+1}{s^2-5s+6} = \mathcal{L}^{-1}\left[1 + \frac{6s-5}{s^2-5s+6}\right]$$

$$= 0 - 7e^{2t} + 13e^{3t}$$

H.W. Find $\mathcal{L}^{-1} \dfrac{2s^4-5s^3+6s^2-5s+3}{s^3-3s^2+3s-1}$

Hint, ① by long division

$$\mathcal{L}^{-1}\left[\frac{2s^4-5s^3+6s^2-5s+3}{s^3-3s^2+3s-1}\right] = \mathcal{L}^{-1}\left[2s+1+\frac{3s^2+4}{(s-1)^3}\right]$$

② $\mathcal{L}^{-1}(2s-1) = 0$

(12) Application of Laplace transformation :-

The Laplace transform is useful in the following applications

(i) solution of ordinary differential equations with const. coeff.
(ii) Solution of ordinary differential equations with variables coeff.

The L.T is useful in solving linear-ordinary differential equations with constant coefficients by the following steps:

(1) Taking the Laplace transform of both sides of O.D.E.
(2) Using initial or boundary conditions to obtain an algebraic equation for determination of: $\mathcal{L} f(t) = F(s)$
(3) Find the inverse Laplace transform of $F(s)$ to obtain the required solution $f(t)$.

ex.(1) Solve $y'' + y = t$, $y(0) = 1$, $y'(0) = -2$

sol. $\mathcal{L} y'' + \mathcal{L} y = \mathcal{L} t$ where $\mathcal{L} y'' = s^2 . y(s) - s . y(0) - y'(0)$
$= s^2 . y(s) - s . 1 + 2$

$$\Rightarrow s^2 . y(s) - s + 2 + y(s) = \frac{1}{s^2}$$

$$\Rightarrow y(s)\left[s^2+1\right] = \frac{1}{s^2} + s - 2 \Rightarrow y(s)(s^2+1) = \frac{-2s^2 + s^3 + 1}{s^2}$$

$$\Rightarrow y(s) = \frac{-2s^2+s^3+1}{s^2(s^2+1)} = \frac{1+s^2(s-2)}{s^2(s^2+1)} = \frac{1}{s^2(s^2+1)} + \frac{\cancel{s^2}(s-2)}{\cancel{s^2}(s^2+1)}$$

$$\Rightarrow \boxed{\mathcal{L}^{-1} y(s) = y(t) = \mathcal{L}^{-1} \frac{1}{s^2(s^2+1)} + \mathcal{L}^{-1} \frac{s-2}{s^2+1}}$$

then $\mathcal{L}^{-1} \frac{1}{s^2(s^2+1)}$ $\Rightarrow \mathcal{L}^{-1} \frac{1}{s(s^2+1)} = \int_0^t \sin u \, du = -\cos u \Big|_0^t = 1 - \cos t$

and $\mathcal{L}^{-1} \frac{1}{s^2(s^2+1)} = \int_0^t (1-\cos u)\, du = u - \sin u \Big|_0^t$
$= t - \sin t$

$$\mathcal{L}^{-1} \frac{s-2}{s^2+1} = \mathcal{L}^{-1} \frac{s}{s^2+1} - \mathcal{L}^{-1} \frac{2}{s^2+1}$$
$$= \cos t - 2 \sin t$$

$$\therefore y(t) = t - \sin t + \cos t - 2 \sin t$$

OR by Paratial fraction for

$$\mathcal{L}^{-1}\frac{1}{s^2(s^2+1)} \Rightarrow \frac{1}{s^2(s^2+1)} = \frac{A}{s}+\frac{B}{s^2}+\frac{Cs+D}{s^2+1}$$

$$= \frac{As(s^2+1)+B(s^2+1)+(Cs+D)s^2}{s^2(s^2+1)}$$

by equating Coeff. $\Rightarrow 1 = As(s^2+1)+B(s^2+1)+(Cs+D)s^2$

$s^3 : 0 = A+C$

$s^2 : 0 = B+D$

$s : 0 = A$

$C.T : 1 = B \quad \Rightarrow D=-1, \; C=0$

$$\mathcal{L}^{-1}\frac{1}{s^2(s^2+1)} = \mathcal{L}^{-1}\frac{1}{s^2}+\mathcal{L}^{-1}\frac{-1}{s^2+1} = t-\sin t$$

ex. 2 Solve $y''-3y'+2y=4e^{2t}$, $y(0)=-3$, $y'(0)=5$

sol.

