Ch. 5: Solution of Simultaneous linear algebraic Equation

The following linear system of equation

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

Where a_{ij} $\begin{array}{l} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n \end{array}$ are the coefficient of n

and $x_1, x_2, \ldots, x_n$ Variable;

$b_1, b_2, \ldots, b_m$ are Constant

The above system can be written in the form:-

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \Longrightarrow A \cdot X = B$$

To solve the above system we have two ~~to~~ types of method:-

1- The direct methods:

A- The matrix Inversion method.

B- The Gauss Elimination method.

C- The Gauss-Jordan Elimination Method.

2- The Indirect Methods:-

A- Jacobi's Method.

B- Gaus-Seidel Method.

C- Relaxation Method.

A- The matrix Inversion Method :-

$\because AX = B \qquad \therefore X = A^{-1} \cdot B$

The inverse of $(A) \Rightarrow A^{-1} = \dfrac{Adj(A)}{|A|}$; $|A| \neq 0$

EX. Use the matrix inversion method to solve the following Linear equation :-

$$2X_1 + 4X_2 - 8X_3 = 6$$
$$-X_1 - 3X_2 + 6X_3 = 4$$
$$5X_1 + 7X_2 - 2X_3 = 24$$

Solution :

$$\begin{bmatrix} 2 & 4 & -8 \\ -1 & -3 & 6 \\ 5 & 7 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 24 \end{bmatrix}$$

$$|A| = 2\begin{vmatrix} -3 & 6 \\ 7 & -2 \end{vmatrix} - 4\begin{vmatrix} -1 & 6 \\ 5 & -2 \end{vmatrix} + (-8)\begin{vmatrix} -1 & -3 \\ 5 & 7 \end{vmatrix} = -24 \neq 0$$

To find A^{-1}, form the matrix $[A \mid I]$ and change it to $[I \mid B]$ as follows

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -8 & 1 & 0 & 0 \\ -1 & -3 & 6 & 0 & 1 & 0 \\ 5 & 7 & -2 & 1 & 0 & 0 \end{array}\right] \quad : New\ R_1 = \frac{R_1}{2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -4 & \frac{1}{2} & 0 & 0 \\ -1 & -3 & 6 & 0 & 1 & 0 \\ 5 & 7 & -2 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} NR_2 = R_2 + R_1 \\ NR_3 = R_3 + R_1(-5) \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -4 & \frac{-1}{2} & 0 & 0 \\ 0 & -1 & 2 & \frac{1}{2} & +1 & 0 \\ 0 & -3 & 18 & -5/2 & 0 & 1 \end{array}\right] \quad NR_2 = \frac{R_2}{-1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -4 & 1/2 & 0 & 0 \\ 0 & 1 & -2 & -\frac{1}{2} & -1 & 0 \\ 0 & -3 & 18 & -5/2 & 0 & 0 \end{array}\right] \quad \begin{array}{l} NR_1 = R_1 + R_2(-2) \\ NR_3 = R_3 + R_2(3) \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & 2 & 0 \\ 0 & 1 & -2 & -12 & -1 & 0 \\ 0 & 0 & 12 & -4 & -3 & 1 \end{array}\right] \longrightarrow NR_3 = \frac{R_3}{12}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & 2 & 0 \\ 0 & 1 & -2 & -\frac{1}{2} & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{4} & \frac{1}{12} \end{array}\right] \longrightarrow R_2 = R_2 + R_3(2) \Rightarrow I|A^{-1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & 2 & 0 \\ 0 & 1 & 0 & -\frac{7}{6} & -\frac{3}{2} & \frac{1}{6} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{4} & \frac{1}{12} \end{array}\right]$$

Now, we have the inverse matrix of A

Thus $X = A^{-1} * B$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 3/2 & 2 & 0 \\ -7/6 & -3/2 & 1/6 \\ -1/3 & -1/4 & 1/12 \end{bmatrix} * \begin{bmatrix} 6 \\ 4 \\ 24 \end{bmatrix}$$

$\therefore X_1 = \frac{3}{2} * 6 + 2 * 4 + 0 * 24 = 9 + 8 + 0 = 17$

$\therefore X_1 = 17, \quad X_2 = -9, \quad X_3 = -1$

B- Gauss Elimination Method :-

- Form the matrix $[a_{ij} | b_i]$ $\quad i = 1, 2, \dots, n$; $j = 1, 2, \dots, n$
- We will get an upper-triangular matrix.

