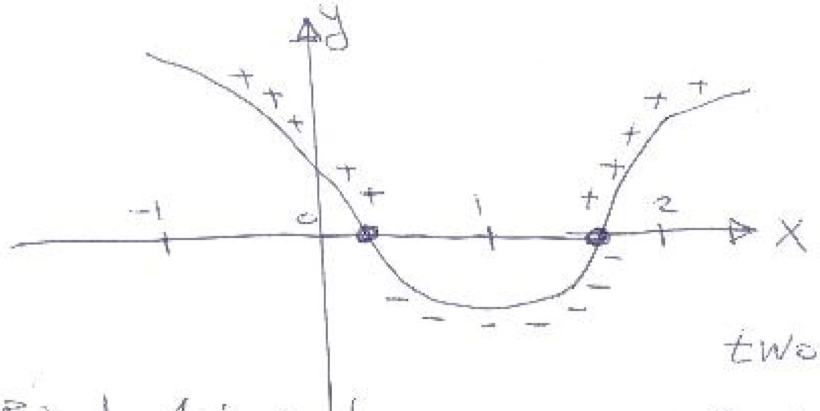
ex. 2 Find the root of the equation $2X^3 - 7X + 2 = 0$ using the simple iterative method.

sol. to find the initial value of iteration X_0 we must graph the function.

then $y = f(x) = 2X^3 - 7X + 2$



X	y
-1	+
0	+
1	−
2	+

two roots

$0 \leqslant X_1^* \leqslant 1$

$1 \leqslant X_2^* \leqslant 2$

* to find 1st root

$0 \leqslant X_1^* \leqslant 1 \Rightarrow X_{i+1} = \frac{2}{7}(X_i^3 + 1)$

and $f'(x) = \frac{6}{7}X^2$

$|f'(0)| = 0 < 1$

$|f'(1)| = \frac{6}{7} < 1$

$\Rightarrow$ the solution is Convergence

i	X_i	X_{i+1}
0	1.000	0.5714
1	0.5714	0.3390
2	0.3390	0.2968
3	0.2968	0.2932
4	0.2932	0.2929
5	0.2929	0.2929

$\Rightarrow X_1^* = 0.2929$

* to find 2nd root

$1 \leqslant X_2^* \leqslant 2$

$\left| f'(1) \right| = \frac{6}{7} < 1$

$\left| f'(2) \right| = \frac{24}{7} > 1$

$\Rightarrow$ the solution is divergence

therefor, the equation must be change

$$X_{i+1} = \sqrt[3]{\frac{7}{2} X_i - 1}$$

and $f'(x) = \frac{1}{3}\left(\frac{7}{2}X - 1\right)^{-2/3} * \frac{7}{2}$

$\left| f'(1) \right| =$ [illegible] < 1

$\left| f'(2) \right| = 0.3533 < 1$

$\Rightarrow$ the solution is convergence

i	X_i	X_{i+1}
0	1.000	1.3572
1	1.3572	1.5536
2	1.5536	1.6433
3	1.6433	1.6812
4	1.6812	1.6967
5	1.6967	1.7029
6	1.7029	1.7054
7	1.7054	1.7064
8	1.7064	1.7068
10	1.7068	1.7070
11	1.7070	1.7071
12	1.7071	1.7071

$\Rightarrow X^* = 1.7071$

ex Find the root of the equation $x = \cos x$ using the simple Iteration method

Sol. $f(x) = x - \cos x$ to graph

x	$f(x)$
-1	-
0	-
1	+
2	+

then $0 \le x^* \le 1$, $x_{i+1} = \cos x_i$

$\Rightarrow f'(x) = -\sin x$

$|f'(0)| = 0 < 1$

$|f'(1)| = 0.841 < 1$

$\Rightarrow$ this form of the equation $x = \cos x$ will be converge

i	x_i	x_{i+1}
0	0.00	1.00
1	1.00	0.54
2	0.54	0.86
3	0.86	0.65
4	0.65	0.79
5	0.79	0.70
6	0.70	0.76
7	0.76	0.72
8	0.72	0.75
9	0.75	0.73
10	0.73	0.74
11	0.74	0.74

$\Rightarrow x^* = 0.74$

ex. find the root of the following equation $x^2 - 4 = \ln x$ use $x_0 = 1.000$

sol. Let $x_{i+1} = \sqrt{\ln x_i + 4} = f(x) = (\ln x + 4)^{1/2}$

$\Rightarrow f'(x) = \frac{1}{2}(\ln x + 4)^{-1/2} * \frac{1}{x}$

$\Rightarrow f'(1) = 0.25 < 1$

i	x_i	x_{i+1}
0	1.000	2.000
1	2.000	2.166
2	2.166	2.185
3	2.185	2.187
4	2.187	2.187

$\Rightarrow x^* = 2.187$

ex. Find one root of the equation $2x^5 - 2x - 1 = 0$ start with $x_0 = 0.000$

Sol. Let $x_{i+1} = \sqrt[5]{\frac{2x_i + 1}{2}} \Rightarrow f'(x) = \frac{1}{5}\left(\frac{2x+1}{2}\right)^{-4/5} * 1$

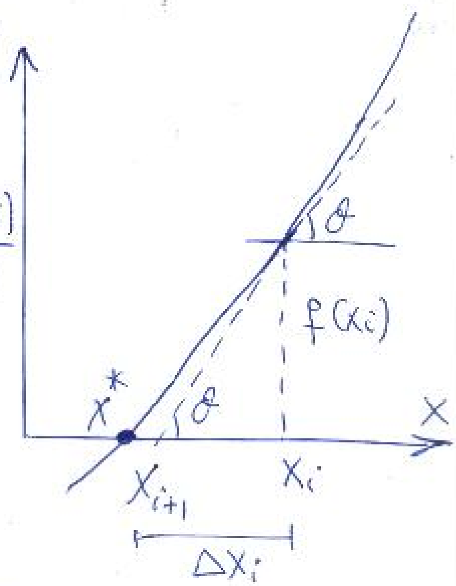
$\Rightarrow f'(0) = 0.348 < 1$

i	x_i	x_{i+1}
0	0.000	0.871
1	0.871	1.065
2	1.065	1.094
3	1.094	1.098
4	1.098	1.098

$\Rightarrow x^* = 1.098$

② Newton-Raphson method

Let $f(x) = 0$



using Taylor series method

$$f(x) = 0 = f(x_i) + \Delta x_i \cdot \frac{f'(x_i)}{1!} + \Delta x_i^2 \cdot \frac{f''(x_i)}{2!} + \Delta x_i^3 \cdot \frac{f'''(x_i)}{3!} + \cdots$$

for more accuracy $\Delta x_i << 0$

then $\Rightarrow f(x_i) + \Delta x_i \cdot \dfrac{f'(x_i)}{1!} = 0$

$$\Rightarrow \Delta x_i = \frac{-f(x_i)}{f'(x_i)}$$

$$x_{i+1} = x_i + \Delta x_i$$

$$\Rightarrow \boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

ex. solve the following equation using Newton Raphson method:

sol. $\Rightarrow f(x) = \dfrac{1}{x} + 1 \quad , \quad f'(x) = -1/x^2$

$$\Rightarrow x_{i+1} = x_i - \frac{(1/x + 1)}{(-1/x^2)}$$

i	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}
0	-0.500	-1.000	-4.000	-0.750
1	-0.750	-0.333	-1.770	-0.937
2	-0.937	-0.067	-1.137	-0.997
3	-0.997	-0.003	-1.006	-1.000

Application of special cases for Newton-Raphson method

(A) Square roots

Let $n > 0 \Rightarrow$ any number , and $x = \sqrt{n}$

$$\Rightarrow x^2 - n = 0 = f(x) \quad , \quad f'(x) = 2x$$

then by N-R-M

$$x_{i+1} = x_i - \frac{x_i^2 - n}{2x_i} = \frac{1}{2}\left[x_i + \frac{n}{x_i}\right]$$

ex. Find the square root of 10 using Newton Raphson method, starting with $x_0 = 3.0000$

sol. $n = 10 \Rightarrow x_{i+1} = x_i - \frac{x_i^2 - 10}{2x_i}$

i	x_i	x_{i+1}
1	3.0000	3.1667
2	3.1667	3.1623
3	3.1623	3.1623

$$\Rightarrow x^* = 3.1623$$

(B) Roots of any arbitrary order

Let $x = \sqrt[k]{n} \Rightarrow x^k - n = f(x)$

$\Rightarrow f'(x) = k\,x^{k-1}$ where n = any number

k = Integar number

then by N-R-M

$$x_{i+1} = x_i - \frac{x_i^k - n}{k\,x_i^{k-1}}$$

ex. Compute $\sqrt[3]{7}$, using Newton-Raphson method starting from $x_0 = 1.5$, take an accuracy 5D places.