$\mathcal{L}y''-3\mathcal{L}y'+2\mathcal{L}y=4\mathcal{L}e^{2t}$ where $\mathcal{L}y''=s^2.y(s)-s.y(0)-y'(0)$

$\mathcal{L}y' = s.y(s)-y(0)$

$$\Rightarrow s^2y(s)-s.(-3)-5-3[s.y(s)-(-3)]+2y(s)=4.\frac{1}{s-2}$$

$$\Rightarrow s^2.y(s)+3s-5-3sy(s)-9+2y(s)=\frac{4}{s-2}$$

$$y(s)[s^2-3s+2]=\frac{4}{s-2}-3s+4 = \frac{4-(3s-14)(s-2)}{(s-2)}$$

$$\Rightarrow y(s)=\frac{4-3s^2+6s+14s-28}{(s-2)(s^2-3s+2)} = \frac{-3s^2+20s-24}{(s-2)^2(s-1)}$$

by partial fraction

$$\frac{-3s^2+20s-24}{(s-2)^2(s-1)} = \frac{A}{(s-1)}+\frac{B}{(s-2)}+\frac{C}{(s-2)^2}$$

$$= \frac{A(s-2)^2+B(s-1)(s-2)+C(s-1)}{(s-1)(s-2)^2}$$

$$\therefore \quad -3s^2+20s-24 = A(s-2)^2 + B(s-1)(s-2) + C(s-1)$$

Let $s-1=0 \Rightarrow s=1$

then $-3(1)^2+20(1)-24 = A + B(0) + C(0) \Rightarrow A=-7$

let $s-2=0 \Rightarrow s=2$

$-3(2)^2+20(2)-24 = A(0)+B(0)+C \Rightarrow C=4$

const. term

C.T : $-24 = 4A+2B+C \Rightarrow B=4$

$$\Rightarrow \mathcal{L}^{-1}\frac{(-3s^2+20s-24)}{(s-1)(s-2)^2} = \mathcal{L}^{-1}\left[\frac{-7}{(s-1)} + \frac{4}{(s-2)} + \frac{4}{(s-2)^2}\right]$$

$$\mathcal{L}^{-1} y(s) = \mathcal{L}^{-1}\frac{-7}{s-1} + \mathcal{L}^{-1}\frac{4}{s-2} + \mathcal{L}^{-1}\frac{4}{(s-2)^2}$$

$$\Rightarrow \quad y(t) = -7e^{t} + 4e^{2t} + 4.t.e^{2t}$$

Ex.(3) solve $y'' + 2y' + 5y = e^{-t}.\sin t \ ; \ y(0)=0, \ y'(0)=1$

sol:

$$\mathcal{L}y'' + 2\mathcal{L}y' + 5\mathcal{L}y = \mathcal{L}[e^{-t}\sin t]$$

$$\left[s^2.y(s) - \cancel{s.y(0)} - y'(0)\right] + 2\left[s.y(s) - \cancel{y(0)}\right] + 5.y(s) = \frac{1}{(s+1)^2+1}$$

$$s^2.y(s) - 1 + 2s.y(s) + 5y(s) = \frac{1}{(s+1)^2+1}$$

$$\Rightarrow y(s)\left[s^2+2s+5\right] = \frac{1}{(s+1)^2+1} + 1 = \frac{1}{s^2+2s+2} + 1$$

$$\Rightarrow y(s) = \frac{1}{s^2+2s+5} + \frac{1}{(s^2+2s+5)(s^2+2s+2)} = \frac{s^2+2s+2+1}{(s^2+2s+5)(s^2+2s+2)}$$

$$\Rightarrow y(s) = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$ by Partial fraction

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$$

$\therefore\ s^2+2s+3 = (As+B)(s^2+2s+5)+(Cs+D)(s^2+2s+2)$

$= s^3(A+C)+s^2(2A+B+2C+D)+s(5A+2B+2C+2D)$
$+ (5B+2D)$

Then

s^3 : $0 = A+C$ — ①

s^2 : $1 = 2A+B+2C+D$ — ②

s^1 : $2 = 5A+2B+2C+2D$ — ③

C.T : $3 = 5B+2D$ — ④

from eq. ① $\Rightarrow A = -C$ — ⑤

" " ④ $\Rightarrow B = \frac{1}{5}(3-2D)$ — ⑥

sub. ⑤,⑥ in eq ②

$\Rightarrow -\cancel{2C}+\frac{1}{5}(3-2D)+\cancel{2C}+D = 1 \quad \Rightarrow D = 2/3$