EX. Find the solution of the following set of simultaneous equtions, using the Gauss Elimintion method work 4D

$2.37 X_1 + 3.06 X_2 - 4.28 X_3 = 1.76$

$1.46 X_1 - 0.78 X_2 + 3.75 X_3 = 4.69$

$-3.6 X_1 + 5.13 X_2 - 1.06 X_3 = 5.74$

Solution :

$$\left[\begin{array}{ccc|c} 2.37 & 3.06 & -4.28 & 1.76 \\ 1.46 & -0.78 & 3.75 & 4.69 \\ -3.69 & 5.13 & -1.06 & 5.74 \end{array}\right] \Rightarrow \begin{array}{l} New R_2 = R_2 - R_1 \frac{a_{21}}{a_{11}} \\ New R_3 = R_3 - R_1 \frac{a_{31}}{a_{11}} \end{array}$$

$$\begin{bmatrix} 2.37 & 3.06 & -4.28 & 1.76 \\ 0 & -2.6650 & 6.3865 & 3.6058 \\ 0 & 9.8944 & -5.604 & 8.4803 \end{bmatrix} \Rightarrow NR_3 = R_3 - R_2 * \frac{a_{32}}{a_{22}}$$

$$\begin{bmatrix} 2.37 & 3.06 & -4.28 & 1.76 \\ 0 & -2.665 & 6.3865 & 3.6058 \\ 0 & 0 & 18.1072 & 21.8676 \end{bmatrix}$$

$\therefore X_3 = \dfrac{21.8676}{18.1072} = 1.2077$ $\quad \therefore X_2 = (3.6058 - 6.3865 * 1.2077)/-2.66$

$\therefore X_2 = 1.5412$

$\therefore X_1 = (1.76 - 3.06 * 1.5412 - (-4.28) * 1.2077)/2.37 = 0.9337$

C- Gauss-Jordan Elimination Method :-

- Form the matrix $[A|B]$, and by same elimination steps chang the matrix to $[I|B]$.

EX. Solve the following linear equations using Gauss-Jordan method.

$$2X_1 + 3X_2 - X_3 = 1$$
$$4X_1 + 4X_2 - 3X_3 = 17$$
$$-2X_1 + 3X_2 - X_3 = -1$$

Solution

$$\begin{bmatrix} 2 & 3 & -1 & 11 \\ 4 & 4 & -3 & 17 \\ -2 & 3 & -1 & -1 \end{bmatrix} \longrightarrow New\ R_1 = \frac{R_1}{a_{11}} = \frac{R_1}{2}$$

$$\begin{bmatrix} 1 & 1.5 & 0.5 & 5.5 \\ 4 & 4 & -3 & 17 \\ -2 & 3 & -1 & -1 \end{bmatrix} \Longrightarrow \begin{matrix} NR_2 = R_2 - a_{21} R_1 = R_2 - 4R_1, \\ NR_3 = R_3 - a_{31} R_1 = R_3 + 2R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1.5 & -0.5 & 5.5 \\ 0 & -2 & -1 & -5 \\ 0 & 6 & -2 & 10 \end{bmatrix} \qquad NR_2 = \frac{R_2}{a_{22}} = \frac{R_2}{-2}$$

$$\begin{bmatrix} 1 & 1.5 & -0.5 & 5.5 \\ 0 & 1 & 0.5 & 2.5 \\ 0 & 6 & -2 & 10 \end{bmatrix} \qquad \begin{aligned} NR_1 &= R_1 - a_{12} R_2 = R_1 - 1.5 R_2 \\ NR_3 &= R_3 - a_{32} R_2 = R_3 - 6 R_2 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & -1.25 & 1.75 \\ 0 & 1 & 0.5 & 2.5 \\ 0 & 0 & -5 & -5.0 \end{bmatrix} \qquad NR_3 = \frac{R_3}{a_{33}} = \frac{R_3}{-5}$$

$$\begin{bmatrix} 1 & 0 & -1.25 & 1.75 \\ 0 & 1 & 0.5 & 2.5 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad \begin{aligned} NR_1 &= R_1 - a_{13} R_3 = R_1 + 1.25 R_3 \\ NR_2 &= R_2 - a_{23} R_3 = R_2 - 0.5 R_3 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad \therefore X_1 = 3 \;;\; X_2 = 2 \;;\; X_3 = 1$$