Sol. $n = 7$, $K = 3 \Rightarrow x_{i+1} = x_i - \dfrac{x_i^3 - 7}{3\,x_i^2}$

i	x_i	x_{i+1}
1	1.50000	2.03704
2	2.03704	1.92034
3	1.92034	1.91296
4	1.91296	1.91293
5	1.91293	1.91293

$\Rightarrow x^* = 1.91293$

(c) The Reciprocal of any number

Let $x = \frac{1}{n} \Rightarrow n = \frac{1}{x} \Rightarrow f(x) = \frac{1}{x} - n = 0$

$\Rightarrow f'(x) = -\frac{1}{x^2}$

by N-R.M $\Rightarrow X_{i+1} = X_i - \frac{\left(\frac{1}{X_i} - n\right)}{\left(-1/X_i^2\right)}$

ex. Find the reciprocal of 2, using Newton Raphson method, starting with $X_0 = 0.1$ work to 4D ?

Sol. $n = 2 \Rightarrow X_{i+1} = X_i - \frac{\left(\frac{1}{X_i} - 2\right)}{\left(-1/X_i^2\right)}$

i	X_i	X_{i+1}
0	0.1000	0.1800
1	0.1800	0.2952
2	0.2952	0.4161
3	0.4161	0.4852
4	0.4852	0.4995
5	0.4995	0.4999
6	0.4999	0.4999

Problems

① By using simple Iteration method, find one root of the following equations

Ⓐ $4X = e^{X}$ use 4D, <u>Ans.</u> $X^{*} = 0.3574$

Ⓑ $e^{2X} \cdot \tan X = e^{-3\pi/2}$ use 5D, <u>Ans.</u> $X^{*} = 0.00883$

Ⓒ $10X = 2^{X^2} + 1$ use 4D, <u>Ans.</u> $X^{*} = 0.2029$

Ⓓ $\sin x = \dfrac{1}{(X^2 - \ln x)^3}$ use 3D, <u>Ans</u> $X^{*} = 0.012$

② find the roots (three only) of the following equation, using simple Iteration method

$e^{X} - 3X^{2} = 0$ use 3D, <u>Ans.</u> $X_1^{*} = -0.459$
$X_2^{*} = 0.910$
$X_3^{*} = 3.733$

③ find the root of the equation

$$e^{2t} \cdot \tan t = e^{-3\pi/2}$$

④ find the smallest positive non-zero root of the following equation

$$\frac{0.625 + 0.3X}{0.625 + 3.27X} - \cos\left[\sqrt{\frac{X}{0.0006}} * 0.0191\right] = 0$$

using Newton-Raphson method with accuracy 0.0001

5. Solve the following non-linear algebraic equation to find one real root of $\tan x - 2\tanh x = 0$

6. Solve the following non-linear algebraic equation to find one real non zero root:

$$f(x) = \sin\left[\sqrt{\sec x + x^3 \cdot e^{5x/\tan x}}\right] - e^{-x}$$

7. for turbulent flow of fluid in a smoth pipe, the following relation exists between the friction factor C_f and Reynolds number Re

$$\sqrt{\frac{1}{C_f}} = -0.4 + 1.74 \ln\left(Re\sqrt{C_f}\right)$$

Compute C_f for $Re = 10^5$, correct to six decimal places.

8. Determine the smallest positive root of the equation

$$\sin(0.573t) - e^{-0.01t} = 0$$

9. Solve using Newton-Raphson method the following equation

$$4000 = \frac{9.8 * 68100}{c}\left[1 - e^{-7c/68100}\right]$$

Special Cases of Newton-Raphson method

① Square Roots إيجاد الجذور التربيعية

Let $f(x) = 0$, $f(x) = x^2 - n$ الرقم المطلوب n

$$\Rightarrow x = \sqrt{n}$$

$$X_{r+1} = X_r - \frac{f(X_r)}{f'(X_r)} = X_r - \frac{X_r^2 - n}{2X_r}$$

$$X_{r+1} = \frac{1}{2}\left[X_r + \frac{n}{X_r}\right] \qquad r = 0, 1, 2 \dots$$

<u>ex</u> Find the square root of 10, using N-R-M starting with 3 as an initial values

r	X_r	X_{r+1}
1	3.0000	3.1667
2	3.1667	3.1623
3	3.1623	3.1623

② Roots of An Arbitrary Order الجذر لاي مرتبة

let $f(x) = x^k - n$ اذا كان

k عدد صحيح

$n > 0$ الرقم المطلوب

$$X_{r+1} = X_r - \frac{X_r^k - n}{k X_r^{k-1} - n}$$

$$\therefore X_{r+1} = \left[1 - \frac{1}{k}\right] X_r + \frac{n}{k} X_r^{1-k} \qquad \text{for } k = 2, 3, 4$$

$$r = 0, 1, 2, \dots$$

<u>ex</u> Compute $\sqrt[3]{7}$, using N.RM, starting with $X_0 = 1.5$, take 5D places

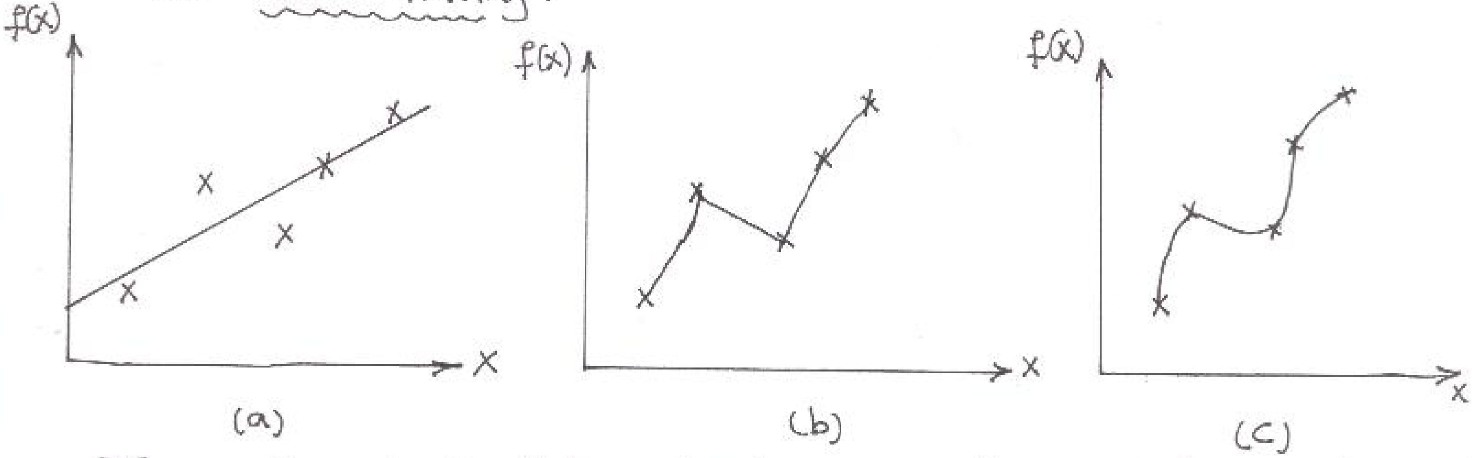
$k = 3$, $n = 7$

r	X_r	X_{r+1}
1	1.50000	2.03704
2	2.03704	1.92034
3	1.92034	1.91296
4	1.91296	1.91293

$\Rightarrow$ <u>1.91293</u>

Curves Fitting

One way to do Fit the data is to compute values of the function at a number of descrete values along the range of interest. Then, a simpler function may be derived to fit these values. Both of these applications are known as curve fitting.



Three attempts to fit a best curve through five data points
ⓐ Least - squares regression ⓑ linear interpolation ⓒ curvilinear interpolation

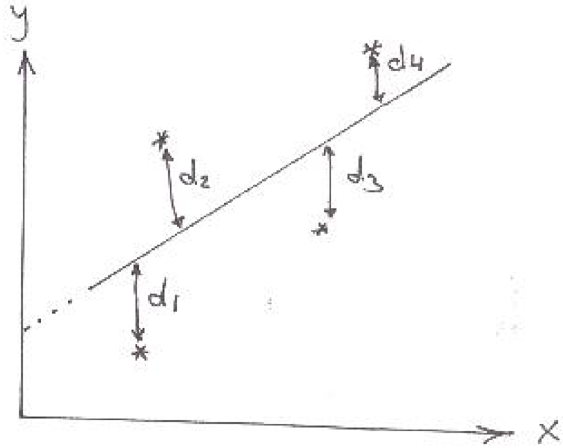
Least - Squares Regression

linear Regression

The simplest example of a least-squares approximation is fitting a straight line to a set of paired observations: $(x_1, y_1), (x_2, y_2)$ ---- (x_n, y_n) the mathmatical expression for the straight line is

$$\boxed{\bar{y} = a_0 + a_1 x}$$

Deviation $= d = y - \acute{y}$



where

$$d_1 = y_1 - \bar{y}_1 = y_1 - f(x_1)$$
$$d_2 = y_2 - \bar{y}_2 = y_2 - f(x_2)$$
$$d_3 = y_3 - \bar{y}_3 = y_3 - f(x_3)$$
$$\vdots \qquad \vdots$$
$$d_m = y_m - \bar{y}_m = y_m - f(x_m)$$

$m =$ No. of points

Now we applied to minimize the sum of the squares of the residuals between the measured y and the y calculated with the linear model

$$S = \sum_{i=1}^{m} d_i^2 = \sum_{i=1}^{m} (y_i - \bar{y}_i)^2 = \sum_{i=1}^{m} (y_i - a_0 - a_1 x_i)^2$$

to find a_0, a_1 must $\frac{\partial S}{\partial a_0} = \frac{\partial S}{\partial a_1} = 0$ minimum values

Then,

$$\frac{\partial S}{\partial a_0} = -2 \sum (y_i - a_0 - a_1 x_i) = 0$$

$$\frac{\partial S}{\partial a_1} = -2 \sum \left[(y_i - a_0 - a_1 x_i) x_i \right] = 0$$

Now, realizing that $\sum_{i=1}^{m} a_0 = m a_0$, we can express the equations as a set of two simultaneous linear equations with two unknowns (a_0 and a_1)

$$n a_0 + a_1 \cdot (\sum x_i) = (\sum y_i)$$
$$(\sum x_i) a_0 + a_1 \cdot (\sum x_i^2) = (\sum x_i y_i)$$

there are called the normal equations they can be solved as

$$a_0 = \frac{\sum y_i \cdot \sum x_i^2 - \sum x_i \cdot \sum x_i \cdot y_i}{m \sum x_i^2 - (\sum x_i)^2} \qquad a_1 = \frac{m \sum x_i y_i - \sum x_i \cdot \sum y_i}{m \sum x_i^2 - (\sum x_i)^2}$$

ex. Use linear regression to fit the following experimental data :

x :	1	3	4	6	8	9	11	14
y :	1	2	4	4	5	7	8	9

sol.