from eq 6. $\Rightarrow B = 1/3$, sub. the values of D and B in eq.③

$2 = 5A+\frac{2}{3}+2C+2*\frac{2}{3} \Rightarrow 5A+2C = 0$ — ⑦

sub. eq. ⑤ in eq. ⑦ $\Rightarrow C = 0 , A = 0$

Then

$$Y(s) = \frac{1/3}{s^2+2s+2} + \frac{2/3}{s^2+2s+5}$$

$$\mathcal{L}^{-1} y(s) = \mathcal{L}^{-1}\frac{1/3}{s^2+2s+2} + \mathcal{L}^{-1}\frac{2/3}{s^2+2s+5}$$

where

$$\mathcal{L}^{-1}\frac{1/3}{s^2+2s+2} = \frac{1}{3}\mathcal{L}^{-1}\frac{1}{s^2+2s+1-1+2} = \frac{1}{3}\mathcal{L}^{-1}\frac{1}{(s+1)^2+1} = \frac{1}{3}e^{-t}.\sin t$$

$$\mathcal{L}^{-1}\frac{2/3}{s^2+2s+5} = \frac{2}{3}\mathcal{L}^{-1}\frac{1}{s^2+2s+1-1+5} = \frac{2}{3}\mathcal{L}^{-1}\frac{2/2}{(s+1)^2+2^2} = \frac{1}{3}e^{-t}\sin 2t$$

Then

$$y(t) = \frac{1}{3}e^{-t}\left[\sin t+\sin 2t\right]$$

H.W Solve , Find $y(s)$ only

① $y''+5y'+2y = e^{-t}.\sin 2t$

② $y'''-y = e^{t}$

Note: The Laplace transform of t^n by gama function when $n > -1$ and $s > 0$ is :-

$$\Rightarrow \boxed{\mathcal{L}\, t^n = \frac{\Gamma(n+1)}{s^{n+1}}} \;;\; \text{and} \quad \mathcal{L}^{-1} \frac{1}{s^{n+1}} = \frac{t^n}{\Gamma(n+1)}$$

ex. (1) $\mathcal{L}\, t^{-1/2}$,

$$n = -\tfrac{1}{2} \;,\; n+1 = \tfrac{1}{2} \;,\; \mathcal{L}\, t^{-1/2} = \frac{\Gamma 1/2}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$$

(2) $\mathcal{L}\, t^{1/2}$,

$$n = 1/2 \;,\; n+1 = \frac{3}{2} \Rightarrow \mathcal{L}\, t^{1/2} = \frac{\Gamma 3/2}{s^{3/2}} \Rightarrow \Gamma 3/2 = \frac{1}{2}\Gamma 1/2 = \frac{\sqrt{\pi}}{2}$$

$$\therefore \mathcal{L}\, t^{1/2} = \frac{\sqrt{\pi}}{2\, s^{3/2}} = 1/2 \cdot \sqrt{\frac{\pi}{s^3}}$$

(3) $\mathcal{L}\, t^{5/2}$,

$$n = 5/2 \;,\; n+1 = \frac{7}{2}$$

$$\therefore \mathcal{L}\, t^{5/2} = \frac{\Gamma 7/2}{s^{7/2}} \qquad \text{where } \Gamma 7/2 = \frac{5}{2}\Gamma 5/2 = \frac{5}{2} * \frac{3}{2}\Gamma 3/2$$

$$= \frac{5}{2} * \frac{3}{2} * \frac{1}{2}\Gamma 1/2$$

$$\Gamma 7/2 = \frac{15}{8}\sqrt{\pi}$$

$$\mathcal{L}\, t^{5/2} = \frac{15}{8}\sqrt{\frac{\pi}{s^7}}$$

(4) $\mathcal{L}\, t^{-3/2}$, $\Rightarrow n = \frac{-3}{2}$, $n+1 = \frac{-1}{2} \Rightarrow \mathcal{L}\, t^{-3/2} = \frac{\Gamma -1/2}{s^{-1/2}} = \frac{-2\sqrt{\pi}}{s^{-1/2}}$

$$\mathcal{L}\, t^{-3/2} = -2\sqrt{s\pi}$$

(5) $\mathcal{L}^{-1} \frac{1}{s^{3/2}}$, we have $\mathcal{L}^{-1}\frac{1}{s^{n+1}} = \frac{t^n}{\Gamma(n+1)}$. $n+1 = 3/2$

$$\therefore \mathcal{L}^{-1}\frac{1}{s^{3/2}} = \frac{t^{1/2}}{\Gamma 3/2} = \frac{\sqrt{t}}{\frac{1}{2}\sqrt{1/2}} = \frac{\sqrt{t}}{\frac{1}{2}\sqrt{\pi}} = 2\sqrt{\frac{t}{\pi}}$$