2- The Indirect Methods :-

In this method we have a sufficient condition for a solution to be found which is :-

$$|a_{ij}| > \sum_{\substack{i=1 \\ j \neq i}}^{n} |a_{ij}| \quad , \quad i = 1, 2, \dots, n$$

A- Jacobi's Method :-

Ex. Solve the following set of linear equation using the Jacobi's method.

$$\begin{aligned} 5X_1 - 2X_2 + X_3 &= 4 \\ X_1 + 4X_2 - 2X_3 &= 3 \\ X_1 + 2X_2 + 4X_3 &= 17 \end{aligned}$$

Solution :-

$$|5| > |-2| + |1| \Rightarrow 5 > 3$$
$$|4| > |1| + |-2| \Rightarrow 4 > 3$$
$$|4| > |1| + |2| \Rightarrow 4 > 3$$

So we have

$$X_1^{k+1} = \left(\frac{4}{5} + \frac{2}{5} X_2^{k} - \frac{1}{5} X_3^{k}\right) \quad \text{--- (1)}$$
$$X_2^{k+1} = \left(\frac{3}{4} - \frac{1}{4} X_1^{k} + \frac{1}{2} X_3^{k}\right) \quad \text{--- (2)}$$
$$X_3^{k+1} = \left(\frac{17}{4} - \frac{1}{4} X_1^{k} - \frac{1}{2} X_2^{k}\right) \quad \text{--- (3)}$$

assume $X_1^{0} = 0$; $X_2^{0} = 0$; $X_3^{0} = 0$ and substuting this values in the last three equation then we will have $X_1^{(1)}$; $X_2^{(1)}$; $X_3^{(1)}$ and so on X_1^{k} ; X_2^{k} ; $X_3^{(k)}$.

i	1	2	3	4	5	6	7	8	9	10
X_1	0.8	0.25	1.14	1.24	1.02	0.92	0.98	1.02	1.01	0.99
X_2	0.75	2.68	2.53	1.89	1.79	1.99	2.07	2.02	1.98	1.99
X_3	4.25	3.68	2.85	2.70	2.99	3.10	3.02	2.97	2.98	3.01

Accuracy : We must satisfied the accuracy condition

$$|X_i^{k+1} - X_i^{k}| < e \quad ; \quad i = 1, 2, 3, \ldots$$

B - Gauss - Seidel Method :-

EX. Solve the following set of linear equation using the Gauss - Seidel method.

$$5X_1 - 2X_2 + X_3 = 4$$
$$X_1 + 4X_2 - 2X_3 = 3$$
$$X_1 + 2X_2 + 4X_3 = 17$$

If $\lambda = 1$ it is Gauss - seidel
If $0 < \lambda < 1$ it is called under relaxation
If $1 < \lambda < 2$ it is called over relaxation

Ex.

Solve the following set of Linear equations using over relaxation with $\lambda = 1.1$

$$10X_1 + X_2 + X_3 = 12$$
$$X_1 + 10X_2 + X_3 = 12$$
$$X_1 + X_2 + 10X_3 = 12$$

we begin our solution by first checking the diagonal coefficients:

$$|10| > |1| + |1| \Rightarrow 10 > 2$$
$$|10| > |1| + |1| \Rightarrow 10 > 2$$
$$|10| > |1| + |1| \Rightarrow 10 > 2$$