Let $\bar{y} = a_0 + a_1 x$ Then $m = 8$

i	x_i	y_i	x_i^2	$x_i \cdot y_i$
1	1	1	1	1
2	3	2	9	6
3	4	4	16	16
4	6	4	36	24
5	8	5	64	40
6	9	7	81	63
7	11	8	121	88
8	14	9	196	126
Σ	56	40	564	364

$$a_0 = \frac{\Sigma y_i \cdot \Sigma x_i^2 - \Sigma x_i \cdot \Sigma x_i y_i}{m \Sigma x_i^2 - (\Sigma x_i)^2}$$

$$\Rightarrow a_0 = \frac{40 * 524 - 56 * 364}{8 * 524 - 56^2}$$

$$\Rightarrow a_0 = 6/11$$

also

$$a_1 = \frac{m \Sigma x_i y_i - \Sigma x_i \Sigma y_i}{m \Sigma x_i^2 - (\Sigma x_i)^2}$$

$$\Rightarrow a_1 = \frac{8 * 364 - 56 * 40}{8 * 524 - 56^2}$$

$\Rightarrow a_1 = 7/11$ $\quad \Rightarrow \bar{y} = \frac{6}{11} + \frac{7}{11} x$ $\quad$ or $11\bar{y} - 7x = 6$

ex: Fit a straight line to the following data

x :	1	2	3	4	5	6	7
y :	0.5	2.5	2.0	4.0	3.5	6.0	5.5

sol: the following quantities can be computed

$m = 7$, $\Sigma x_i y_i = 119.5$, $\Sigma x_i^2 = 140$, $\Sigma x_i = 28$

$\Sigma y_i = 24$

Then $a_0 = 0.07142857$

$a_1 = 0.8392857$

Linearization of nonlinear relationships

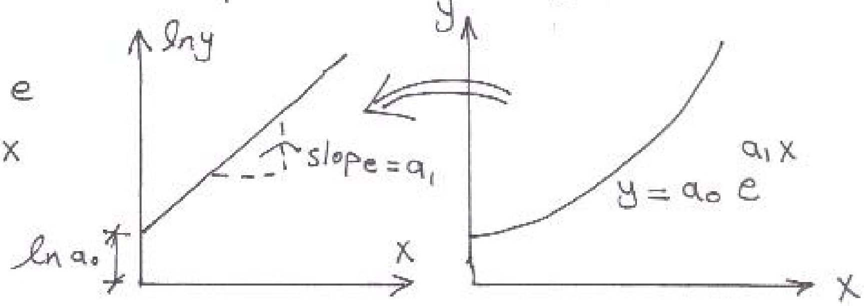
Linear regression provides a powerful technique for fitting a best line to data.

Case ① $\Rightarrow$ Exponential equation $\quad y = a_0 e^{a_1 x}$

then,

$$\ln y = \ln a_0 + a_1 x \ln e$$

$$\Rightarrow \ln y = \ln a_0 + a_1 x$$

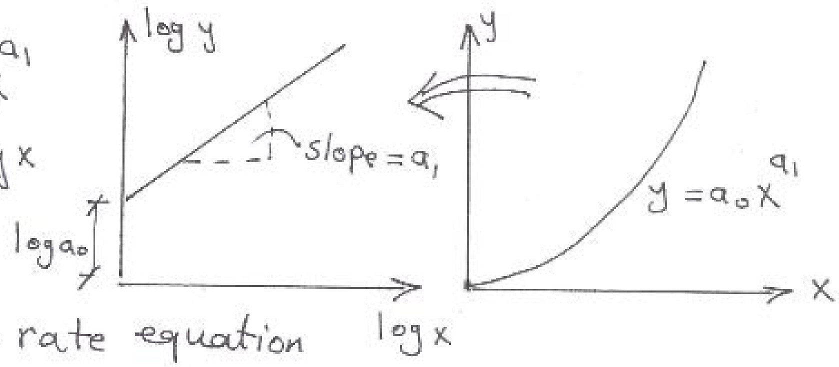


Case ② $\Rightarrow$ power equation $\quad y = a_0 x^{a_1}$

then

$$\log y = \log a_0 + \log x^{a_1}$$

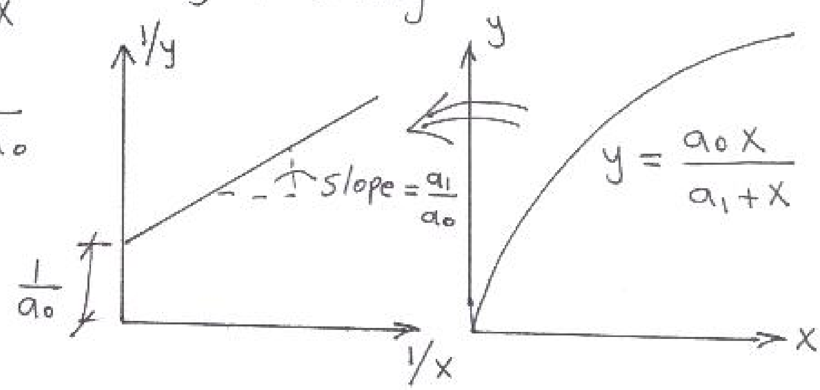
$$\log y = \log a_0 + a_1 \log x$$



Case ③ $\Rightarrow$ Growth-rate equation

$$y = a_0 \cdot \frac{x}{a_1 + x} \quad \text{by inverting}$$

$$\Rightarrow \frac{1}{y} = \frac{a_1}{a_0} \quad + \frac{1}{a_0}$$



ex: Fit the data in the following table using a logarithmic transformation of the data

x :	1	2	3	4	5
y :	0.5	1.7	3.4	5.7	8.4

sol: logarithmic transformation $\Rightarrow$ applied for power eq.

$$y = a_0 x^{a_1} \xrightarrow[\text{to}]{\text{Linearization}} \log y = \log a_0 + a_1 \log x$$

then

$$Y = a_0 + a_1 X$$

x	y	log x	log y	$(\log x)^2$	log x · log y
1	0.5	0.000	-0.301	0.00	0.00
2	1.7	0.301	0.226	0.090	0.080
3	3.4	0.477	0.534	0.227	0.254
4	5.7	0.602	0.753	0.362	0.453
5	8.4	0.699	0.922	0.488	0.644
Σ		2.079	2.134	1.167	1.431

then

$$a_0 = \frac{\Sigma \log y \cdot \Sigma \log x^2 - \Sigma \log x \cdot \Sigma \log x \cdot \log y}{m \cdot \Sigma \log x^2 - (\Sigma \log x)^2} = \frac{2.134 \times 1.167 - 2.079 \times 1.431}{5 \times 1.167 - 2.079^2}$$

$$\Rightarrow a_0 = -0.320 = \log a_0 \Rightarrow a_0 = 0.478$$

also

$$a_1 = \frac{m \cdot \Sigma \log x \cdot \log y - \Sigma \log x \cdot \Sigma \log y}{m \; \Sigma \log x^2 - (\Sigma \log x)^2} = \frac{5 \times 1.431 - 2.079 \times 2.134}{5 \times 1.167 - 2.079^2}$$

$$\Rightarrow a_1 = 1.796 = a_1$$

then $y = 0.478\, x^{1.796}$ or $\log y = -0.32 + 1.796 \log x$

Polynomial Regression

Let $\bar{y} = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

$$\Rightarrow S = \sum_{i=1}^{m} [y_i - \bar{y}] = \sum_{i=1}^{m} [y_i - a_0 - a_1 x_i - a_2 x_i^2 - \cdots - a_n x_i^n]$$

$$\Rightarrow \frac{\partial S}{\partial a_0} = -2 \sum [y_i - a_0 - a_1 x_i - a_2 x_i^2 - \cdots - a_n x_i^n] = 0$$

$$\Rightarrow \frac{\partial S}{\partial a_1} = -2 \sum [x_i (y_i - a_0 - a_1 x_i - a_2 x_i^2 - \cdots - a_n x_i^n)] = 0$$

$$\Rightarrow \frac{\partial S}{\partial a_2} = -2 \sum [x_i^2 (y_i - a_0 - a_1 x_i - a_2 x_i^2 - \cdots - a_n x_i^n)] = 0$$

$$\vdots \qquad \vdots$$

$$\frac{\partial S}{\partial a_n} = -2 \sum [x_i^n (y_i - a_0 - a_1 x_i - a_2 x_i^2 - \cdots - a_n x_i^n)] = 0$$

then

$$a_0 m + a_1 \sum x_i + a_2 \sum x_i^2 + \cdots + a_n \sum x_i^n = \sum y_i$$
$$a_0 \sum x_i + a_1 \sum x_i^2 + a_2 \sum x_i^3 + \cdots + a_n \sum x_i^{n+1} = \sum x_i y_i$$
$$a_0 \sum x_i^2 + a_1 \sum x_i^3 + a_2 \sum x_i^4 + \cdots + a_n \sum x_i^{n+2} = \sum x_i^2 \cdot y_i$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$\Rightarrow a_0 \sum x_i^n + a_1 \sum x_i^{n+1} + a_2 \sum x_i^{n+2} + \cdots + a_n \sum x_i^{2n} = \sum x_i^n \cdot y_i$$