	Laplace Transformation (L)				Inverse Laplace Transformation (L^{-1})		
	Name	Case	Example		Name	Case	Example
1.	Linearity property	If : $f(t)=c_1 f_1(t)+c_2 f_2(t)+\dots$, **then :** $L f(t)=c_1 L f_1(t)+$ $c_2 L f_2(t)+\dots\dots$	If : $f(t)=2t^2+5\sinh 4t$, *then :* $L f(t)=\frac{2*2!}{s^3}+\frac{5*4}{s^2+16}$	1.	*Linearity property*	If : $F(s)=c_1 F_1(s)+c_2 F_2(s)$ $+\dots$, *then :* $L^{-1}F(s)=c_1 L^{-1}F_1(s)+$ $c_2 L^{-1}F_2(s)+\dots$	$L^{-1}\left\{\frac{2}{s}+\frac{7}{s-3}\right\}=2+7e^{3}$
2.	First Shifting property	If : $L f(t)=F(s)$ $L\{e^{at}f(t)\}=F(s-a)$	If : $f(t)=\cos 4t$, *then :* $L\{e^{-t}\cos 4t\}=$ $\frac{s+1}{(s+1)^2+16}$	2.	*First Shifting property*	If : $L^{-1}F(s)=f(t)$, *then :* $L^{-1}F(s-a)=e^{at}f(t)$	If: $F(s)=\frac{s+1}{(s+1)^2+16}$ *then :* $L^{-1}F(s)=e^{-t}\cos 4t$
3.	Second Shifting property	If : $L f(t)=F(s)$, *and we have:* $g(t)=\begin{cases} f(t-a) & t>a \\ 0 & t<a \end{cases}$, *then:* $L\,g(t)=e^{-as}F(s)$	If : $g(t)=\begin{cases} 4(t-2)^2 & t>2 \\ 0 & t<2 \end{cases}$, *then :* $L g(t)=e^{-2t}\frac{4*2!}{s^3}$	3.	*Second Shifting property*	If : $L^{-1}F(s)=f(t)$, *then :* $L^{-1}e^{-as}F(s)=$ $\begin{cases} f(t-a) & t>a \\ 0 & t<a \end{cases}$	If : $L^{-1}e^{-3s}\frac{1}{s-2}$ $=\begin{cases} e^{2(t-3)} & t>3 \\ 0 & t<3 \end{cases}$
4.	Change of Scale	If : $L f(t)=F(s)$, *then :* $L\ f(at)=\frac{1}{a}F(\frac{s}{a})$	If : $f(t)=\cosh 2t$, *then :* $L\cosh 2t=\frac{1}{2}\frac{s/2}{(s/2)^2-4}$ $=\frac{1}{4}\frac{s}{\frac{s^2-4}{4}}=\frac{s}{s^2-4}$	4.	*Change of Scale*	If : $L^{-1}F(s)=f(t)$ $L^{-1}F(as)=\frac{1}{a}f(\frac{t}{a})$	$F(s)=\frac{1}{3s+7}$;$a=3$ *then :* $L^{-1}F(s)=\frac{1}{3}e^{-\frac{7}{3}t}$

$$\therefore 6s-5 = A(s-3) + B(s-2)$$

Let $s-2=0 \Rightarrow s=2$

$$7 = -A + B(0) \Rightarrow \boxed{A=-7}$$

Let $s-3=0 \Rightarrow s=3$

$$13 = A(0) + B \Rightarrow \boxed{B=13}$$

$$\therefore \mathcal{L}^{-1} \frac{6s-5}{s^2-5s+6} = \mathcal{L}^{-1} \frac{-7}{(s-2)} + \mathcal{L}^{-1} \frac{13}{(s-3)}$$

$$= -7e^{2t} + 13e^{3t}$$

$$\therefore \mathcal{L}^{-1} \frac{s^2+s+1}{s^2-5s+6} = \mathcal{L}^{-1}\left(1 + \frac{6s-5}{s^2-5s+6}\right)$$

$$= 0 - 7e^{2t} + 13e^{3t}$$

$$= \underline{\underline{13e^{3t} - 7e^{2t}}}$$

H.W:-

Find $\mathcal{L}^{-1} \frac{2s^4-5s^3+6s^2-5s+3}{s^3-3s^2+3s-1}$

Hint: ① by long division:-

$$\mathcal{L}^{-1} \frac{2s^4-5s^3+6s^2-5s+3}{s^3-3s^2+3s-1} = \mathcal{L}^{-1}\left(2s+1+\frac{3s^2+4}{(s-1)^3}\right)$$