<u>ex</u>: fit a second-order polynomial to the data in the following table:

x :	0	1	2	3	4	5
y :	2.1	7.7	13.6	27.2	40.9	61.1

<u>sol</u> $\bar{y} = a_0 + a_1 x + a_2 x^2$

then $n = 2$, $m = 6$

$\sum x_i = 15$, $\sum y_i = 152.6$, $\sum x_i^2 = 55$, $\sum x_i^3 = 225$

$\sum x_i^4 = 979$, $\sum x_i y_i = 585.6$, $\sum x_i^2 \cdot y_i = 2488.8$

$\Rightarrow$

$$\Rightarrow \quad 6a_0 + 15a_1 + 55a_2 = 152.6$$
$$15a_0 + 55a_1 + 225a_2 = 585.6$$
$$55a_0 + 225a_1 + 979a_2 = 2488.8$$

by Guass method

$$a_0 = 2.47857$$
$$a_1 = 2.35929 \qquad \Rightarrow \quad y = 2.47857 + 2.35929X + 1.8607X^2$$
$$a_2 = 1.86071$$

Dr. Sattar

(1)

Sheet No. 1

Numerical Analysis For Engineering

Q1: Find the Laplace transforms of the following function:

1. $\cosh at \cdot \cos at$
2. $\dfrac{\sinh t}{t}$
3. $\dfrac{\cos at - \cos bt}{t}$
4. $\sin(at + b)$

Q2:- Find the inverse Laplace transform of:

1. $\dfrac{3s - 8}{4s^2 + 25}$
2. $\dfrac{s^2 + 6}{(s^2+1)(s^2+4)}$
3. $\dfrac{1}{(s-2)^2} + \dfrac{1}{(s-2)^5}$
4. $\dfrac{2s + 1}{s^2 + 4s + 13}$
5. $\dfrac{54}{s^3 (s-1)}$
6. $\dfrac{2s^3 + 2s^2 + 4s + 1}{(s^2+1)(s^2+s+1)}$

Q3:- Solve by Laplace transformation method the following D.E.

1. $y'' - 3y' + 2y = 4e^{2t}$ given that $y(0) = -3$, $y'(0) = 5$
2. $x'' - x' - 2x = 20 \sin 2t$ when $x(0) = -1$, $x'(0) = 2$
3. $y'' + 2y' + y = t \cdot e^{-t}$ given that $y(0) = 1$, $y'(0) = 2$
4. $y''' + 8y = 32t^3 - 16t$ IF $y(0) = 3$, $y'(0) = y''(0) = 0$
5. $y'' - 4y' + 13y = \frac{1}{3} e^{-2t} \cdot \sin 3t$ IF $y(0) = 1$, $y'(0) = 2$
6. $y^{iv} + 2y''' + 2y'' + 2y' + y = e^{-t}$ given that $y(0) = y'(0) = y''(0) = y'''(0) = 0$

Q4: Solve in series

1. $y'' - x y' + x^2 y = 0$
2. $(1+x^2)\, y'' + x\, y' - y = 0$
3. $2x(1-x)\, y'' + (1-x)\, y' + 3y = 0$
4. $2(x^2+x^3)\, y'' - (x-3x^2)\, y' + y = 0$
5. $4x\, y'' + 2(1-x)\, y' - y = 0$
6. $y'' + \frac{1}{x} y' = \frac{y}{4x^2} - y$
7. $x^2 y'' + 6x y' - (6-x^2) y = 0$
8. $y'' - \frac{1}{x} y' + \frac{3}{4x^2} y = y$

Q5:

Using separation of variables method, solve:

1. $\dfrac{\partial^2 u}{\partial x^2} - 4 \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial x} = 0$

2. $\dfrac{\partial u}{\partial t} = 4 \cdot \dfrac{\partial^2 u}{\partial x^2}$

3. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Partial Differential Equations :-

1. Introduction
2. Wave equation
3. Heat Conduction "one-dimension unsteady"
4. Heat conduction "two-dimension steady" Laplace's equation

① Introduction

1.1 Equations of the form

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

Auxiliary equation $am^2 + bm + C = 0$ solutions depend on the roots of this equation.

ⓐ Real and different roots $m = m_1$ & $m = m_2$

solution, $y = A e^{m_1 x} + B e^{m_2 x}$ --- ①

ⓑ Real and equal roots $m = m_1 = m_2$

solution, $y = e^{m_1 x}(A + Bx)$ ----- ②

ⓒ Complex roots $m = \alpha \mp \beta j$

solution, $y = e^{\alpha x}(A \cos \beta x + B \sin \beta x)$ -- ③

1.2 Equations of the form $\frac{d^2y}{dx^2} \mp n^2 y = 0$

If we take the general equation

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + c\, y = 0$$

And consider the case when $b=0$; then dividing through by a, we have $\frac{d^2y}{dx^2} + \frac{c}{a} y = 0$ which we write as $\frac{d^2y}{dx^2} \mp n^2 y = 0$ to cover separating the two cases when $\frac{c}{a}$ is positive or negative

(a) $\frac{c}{a}$ is positive $\quad \frac{d^2y}{dx^2} + n^2 y = 0$

$\Rightarrow m^2 + n^2 = 0 \Rightarrow m^2 = -n^2 \Rightarrow m = \mp nj$

solution, $\quad y = A \cos nx + B \sin nx$ ---- (4)

(b) $\frac{c}{a}$ is negative $\quad \frac{d^2y}{dx^2} - n^2 y = 0$

$\Rightarrow m^2 - n^2 = 0 \Rightarrow m^2 = n^2 \Rightarrow m = \mp n$

solution, $\quad y = A \cosh nx + B \sinh nx$

or $\quad y = A e^{nx} + B e^{-nx}$

or $\quad y = A \sinh n(x - \phi)$

In each case, A & B are arbitrary constants depending on the initial condition

A partial differential equation is a relationship between a dependent variable U and two or more independent variables $(x, y, t, \ldots)$ and partiad differential coefficients of U with respect to these independent variables, the solution is therefore of the form

$$U = f(x, y, t, \ldots)$$

1.3 Solution by direct integration :-

The simplest form of partial differential equation is such that a solution can be determined by direct partial integration.

Example: Solve the equation $\frac{\partial^2 u}{\partial x^2} = 12x^2(t+1)$ given that at

$x = 0$; $U = \cos 2t$; $\frac{\partial u}{\partial x} = \sin t$

Sol.

$$\frac{\partial^2 u}{\partial x^2} = 12x^2(t+1) \quad \text{Integrat}$$

$$\Rightarrow \frac{\partial u}{\partial x} = 4x^3(t+1) + \phi t \quad \text{where } \phi t = \text{arbitrary function}$$
$$\theta t$$

integrat again $\Rightarrow U = x^4(t+1) + x\phi t + \theta t$

applied initial conditions that at $x = 0$

$\frac{\partial u}{\partial x} = \sin t$, $U = \cos 2t$

substituting

$$\frac{\partial u}{\partial x} = 4x^3(t+1) + \phi t \Rightarrow \sin t = 0 + \phi t \Rightarrow \phi t = \sin t$$

$$U = x^4(t+1) + x \sin t + \theta t \Rightarrow \cos 2t = 0 + 0 + \theta t \Rightarrow \theta t = \cos 2t$$

$$\therefore U = x^4(t+1) + x \sin t + \cos 2t$$

ex. Solve the equation $\frac{\partial^2 u}{\partial x \partial y} = \sin(x+y)$, given that at $y=0$; $\frac{\partial u}{\partial x} = 1$ and at $x=0$; $u = (y-1)^2$

Sol.

$$\frac{\partial^2 u}{\partial x \partial y} = \sin(x+y) \Rightarrow \frac{\partial u}{\partial x} = -\cos(x+y) + \phi x$$

at $y=0$; $\frac{\partial u}{\partial x} = 1 \Rightarrow 1 = -\cos x + \phi x \Rightarrow \phi x = 1 + \cos x$

$$\therefore \frac{\partial u}{\partial x} = -\cos(x+y) + 1 + \cos x \Rightarrow u = -\sin(x+y) + x + \sin x + \theta y$$