② $\mathcal{L}^{-1}(2s-1) = 0$

5.	L.T. of derivative	If : $L\,f(t)=F(s)$, then : $L\ f^{n}(t)=[s^{n}F(s)-s^{n-1}f(0)-\ldots-f^{n-1}(0)]$	If : $f(t)=\sin t$, then : $L\sin t=-L\frac{d}{dt}\cos t=$ $-Lf'(t)=-[s\frac{s}{s^2+1}-1]$ $=-[\frac{s^2-s^2-1}{s^2+1}]$ $=\frac{1}{s^2+1}=L\sin t$	5.	Inv. L.T. of derivative	If : $L^{-1}F(s)=f(t)$, then: $L^{-1}F^{n}(s)=L^{-1}\left[\frac{d^n}{ds^n}F(s)\right]$ $=(-1)^{n}t^{n}f(t)$	$L^{-1}\frac{-2s}{(s^2+1)^2}$ $=L^{-1}\left[\frac{d}{ds}\frac{1}{s^2+1}\right]$ $=-t\sin t$
6.	L.T. of Integral	If : $L\,f(t)=F(s)$, then : $L\int_0^t f(u)du=\frac{F(s)}{s}$	If : $L\int_0^t \sin 2u\,du=\frac{\frac{2}{s^2+4}}{s}$ $=\frac{2}{s(s^2+4)}$	6.	Inv. L.T. of Integral	If : $L^{-1}F(s)=f(t)$, then: $L^{-1}\int_s^{\infty}F(u)\,du=\frac{f(t)}{t}$	$L^{-1}\int_s^{\infty}\frac{1}{u^2}\,du=L^{-1}(\frac{1}{s^2})$ $=\frac{t}{t}=1$
7.	L.T. of Multiplication by t^n	If : $L\,f(t)=F(s)$, then : $L\{t^{n}f(t)\}=(-1)^{n}*$ $*\frac{d^n}{ds^n}F(s)=(-1)^{n}F^{n}(s)$	If : $L\{t^{n}e^{2t}\}=-\frac{d}{ds}\frac{1}{s-2}$ $=-\frac{-1}{(s-2)^2}=\frac{1}{(s-2)^2}$	7.	Inv. L.T. of Multiplication By (s^n)	If : $L^{-1}F(s)=f(t)$, then: $L^{-1}sF(s)=f'(t)$	$L^{-1}\frac{s}{(s^2+4)}=\frac{1}{2}\frac{d}{dt}\sin 2t$ $=\cos 2t$
8.	L.T. of Division by t	If : $L\,f(t)=F(s)$, then : $L\frac{f(t)}{t}=\int_s^{\infty}F(u)du$	If : $L\frac{\sin t}{t}=\int_s^{\infty}\frac{1}{u^2+1}du$ $=\tan^{-1}]_s^{\infty}=\frac{\pi}{2}-\tan^{-1}s$ $=\tan^{-1}\frac{1}{s}$	8.	Inv. L.T. of Division by(s^n)	If : $L^{-1}F(s)=f(t)$, then : $L^{-1}\frac{F(s)}{s}=\int_0^t f(u)du$	$L^{-1}\frac{1}{s^2}=\int_0^t 1du=u]_0^t$ $=t-0=t$

9.	L.T. of Periodic Function	If : $f(t)$ have period $T>0$, so that: $Lf(t)=\frac{\int_{0}^{T} e^{-st} f(t)\,dt}{1-e^{-sT}}$		9.	Inv. L.T. By Partial Fraction		
10.	L.T. of Unit Step Function	If : $Lf(t)=F(s)$, and we have: $u(t-a)=\begin{cases}1 & t>a\\ 0 & t<a\end{cases}$, then: $L\,u(t-a)=\frac{e^{-as}}{s}$	$u(t-2)=\begin{cases}1 & t>2\\ 0 & t<2\end{cases}$, then: $L\,u(t-2)=\frac{e^{-2s}}{s}$	10.	Inv. L.T. By Long Division		

(13) The Gama function :-

IF $n > 0$, we define the gamma function

$$\boxed{\Gamma(n) = \int_0^{\infty} u^{n-1} e^{-u} \, du}$$

the following are some important properties of the gama function

1. $\Gamma(n+1) = n\Gamma(n) \quad , n > 0$
2. $\Gamma(1) = 1$
3. $\Gamma(1/2) = \sqrt{\pi}$
4. for $n < 0$ we define $\Gamma(n)$ by $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$

Examples :- Find

1. $\Gamma(2) \Rightarrow n+1 = 2 \Rightarrow n = 1 > 0 \Rightarrow \therefore \Gamma(2) = 1 \cdot \Gamma 1 = 1.1 = 1!$
2. $\Gamma(3) \Rightarrow n+1 = 3 \Rightarrow n = 2 > 0 \Rightarrow \therefore \Gamma(3) = 2 \cdot \Gamma 2 = 2!$
3. $\Gamma(4) \Rightarrow n+1 = 4 \Rightarrow n = 3 > 0 \Rightarrow \therefore \Gamma(4) = 3 \cdot \Gamma 3 = 3 \cdot 2! = 6!$

and in general $\Gamma(n+1) = n\Gamma n = n!$ if $n = 1, 2, 3, \ldots$

4. $\Gamma(-1/2)$ by using $\Gamma n = \dfrac{\Gamma(n+1)}{n}$ for $n < 0$

$n = -1/2$ and $n+1 = 1/2 \Rightarrow \Gamma(-1/2) = \dfrac{\Gamma 1/2}{-1/2} = -2\sqrt{\pi}$

5. $\Gamma(-3/2) \Rightarrow n = \dfrac{-3}{2}$, $n+1 = -1/2 \Rightarrow \Gamma -3/2 = \dfrac{\Gamma -1/2}{-3/2} = \dfrac{-2\sqrt{\pi}}{-3/2} = \dfrac{4\sqrt{\pi}}{3}$
6. $\Gamma(-5/2) \Rightarrow n = -5/2$, $n+1 = -3/2$

$\Rightarrow \therefore \Gamma -5/2 = \dfrac{\Gamma -3/2}{-5/2} = \dfrac{4\sqrt{\pi}/3}{-5/2} = -\dfrac{8}{15}\sqrt{\pi}$

7. $\Gamma(0) \Rightarrow n = 0$, $n+1 = 1 \Rightarrow \Gamma 0 = \dfrac{\Gamma 1}{0} = \dfrac{1}{0} = \infty$
8. $\Gamma(-1) \Rightarrow n = -1$, $n+1 = 0 \Rightarrow \Gamma -1 = \dfrac{\Gamma 0}{-1} = \infty$
9. $\Gamma(-2) \Rightarrow n = -2$, $n+1 = -1$

$\Rightarrow \Gamma -2 = \dfrac{\Gamma -1}{-2} = \dfrac{\infty}{-2} = \infty$

In general $\Gamma(-P) = \infty$ if P any positive const. integer

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7. $\Gamma(0) \Rightarrow n = 0$, $n+1 = 1 \Rightarrow \Gamma_0 = \dfrac{\Gamma_1}{0} = \dfrac{1}{0} = \infty$
8. $\Gamma(-1) \Rightarrow n = -1$, $n+1 = 0 \Rightarrow \Gamma_{-1} = \dfrac{\Gamma_0}{-1} = \infty$
9. $\Gamma(-2) \Rightarrow n = -2$, $n+1 = -1 \Rightarrow \Gamma_{-2} = \dfrac{\Gamma_{-1}}{-2} = \dfrac{\infty}{-2} = \infty$

In general $\Gamma(-P) = \infty$ if P any positive const. integer

ex.(4) Find the general solution of the diff. eq.

$$y''' - 3y'' + 3y' - y = t^2 \cdot e^t \quad , \quad y(0) = 1 , \ y'(0) = 0 , \ y''(0) = -2$$

Boundary and initial condition

Sol.

$$\mathcal{L} y''' = s^3 y(s) - s^2 y(0) - s \cdot y'(0) - y''(0)$$
$$\mathcal{L} y''' = s^3 \cdot y(s) - s^2 + 2$$

$$\mathcal{L} y'' = s^2 y(s) - s\, y(0) - y'(0)$$
$$\mathcal{L} y'' = s^2 \cdot y(s) - s$$

$$\mathcal{L}\, t^2 \cdot e^t = \frac{2}{(s-1)^3}$$

$$\mathcal{L} y' = s \cdot y(s) - y(0)$$
$$\mathcal{L} y' = s \cdot y(s) - 1$$

Then

$$s^3 \cdot y(s) - s^2 + 2 - 3 s^2 y(s) + 3s + 3 s \cdot y(s) - 3 - y(s) = \frac{2}{(s-1)^3}$$

$$y(s)\left[s^3 - 3s^2 + 3s - 1\right] = \frac{2}{(s-1)^3} + s^2 - 3s + 1$$

by simplify

$$y(s) = \frac{2}{(s-1)^6} + \frac{s^2 - 3s + 1}{(s-1)^3}$$

$$\to s^2 - 3s + 1$$
$$= s^2 - 2s + 1 - s$$
$$= (s-1)^2 - s + 1 - 1$$
$$= (s-1)^2 - (s-1) - 1$$

$$y(s) = \frac{2}{(s-1)^6} + \frac{(s-1)^2 - (s-1) - 1}{(s-1)^3} = \frac{2}{(s-1)^6} + \frac{1}{(s-1)} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3}$$

$$\Rightarrow \mathcal{L}^{-1} y(s) = y(t) = \frac{1}{60} t^5 \cdot e^t + e^t - t \cdot e^t - t^2 \cdot e^t / 2.$$