Put at $x=0$, $u = (y-1)^2 \Rightarrow (y-1)^2 = -\sin y + \theta y$

$\Rightarrow \theta y = (y-1)^2 + \sin y$

$$\therefore u = -\sin(x+y) + x + \sin x + \sin y + (y-1)^2$$

1.4 Initial Conditions and Boundary conditions

As with any differential equation, the arbitrary constants or arbitrary functions in any particular case are determined from the additional information given concerning the variables of equation. These extra facts are called the initial conditions or more generally, the boundary conditions since they do not always refer to zero values of the independent variables.

ex. Solve the equation $\frac{\partial^2 u}{\partial x \partial y} = \sin x \sin y$ subject to the boundary conditions

that at $y = \frac{\pi}{2}$; $\frac{\partial u}{\partial x} = 2x$

at $x = \pi$, $u = 2\sin y$

Sol. $\dfrac{\partial^2 u}{\partial x \partial y} = \sin x \sin y \Rightarrow \dfrac{\partial u}{\partial x} = \sin x \underset{\uparrow}{\overset{-\cos y}{\sin y}} + \phi x$

put at $y = \frac{\pi}{2}$; $\frac{\partial u}{\partial x} = 2x \Rightarrow 2x = \sin x \cdot \sin\frac{\pi}{2} + \phi x$

$\Rightarrow 2x = \sin x + \phi x \Rightarrow \phi x = 2x - \sin x$

$\Rightarrow \frac{\partial u}{\partial x} = \sin x \sin y + 2x - \sin x$

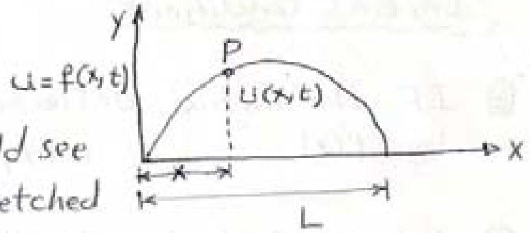
$\Rightarrow \frac{\partial u}{\partial x} = 2x + \sin x(\sin y - 1) \Rightarrow u = x^2 - \cos x(\sin y - 1) + \theta y$

put at $x = \pi$, $u = 2\sin y \Rightarrow 2\sin y = \pi^2 - \cos\pi(\sin y - 1) + \theta y$

$\Rightarrow 2\sin y = \pi^2 - 1 + \sin y + \theta y \Rightarrow \theta y = 1 - \pi^2 + \sin y$

$\therefore U = x^2 + \cos x(1 - \sin y) + \sin y + 1 - \pi^2$

② Wave Equation :-



In this equation we could see

* flexible elastic string stretched between two points at $x = 0$ and $x = L$ with uniform tension T
* The end points remaining fixed
* The string will vibrate
* Its displacements u at any time t can be expressed as $u = f(x, t)$

where x is its distance from the left-hand end the equation of motion is given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial^2 u}{\partial t^2}$$

where $c^2 = T/\rho$

T = tension in the string

ρ = mass per unit length of the string

2.1 solution of the wave equation

The wave equation is $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial^2 u}{\partial t^2}$
have a solution $u = f(x,t)$ written $u(x,t)$.

Boundary Conditions

(a) The string is fixed at both ends

i.e $\left.\begin{matrix} x = 0 \\ x = L \end{matrix}\right\}$ for all values of time $t > 0$

$u(x,t)$ becoms $\begin{matrix} u(0,t) = 0 \\ u(L,t) = 0 \end{matrix}$ for $t > 0$

Initial Conditions

(b) IF the initial deflection of P at $t=0$ is denoted by $f(x)$ $\therefore$ $u(x,0) = f(x)$ for $t=0$

(c) Let the initial velocity of P at $t=0$ is denoted by $g(x)$ $\left[\frac{\partial u}{\partial t}\right]_{t=0} = g(x)$

2.2 Solution by separation of variables :-

$u(x,t) = X_{(x)} \cdot T_{(t)}$ where $X(x)$ is a function of x only
$T(t)$ is a function of t only

$\therefore$ $u = X \cdot T$

$\therefore$ $\frac{\partial u}{\partial x} = X' \cdot T \Rightarrow \frac{\partial^2 u}{\partial x^2} = X'' \cdot T$

$\frac{\partial u}{\partial t} = X \cdot T' \Rightarrow \frac{\partial^2 u}{\partial t^2} = X \cdot T''$

The wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ can be written

$$X''.T = \frac{1}{c^2}.X.T'' \Rightarrow \boxed{\frac{X''}{X} = \frac{1}{c^2}.\frac{T''}{T}}$$

Denote this arbitrary constant by K, we have

$$\frac{X''}{X} = K \quad \text{and} \quad \frac{1}{c^2}.\frac{T''}{T} = K$$

$$\Rightarrow X'' - KX = 0 \quad , \quad T'' - c^2K.T = 0$$

Let us consider the first of these two equations for different values of K

(i) IF $K = 0$ then, $X'' = 0 \Rightarrow X' = a \Rightarrow X = ax + b$

at $x = 0$, $X = 0 \Rightarrow b = 0$

$x = L$, $X = 0 \Rightarrow a = 0$

$\therefore X = 0$ which is not oscillatory as the proplem required

(ii) IF K is positive Let $K = P^2$

$\Rightarrow X'' - KX = 0 \Rightarrow X'' - P^2X = 0$ the auxiliary equation is therefore $m^2 - P^2 = 0 \Rightarrow m^2 = P^2 \Rightarrow m = \mp P$

the solution is $X = A.e^{Px} + B.e^{-Px}$

at $x = 0 \Rightarrow X = 0 \Rightarrow 0 = A + B \Rightarrow A = -B$

$x = L \Rightarrow X = 0 \Rightarrow 0 = Ae^{PL} + Be^{-PL}$

$\Rightarrow 0 = -B(e^{PL} - e^{-PL}) \Rightarrow B = 0 = A$

Here again $X = 0$ which is oscillatory

(iii) If k is negative, let $k=-p^2$

$\therefore X'' - kX = 0 \Rightarrow X'' + p^2X = 0$, the solution is

$\Rightarrow X = A\cos px + B\sin px$ — $*$

the second equation $T'' - c^2kT = 0 \Rightarrow T'' + c^2p^2T = 0$

the solution is $T = C\cos cpt + D\sin cpt$ — $*$

$\therefore$ the general solution becoms

$$U = X \cdot T$$

$$\Rightarrow u(x,t) = [A\cos px + B\sin px][C\cos cpt + D\sin cpt]$$

if we put $cp = \lambda \Rightarrow p = \frac{\lambda}{c}$ sub. in above eq.

$$\Rightarrow u(x,t) = \left[A\cos\frac{\lambda}{c}x + B\sin\frac{\lambda}{c}x\right][C\cos\lambda t + D\sin\lambda t] \quad --- * (A)$$

where A, B, C, and D are arbitrary constants.

the results of course must be satisfy the set of boundary conditions which we now turn to.

(a) $u(0,t) = 0$
$u(L,t) = 0$ for $t > 0$

Then

at $x=0$, $u=0$ sub. in eq. (A) we get

$$u(x,t) = \left[A\cos\frac{\lambda}{c}x + B\sin\frac{\lambda}{c}x\right][C\cos\lambda t + D\sin\lambda t]$$

$$\Rightarrow 0 = [A\times 1 + B\times 0][C\cdot\cos\lambda t + D\sin\lambda t] \Rightarrow A = 0$$

sub. in eq. (A)

$$\Rightarrow u(x,t) = B\cdot\sin\frac{\lambda}{c}x\,[C\cos\lambda t + D\sin\lambda t] \quad --- \text{eq. (B)}$$

at $x = L$, $u = 0$ Then eq. (B) becoms

$$0 = B \sin \frac{\lambda L}{c} \left[C \cos \lambda t + D \sin \lambda t \right]$$

Now

$B \neq 0$, $\therefore \sin \frac{\lambda L}{c} = 0 \Rightarrow \frac{\lambda L}{c} = n\pi$ where $n = 1, 2, \ldots$

$$\Rightarrow \lambda = \frac{n \cdot c \pi}{L} \quad \text{for } n = 1, 2, 3, \ldots$$

there is an infinite set of values of λ and each separate value gives aparticular solution of $u(x,t)$
The values of λ are called the Eigen values and each corresponding solution the Eigen function
putting $n = 1, 2, 3, \ldots$ we have that solution.