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Boundary and initial condition

Sol.

$$\mathcal{L}\,y''' = s^3\, y(s) - s^2 y(0) - s \cdot y'(0) - y''(0)$$
$$\mathcal{L}\,y''' = s^3 \cdot y(s) - s^2 + 2$$

$$\mathcal{L}\,y'' = s^2\, y(s) - s\, y(0) - y'(0) \qquad\qquad \mathcal{L}\, t^2 \cdot e^t = \frac{2}{(s-1)^3}$$
$$\mathcal{L}\,y'' = s^2 \cdot y(s) - s$$

$$\mathcal{L}\,y' = s \cdot y(s) - y(0)$$
$$\mathcal{L}\,y' = s \cdot y(s) - 1$$

Then

$$s^3 \cdot y(s) - s^2 + 2 - 3 s^2 y(s) + 3s + 3s \cdot y(s) - 3 - y(s) = \frac{2}{(s-1)^3}$$

$$y(s)\left[s^3 - 3s^2 + 3s - 1\right] = \frac{2}{(s-1)^3} + s^2 - 3s + 1$$

by simplify

$$y(s) = \frac{2}{(s-1)^6} + \frac{s^2 - 3s + 1}{(s-1)^3}$$

$$s^2 - 3s + 1 = s^2 - 2s + 1 - s = (s-1)^2 - s + 1 - 1 = (s-1)^2 - (s-1) - 1$$

$$y(s) = \frac{2}{(s-1)^6} + \frac{(s-1)^2 - (s-1) - 1}{(s-1)^3} = \frac{2}{(s-1)^6} + \frac{1}{(s-1)} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3}$$

$$\Rightarrow \mathcal{L}^{-1}\, y(s) = y(t) = \frac{1}{60} t^5 \cdot e^t + e^t - t \cdot e^t - t^2 \cdot e^t / 2 .$$

Numerical Anylsis

Solution of non-linear equations

1 simple Iterative method

As mentioned above, open methods employ a formula to predict the root. such a formula can be developed for simple Iterative by rearanging the function $f(x) = 0$ so that x is one the left-hand side of the equation:

$$x = g(x) \quad \text{---} (1)$$

this transformation can be accomplished either by algebraic manipulation or by simply adding x to both sides of the original ~~equation~~.

for example, $x^2 - 2x + 3 = 0$

can be simply mainipulated to yield $x = \frac{x^2+3}{2}$

whereas $\sin x = 0$ would be put into the form of eq. (1) by adding x to both sides to yield

$x = \sin x + x$

the utility of eq. (1) is that it provides a formula to predict a value of x as a function of x, Thus, given an initial guess at the root x_i, eq. (1) can be used to compute a new estimate x_{i+1}, as expressed by the Iterative

formula : $X_{i+1} = g(X_i)$

And the approximate error for this equation can be determined using the error in the following equation

$$|\epsilon| = \left| \frac{X_{i+1} - X_i}{X_{i+1}} \right| 100\%$$

ex.1 use simple Iterative method to locate the root of $f(x) = e^{-x} - x = 0$

sol. For $x_{i+1} = g(x_i) \Rightarrow X_{i+1} = e^{-X_i}$

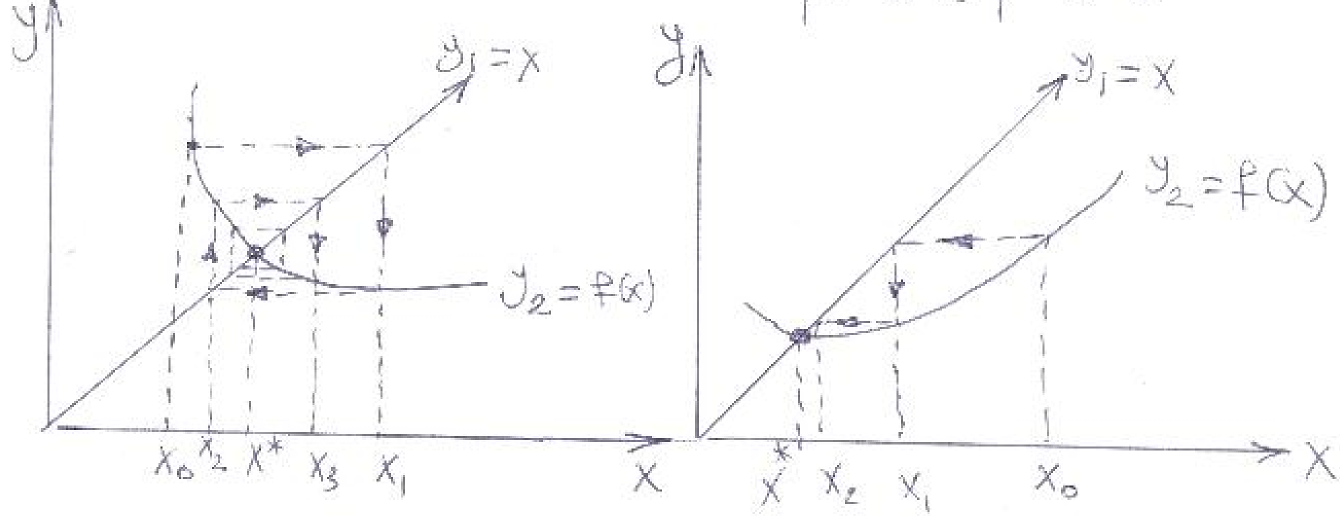
and starting with initial value of $x_0 = 0$