	Eigen values	Eigen function
n	$\lambda = \frac{n \cdot c \cdot \pi}{L}$	$u(x,t) = B \sin \frac{\lambda x}{c} \left[C \cos \lambda t + D \sin \lambda t \right]$
1	$\lambda_1 = \frac{1 \cdot c \cdot \pi}{L}$	$u_1 = \sin \frac{\pi x}{L} \left[C_1 \cos \frac{c \pi t}{L} + D_1 \sin \frac{c \pi t}{L} \right]$
2	$\lambda_2 = \frac{2 \cdot c \cdot \pi}{L}$	$u_2 = \sin \frac{2\pi x}{L} \left[C_2 \cos \frac{2 c \pi t}{L} + D_2 \sin \frac{2 c \pi t}{L} \right]$
3	$\lambda_3 = \frac{3 \cdot c \cdot \pi}{L}$	$u_3 = \sin \frac{3\pi x}{L} \left[C_3 \cos \frac{3 c \pi t}{L} + D_3 \sin \frac{3 c \pi t}{L} \right]$
$\vdots$	$\vdots$	
r	$\lambda_r = \frac{r \cdot c \pi}{L}$	$u_r = \sin \frac{r \pi x}{L} \left[C_r \cos \frac{r c \pi t}{L} + D_r \sin \frac{r c \pi t}{L} \right]$

where $c_1, c_2, c_3, \ldots$ and $D_1, D_2, D_3, \ldots$ are arbitrary constants.

$\therefore\ u = u_1 + u_2 + u_3 + \ldots$

the more general solution is therefore

$$u(x,t) = \sum_{r=1}^{\infty} u_r$$

$$\Rightarrow\ u(x,t) = \sum_{r=1}^{\infty} \sin\frac{r\pi x}{L}\left[C_r \cos\frac{rc\pi t}{L} + D_r \sin\frac{rc\pi t}{L}\right] - *$$

We use the initial conditions

C at $t=0$; $u(x,0) = f(x)$ for $0 \leqslant x \leqslant L$

sub. in eq. $*$

$$\Rightarrow\ u(x,0) = f(x) = \sum_{r=1}^{\infty} C_r \cdot \sin\frac{r\pi x}{L}$$

D Also at $t=0$; $\left[\frac{\partial u}{\partial t}\right]_{t=0} = g(x)$ for $0 \leqslant x \leqslant L$

from eq. $*$

$$u(x,t) = \sum_{r=1}^{\infty} \sin\frac{r\pi x}{L}\left[C_r \cdot \cos\frac{rc\pi t}{L} + D_r \cdot \sin\frac{rc\pi t}{L}\right]$$

for

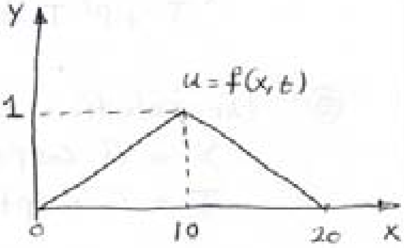
$$\frac{\partial u}{\partial t} = \sum_{r=1}^{\infty} \sin\frac{r\pi x}{L}\left[-C_r \cdot \frac{rc\pi}{L}\sin\frac{rc\pi t}{L} + D_r \cdot \frac{rc\pi}{L}\cos\frac{rc\pi t}{L}\right]$$

with $t=0$, $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$

$$\Rightarrow\ g(x) = \sum_{r=1}^{\infty} D_r \cdot \frac{rc\pi}{L}\sin\frac{r\pi x}{L}$$

$$\therefore\ g(x) = \frac{c\pi}{L}\sum_{r=1}^{\infty} D_r \cdot r \cdot \sin\frac{r\pi x}{L}$$

Example:- A stretched string of length 20 cm is set oscillating by displacing its mid-point a distance 1 cm from its rest position and releasing it with zero initial velocity. Solve the wave equation where $c^2 = 1$ to determine the resulting motion, $u(x,t)$.



Sol. ① to find the boundary conditions from the data given in the question

$$\left.\begin{aligned} u(0,t) &= 0 \\ u(20,t) &= 0 \end{aligned}\right\} \text{fixed end points}$$

$$u(x,0) = f(x) = \begin{cases} \dfrac{x}{10} & \text{for } 0 \leqslant x \leqslant 10 \\[2ex] \dfrac{20-x}{10} & \text{for } 10 \leqslant x \leqslant 20 \end{cases}$$

$$\left.\frac{\partial u}{\partial t}\right]_{t=0} = 0 \quad (\text{zero initial velocity})$$

② where $c^2 = 1 \Rightarrow$ the equation $\Rightarrow \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$

for $U = X \cdot T$

$$\therefore \frac{\partial u}{\partial x} = {}'X \cdot T \quad , \quad \frac{\partial u}{\partial t} = X \cdot {}'T$$

$$\frac{\partial^2 u}{\partial x^2} = {}''X \cdot T \quad , \quad \frac{\partial^2 u}{\partial t^2} = X \cdot {}''T$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \Rightarrow {}''X \cdot T = X \cdot {}''T$$

③ We rearrange the equation to separate the varables

$$\frac{{}''X}{X} = \frac{{}''T}{T}$$

4. Since the two sides are equal for a values of the variables, each must be equal to constant K and to give an oscillatory solution we put $K = -P^2$

$$X'' + P^2 X = 0$$
$$T'' + P^2 T = 0$$

5. the solution of the above eq. are

$$X = A \cos px + B \sin px$$
$$T = C \cos pt + D \sin pt$$

$$\therefore u(x,t) = \left[A \cos px + B \sin px\right]\left[C \cdot \cos pt + D \sin pt\right]$$

6. put $cp = \lambda$, in this case $c = 1 \Rightarrow p = \lambda$

$$u(x,t) = \left[A \cos \lambda x + B \sin \lambda x\right]\left[C \cdot \cos \lambda t + D \sin \lambda t\right]$$

7. Now we determine A & B from B.C

(i) $u(0,t) = 0$

$$u(x,t) = \left[A \cos \lambda x + B \sin \lambda x\right]\left[C \cos \lambda t + D \sin \lambda t\right]$$
$$0 = \left[A \cdot 1 + B \cdot 0\right]\left[C \cdot \cos \lambda t + D \sin \lambda t\right]$$
$$\Rightarrow A = 0$$
$$\therefore u(x,t) = B \sin \lambda x\left[C \cdot \cos \lambda t + D \cdot \sin \lambda t\right]$$

(ii) $u(20,t) = 0$

$$u(x,t) = B \sin \lambda x\left[C \cdot \cos \lambda t + D \cdot \sin \lambda t\right]$$
$$0 = B \sin 20\lambda\left[C \cdot \cos \lambda t + D \sin \lambda t\right]$$
$$\Rightarrow B \neq 0$$
$$\therefore \sin 20\lambda = 0 \Rightarrow 20\lambda = n\pi \Rightarrow \lambda = \frac{n\pi}{20}$$

$$\therefore u(x,t) = B \sin \lambda x\left[C \cdot \cos \lambda t + D \cdot \sin \lambda t\right]$$ Let $B \cdot C = G$, $B \cdot D = \varphi$

$$\therefore u(x,t) = \sin \frac{n\pi}{20} x\left[G \cos \frac{n\pi}{20} t + \varphi \sin \frac{n\pi}{20} t\right]$$

⑧ to find the eigen values and eigen functions

	Eigen values	Eigen functions
n	$\lambda = \frac{n\pi}{20}$	$u(x,t) = \sin \lambda x \left[G \cos \lambda t + \varphi \sin \lambda t \right]$
1	$\lambda_1 = \frac{\pi}{20}$	$u_1 = \sin \frac{\pi x}{20} \left[G_1 \cos \frac{\pi t}{20} + \varphi_1 \sin \frac{\pi t}{20} \right]$
2	$\lambda_2 = \frac{2\pi}{20}$	$u_2 = \sin \frac{2\pi x}{20} \left[G_2 \cos \frac{2\pi t}{20} + \varphi_2 \sin \frac{2\pi t}{20} \right]$
3	$\lambda_3 = \frac{3\pi}{20}$	$u_3 = \sin \frac{3\pi x}{20} \left[G_3 \cos \frac{3\pi t}{20} + \varphi_3 \sin \frac{3\pi t}{20} \right]$
$\vdots$	$\vdots$	$\vdots$
r	$\lambda_r = \frac{r\pi}{20}$	$u_r = \sin \frac{r\pi x}{20} \left[G_r \cdot \cos \frac{r\pi t}{20} + Q_r \cdot \sin \frac{r\pi t}{20} \right]$

where $\quad U = u_1 + u_2 + u_3 + \cdots$

$$\Rightarrow u(x,t) = \sum_{r=1}^{\infty} \sin \frac{r\pi x}{20} \left[G_r \cos \frac{r\pi t}{20} + \varphi_r \cdot \sin \frac{r\pi t}{20} \right]$$

⑨ Now we apply the remaining initial conditions

① $u(x,0) = f(x) = \frac{x}{10} \quad$ for $\quad 0 \leq x \leq 10$

$= \frac{20-x}{10} \quad$ for $\quad 10 \leq x \leq 20$

$\therefore \quad G_r = 2 \times$ mean value of $f(x) \sin \frac{r\pi x}{20}$

$$\Rightarrow G_r = \frac{2}{20} \int_0^{20} f(x) \sin \frac{r\pi x}{20}$$

$$\Rightarrow 10\, G_r = \underbrace{\int_0^{10} \frac{x}{10} \sin \frac{r\pi x}{20} dx}_{I_1} + \underbrace{\int_{10}^{20} \frac{20-x}{10} \sin \frac{r\pi x}{20} dx}_{I_2}$$

Note

$f(x) = \sum G_r \sin \lambda x \Big] \times \sin\lambda x \; dx$

$f(x) \cdot \sin \lambda x \, dx = \sum G_r \sin^2 \lambda x \, dx$

$f(x) \cdot \sin \lambda x \, dx = G_r \cdot \underbrace{\int_0^L \sin^2 \lambda x \, dx}_{\frac{L}{2}}$

$$I_1 = \int_0^{10} \frac{x}{10} \sin\frac{r\pi x}{20}\, dx \quad \text{integrating by parts}$$

$$I_1 = -\frac{20}{r\pi}\cos\frac{r\pi}{2} + \frac{40}{r^2\pi^2}\sin\frac{r\pi}{2}$$

similarly $\Rightarrow I_2 = \int_{10}^{20} \frac{20-x}{10} \sin\frac{r\pi x}{20}\, dx$

$$\Rightarrow I_2 = \frac{20}{r\pi}\cos\frac{r\pi}{2} - \frac{40}{r^2\pi^2}\sin r\pi$$

then

$$10\, G_r = -\cancel{\frac{20}{r\pi}\cos\frac{r\pi}{2}} + \frac{40}{r^2\pi^2}\sin\frac{r\pi}{2} + \cancel{\frac{20}{r\pi}\cos\frac{r\pi}{2}} - \frac{40}{r^2\pi^2}\sin r\pi$$

for $r = 1, 2, 3, \ldots \quad \Rightarrow G_r = \frac{4}{r^2\pi^2}\sin\frac{r\pi}{2}$

$$\therefore u(x,t) = \sum_{r=1}^{\infty} \sin\frac{r\pi x}{20}\left[\frac{4}{r^2\pi^2}\sin\frac{r\pi}{2} + Q_r \sin\frac{r\pi t}{20}\right]$$

(ii) Also at $t=0$; $\frac{\partial u}{\partial t} = 0$

$$\Rightarrow \frac{\partial u}{\partial t} = \sum_{r=1}^{\infty} \sin\frac{r\pi x}{20}\left[\left(\frac{4}{r^2\pi^2}\sin\frac{r\pi}{2}\right)\left(-\frac{r\pi}{20}\sin\frac{r\pi t}{20}\right) + Q_r \cdot \frac{r\pi}{20}\cos\frac{r\pi t}{20}\right]$$

at $t = 0$

$$\Rightarrow 0 = \sum_{r=1}^{\infty} \sin\frac{r\pi x}{20} \times Q_r \cdot \frac{r\pi}{20} \times 1 \quad \Rightarrow Q_r = 0$$

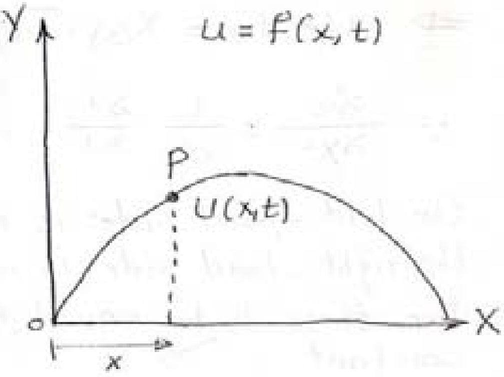
$$\therefore \boxed{u(x,t) = \frac{4}{\pi^2}\sum_{r=1}^{\infty} \frac{1}{r^2} \sin\frac{r\pi x}{20} \sin\frac{r\pi}{2} \cos\frac{r\pi t}{20}}$$

③ Heat conduction Equation for a uniform finite bar :-

the one-dimensional heat equation is then of the form

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial u}{\partial t}$$

where $c^2 = \frac{k}{\sigma \cdot \rho}$



K = thermal conductivity of the material

σ = specific heat of the material

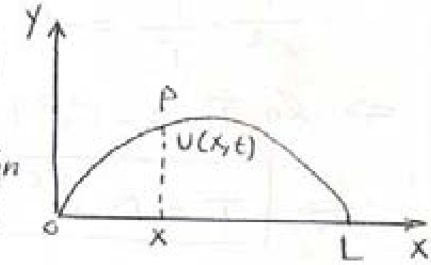
ρ = mass per unit length of the bar

3.1 Solutions of the heat conduction equation :-

ⓐ the bar extends from $x=0$ to $x=L$

ⓑ the temperature of the ends of the bar is maintained at zero

ⓒ the initial temperature distribution a long the bar is defined by $f(x)$



the boundary conditions can be expressed

$$\left.\begin{array}{l} u(0,t) = 0 \\ u(L,t) = 0 \end{array}\right\} \text{ for all } t \geq 0$$

$$u(x,0) = f(x) \quad \text{for } 0 \leq x \leq L \quad \text{at } t = 0$$

the solution of the form $u(x,t)$

$$\Rightarrow \quad u(x,t) = X_{(x)} \cdot T_{(t)} \quad \text{where} \quad \begin{array}{l} X \text{ is a function of } x \text{ only} \\ T \text{ is a function of } t \text{ only} \end{array}$$

$$\because \quad \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial u}{\partial t} \quad \Rightarrow \quad \boxed{\frac{X''}{X} = \frac{1}{c^2} \cdot \frac{T'}{T}}$$

the left - hand side is a function of x only
the right - hand side is a function of t only
for these to be equal each side must be equal the same constant

$$\therefore \quad \frac{X''}{X} = -p^2 \quad \Rightarrow X'' + p^2 X = 0 \quad \Rightarrow \text{giving the solution}$$

$$\boxed{X = A\cos px + B\sin px}$$

And

$$\frac{1}{c^2} \cdot \frac{T'}{T} = -p^2 \quad \Rightarrow \quad T' + p^2c^2\, T = 0 \quad \Rightarrow \frac{T'}{T} = -p^2c^2$$

$$\Rightarrow \quad \ln T = -p^2c^2t + c_1 \quad \Rightarrow \quad T = e^{-p^2c^2t + c_1} = e^{-p^2c^2 \cdot t} \cdot \underbrace{e^{c_1}}_{D}$$

$$\Rightarrow \quad \boxed{T = D\, e^{-p^2c^2t}}$$

$$\Rightarrow \quad u(x,t) = X \cdot T$$

$$u(x,t) = \left[A\cos px + B\sin px\right] D\, e^{-p^2c^2t}$$

$$u(x,t) = \left[G\cos px + Q\sin px\right] e^{-p^2c^2t}$$

Now put $\lambda = p.c \quad \Rightarrow p = \frac{\lambda}{c}$

$$\therefore \quad \boxed{u(x,t) = e^{-\lambda^2 t}\left[G\cos\frac{\lambda}{c}x + Q\sin\frac{\lambda}{c}x\right]}$$

Applying the boundary condition

$u(0,t)=0$

$$\Rightarrow u(x,t) = e^{-\lambda^2 t}\left[G\cos\frac{\lambda}{c}x + \varphi\sin\frac{\lambda}{c}x\right]$$

$$\Rightarrow 0 = e^{-\lambda^2 t}\left[G\cdot 1 + \varphi\cdot 0\right] \Rightarrow G = 0$$

$$\therefore \boxed{u(x,t) = \varphi\, e^{-\lambda^2 t}\sin\frac{\lambda}{c}x}$$

Also $u(L,t)=0$

$$\Rightarrow 0 = \varphi\, e^{-\lambda^2 t}\cdot\sin\frac{\lambda}{c}L \Rightarrow \varphi \neq 0$$

$$\therefore \sin\frac{\lambda}{c}L = 0 \Rightarrow \frac{\lambda}{c}L = n\pi \quad , n = 1,2,3,\cdots \Rightarrow \lambda = \frac{nc\pi}{L}$$

n	$\lambda = \frac{nc\pi}{L}$	$u(x,t) = \varphi\, e^{-\lambda^2 t}\sin\frac{n\pi x}{L}$
1	$\lambda_1 = \frac{c\pi}{L}$	$u_1 = \varphi_1 e^{-\lambda_1^2 t}\cdot\sin\frac{\pi x}{L}$
2	$\lambda_2 = \frac{2c\pi}{L}$	$u_2 = \varphi_2 e^{-\lambda_2^2 t}\cdot\sin\frac{2\pi x}{L}$
3	$\lambda_3 = \frac{3c\pi}{L}$	$u_3 = \varphi_3\cdot e^{-\lambda_3^2 t}\cdot\sin\frac{3\pi x}{L}$
$\vdots$	$\vdots$	$\vdots$
r	$\lambda_r = \frac{rc\pi}{L}$	$U_r = \varphi_r\cdot e^{-\lambda_r^2 t}\cdot\sin\frac{r\pi x}{L}$

$$\Rightarrow u = u_1 + u_2 + u_3 + \cdots$$

$$\boxed{u(x,t) = \sum_{r=1}^{\infty}\varphi_r\, e^{-\lambda_r^2 t}\sin\frac{r\pi x}{L}}$$

also apply the remaining boundary condition

$$u(x,t) = f(x) \quad \text{at} \quad t=0$$
$$u(x,0) = f(x)$$

$$\Rightarrow \quad f(x) = \sum_{r=1}^{\infty} Q_r \cdot \sin \frac{r\pi x}{L}$$

where $\quad Q_r = \frac{2}{L} \times$ mean value of $f(x) \sin \frac{r\pi x}{L}$

$$\Rightarrow \quad Q_r = \frac{2}{L} \int_0^L f(x) \sin \frac{r\pi x}{L} \, dx \quad \text{for } L=1$$

$$Q_r = 2 \int_0^1 f(x) \cdot \sin \frac{r\pi x}{1} \, dx$$

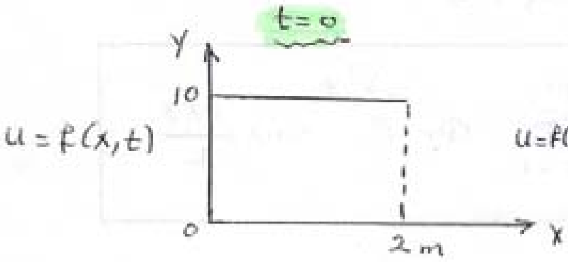
substitution Q_r in the equation $u(x,t)$ for $L=1$