i	x_i	$\|\epsilon\|$
0	0	
1	1.000000	100
2	0.367879	171.8
3	0.692201	46.9
4	0.500473	38.3
5	0.606244	17.4
6	0.545396	11.2
7	0.579612	5.9
8	0.560115	3.48
9	0.571143	1.93
10	0.564879	1.11
⋮	⋮	⋮

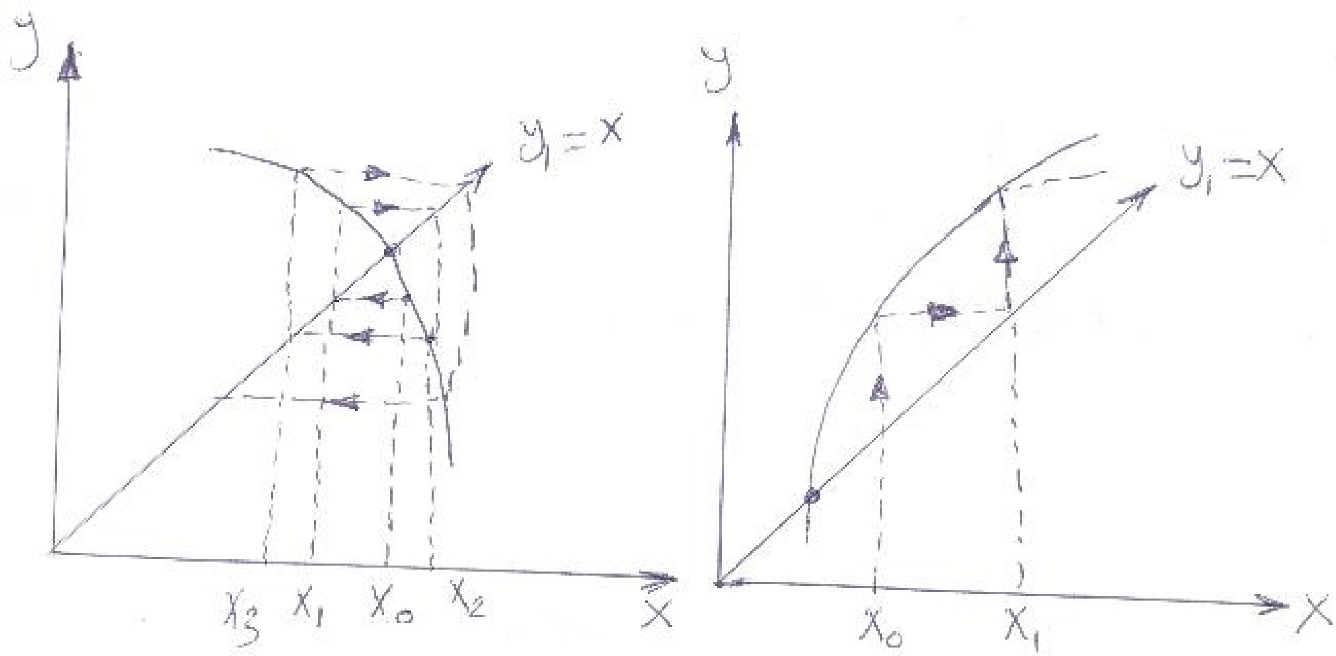
the true value $X^* = 0.567143$

Condition of convergence and divergence for simple Iteration method

the function $x_{i+1} = g(x_i)$ to converge must be satisfied $|f'(x_i)| < 1$



Convergence cases



divergence cases