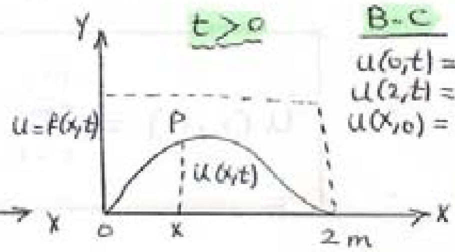
$$u(x,t) = 2 \sum_{r=1}^{\infty} \left[\int_0^1 f(x) \sin r\pi x \, .dx \right] e^{-\lambda_r^2 t} \cdot \sin r\pi x$$

where $\quad \lambda_r = \frac{rc\pi}{L} = rc\pi \quad ; r = 1, 2, 3, \ldots$

Example :- A bar of length 2 m is fully insulated along its sides. It is initially at a uniform temperature of $10°C$ at $t=0$. the ends are plunged into ice and maintained at a temperature of $0\,C°$. Determine an expression for temperature of a point P at a distance x from one end at any subsequent time t seconds after $t=0$.

Sol.





B.C
$u(0,t) = 0$
$u(2,t) = 0$
$u(x,0) = 10$

the solution is $u(x,t) = e^{-\lambda^2 t}\left[G \cos \frac{\lambda}{c} x + Q \sin \frac{\lambda}{c} x\right]$

where $X = A \cos px + B \sin px$
$T = D e^{-p^2c^2t}$ $\Rightarrow$ Let $D.A = G$, $D.B = Q$ and $pc = \lambda$; $p = \frac{\lambda}{c}$

$$\Rightarrow u(x,t) = e^{-\lambda^2 t}\left[G \cos \frac{\lambda}{c} x + Q \sin \frac{\lambda}{c} x\right]$$

Applying B.C

for $u(0,t) = 0 \Rightarrow 0 = e^{-\lambda^2 t}\left[G \times 1 + Q \times 0\right] \Rightarrow G = 0$

$$\Rightarrow u(x,t) = e^{-\lambda^2 t} \cdot Q \sin \frac{\lambda}{c} x$$

Also $u(2,t) = 0$

$$\Rightarrow 0 = e^{-\lambda^2 t} \cdot Q \sin \frac{2\lambda}{c} \Rightarrow Q \neq 0 \Rightarrow \therefore \sin \frac{2\lambda}{c} = 0$$

$$\therefore \frac{2\lambda}{c} = n\pi \Rightarrow \lambda = \frac{nc\pi}{2} \quad ; n = 1, 2, 3, \cdots$$

$$\therefore u(x,t) = e^{-\lambda^2 t} \cdot Q \cdot \sin \frac{n\pi x}{2}$$

when $t = 0$; $u(x,0) = 10 = f(x)$

$$\Rightarrow 10 = \sum_{r=1}^{\infty} Q_r \cdot \sin \frac{r\pi x}{2}$$ where $Q_r = 2 \times$ mean value of $10 \sin \frac{r\pi x}{2}$ from 0 to 2

$$\Rightarrow Q_r = \frac{2}{2}\int_0^2 10 \sin \frac{r\pi x}{2}\, dx$$

$$\Rightarrow Q_r = 10 \int_0^2 \sin \frac{r\pi x}{2}\, dx = \frac{-20}{r\pi}\left[\cos \frac{r\pi x}{2}\right]_0^2 = \frac{20}{r\pi}\left[1 - \cos r\pi\right]$$

$$\Rightarrow Q_r = \begin{cases} 0 & \text{for } r = \text{even} \\ \frac{40}{r\pi} & \text{for } r = \text{odd} \end{cases}$$

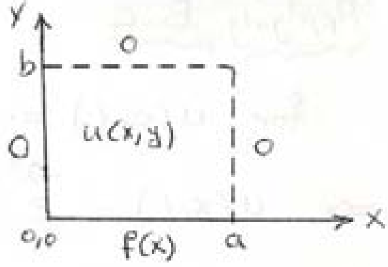
$$\Rightarrow \therefore u(x,t) = \frac{40}{\pi} \sum_{r=1,3,5,7,\cdots \text{odd}} \frac{1}{r} \sin \frac{r\pi x}{2} \cdot e^{-\lambda^2 t} \quad \text{where } \lambda = \frac{rc\pi}{2}$$

④ Heat conduction "two-dimension ; Laplace equation"

the solution of the Laplace equation two-dimension equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad u = f(x,y)$$

4.1 Solution of the Laplace eq.



to determine solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for the rectangle bounded by the lines $x=0$, $x=a$
$y=0$, $y=b$

Boundary Condition

① at $x=0$, $u=0$ for $0 \leq y \leq b$
② at $x=a$, $u=0$ for $0 \leq y \leq b$
③ at $y=b$, $u=0$ for $0 \leq x \leq a$
④ at $y=0$, $u=f(x)$ for $0 \leq x \leq a$

i.e $u(0,y)=0$, $u(a,y)=0$ for $0 \leq y \leq b$
$u(x,b)=0$, $u(x,0)=f(x)$ for $0 \leq x \leq a$

the solution $u=f(x,y)$ will give the potential at any point within the rectangle domain, we start off, as used by assuming a solution of the form

$$u(x,y) = X_{(x)} \cdot Y_{(y)} \quad \text{where}$$

X is a function of x only
Y is a function of y only

The equation in terms of X and Y is separate the variables to give

for $U = X . Y$ where $\frac{\partial u}{\partial x} = X'.Y$ & $\frac{\partial^2 u}{\partial x^2} = X''.Y$

then, $\frac{\partial u}{\partial y} = X.Y'$ & $\frac{\partial^2 u}{\partial y^2} = X.Y''$

$$X''.Y = -X.Y''$$

$$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y}$$ putting each side equal to a constant $-p^2$ gives two equations

$\Rightarrow X'' + p^2 X = 0 \Rightarrow$ has a solution $X = A\cos px + B\sin px$

$Y'' - p^2 Y = 0 \Rightarrow$ " " " $Y = C\cosh py + D\sinh py$

which can also be expressed as (For Y equation)

$$Y = E\sinh p(y+\phi)$$

$$\therefore u(x,y) = \left[A\cos px + B\sin px\right] * E\sinh p(y+\phi)$$

$$\boxed{\therefore u(x,y) = \left[G\cos px + \varphi\sin px\right] * \sinh p(y+\phi)}$$

Now we apply the first of the boundary conditions

$u(0,y) = 0$

$$\Rightarrow 0 = \left[G*1 + \varphi*0\right]\sinh p(y+\phi) \Rightarrow G = 0$$

$$\therefore u(x,y) = \varphi\sin px . \sinh p(y+\phi)$$

from the second boundary condition

$u(a,y) = 0$

$$\Rightarrow 0 = \varphi\sin pa . \sinh p(y+\phi) \Rightarrow \varphi \neq 0$$

$$\therefore \sin pa = 0 \Rightarrow pa = n\pi$$

Let $\lambda = P \Rightarrow \lambda = \frac{n\pi}{a}$

$$\therefore u(x,y) = \varphi \sin \lambda x \cdot \sinh \lambda (y + \phi)$$

from the third condition

$u(x,b) = 0$

$$\Rightarrow 0 = \varphi \sin \lambda x \cdot \sinh \lambda (b + \phi)$$

$$\Rightarrow \sinh \lambda (b + \phi) = 0 \quad \Rightarrow \phi = -b$$

$$\therefore u(x,y) = \varphi \sin \lambda x \cdot \sinh \lambda (y - b)$$

for λ_r, $u_r = u_1 + u_2 + u_3 + \cdots$

$$\Rightarrow u(x,y) = \sum_{r=1}^{\infty} \varphi_r \cdot \sin \lambda_r x \cdot \sinh \lambda_r (y - b)$$

from fourth B.C

$u(x,0) = f(x) \quad \Rightarrow f(x) = \sum_{r=1}^{\infty} - \varphi_r \cdot \sin \lambda_r x \cdot \sinh \lambda_r b$

$$\Rightarrow \varphi_r \cdot \sinh \lambda_r b = \frac{2}{a} \int_0^a - f(x) \sin \lambda x \cdot dx$$

then

$$\boxed{u(x,y) = \sum_{r=1}^{\infty} \left[\frac{-2}{a} \int_0^a f(x) \sin \lambda_r x \, dx \right] \cdot \sin \lambda_r x \cdot \frac{\sinh \lambda_r (y - b)}{\sinh \lambda_r b}}$$