So we have $X_1^{(k+1)} = 1.2 - 0.1\, X_2^{(k)} - 0.1\, X_3^{(k)}$

$$X_1^{(k+1)*} = \lambda X_1^{(k+1)} + (1-\lambda) X_1^{(k)}$$

and $X_2^{k+1} = 1.2 - (0.1)\, X_1^{(k+1)*} - (0.1)\, X_3^{(k)}$

$$X_2^{(k+1)*} = \lambda X_2^{(k+1)} + (1-\lambda) X_2^{k}$$

also $X_3^{(k+1)} = 1.2 - (0.1)\, X_1^{(k+1)*} - (0.1)\, X_2^{(k+1)*}$

$$X_3^{(k+1)*} = \lambda X_3^{(k+1)} + (1-\lambda) X_3^{(k)}$$

Now assuming an initial value of $X_2 = X_3 = 0$

so $X_1^{(1)} = 1.2$; $X_1^{(1)*} = \lambda X_1^{(1)} + (1-\lambda) X_1^{(0)}$

$$X_1^{(1)*} = (1.1)(1.2) + (1-1.1)(0) = 1.32$$

Solution : We begin our solution by checking

$|5| > |-2| + |1| \Rightarrow 5 > 3$

$|4| > |+1| + |-2| \Rightarrow 4 > 3$

$|4| > |1| + |2| \Rightarrow 4 > 3$

$$X_1^{k+1} = \frac{4}{5} + \frac{2}{5} X_2^{k} - \frac{1}{5} X_3^{k} \quad \text{--- (1)}$$

$$X_2^{k+1} = \frac{3}{4} - \frac{1}{4} X_1^{k+1} + \frac{1}{2} X^{k} \quad \text{--- (2)}$$

$$X_3^{k+1} = \frac{17}{4} - \frac{1}{4} X_1^{k+1} - \frac{1}{2} X_2^{k+1} \quad \text{--- (3)}$$

assume $X_2^{(0)} = 0$ and $X_3^{(0)} = 0$ and Sub. into eq. (1,3)

$X_1^{(1)} = 0.8 \Rightarrow$ subst. in eq. (2)

$X_2^{(1)} = 0.75 - 0.2\,(0.8) + 0 = 0.55$ Sub. in eq. (3

$X_3^{(1)} = 4.25 - 0.25\,(0.8) - 0.5\,(0.55) = 3.775$

and go on until $|X_i^{k+1} - X_i^{k}| < e$

So we will have the following values :-

i	1	2	3	4	5	6	7
X_1	0.8	0.265	1.249	0.956	1.002	1.001	0.999
X_2	0.55	2.571	1.887	2.008	2.003	1.999	2.000
X_3	3.775	2.898	2.994	3.007	3.007	3.000	3.000

C- Relaxation Method :-

After each new value of (x) is computed using Gauss-Seidel method that value is modified by

$$X_i^{(new)*} = \lambda X_i^{new} + (1-\lambda) X_i^{old}$$

where (λ) is corrected term its value $0 < \lambda < 2$

Now $X_2^{(1)} = 1.2 - (0.1)(1.32) - (0.1)(0) = 1.068$

$$X_2^{(1)*} = \lambda X_2^{(1)} + (1-\lambda) X_2^{(0)}$$

$$= (1.1)(1.068) + (1-1.1)(0) = 1.1748$$

Now $X_3^{(1)} = 1.2 - (0.1)(1.1748) - (0.1)(1.32) = 0.95052$

$$X_3^{(1)*} = \lambda X_3^{(1)} - (1-\lambda) X_3^{(0)}$$

$$= (1.1)(0.95052) + (1-1.1)(0) = 1.04572$$

Thus we get $X_1^{(1)*} = 1.32$

$X_2^{(1)*} = 1.1748$

$X_3^{(1)*} = 1.0457$

i Iteration	1	2	3	4	5
X_1	1.32	0.955	1.005	0.996	1.000
X_2	1.1748	0.9931	1.000	1.001	1.000
X_3	1.0456	1.001	0.9993	1.001	1.000

Q1 :- Solve the following system of linear equation by Gauss elimintion method :-

$X_1 - X_2 + 3X_3 = 10$

$2X_1 + 3X_2 + X_3 = 15$

$4X_1 + 2X_2 - X_3 = 6$

Ans. $X_1 = 1;\ X_2 = 3;\ X_3 = 4$

Q2 :- Solve the following system ~~of~~ linear eq. by :-

1- Gauss-Seidel ($\lambda = 1$) 2- Relaxation method ($\lambda = 1.2$)

3- Relaxation method ($\lambda = 1.7$).

① $x - 3y + 2z = 1$

$2x - 2y = k^2$ $\qquad k = \mp 2.$

$3x - 5y + z = 0$

$-2x + 8y + 4z = 49$

② $10x_1 + x_2 + 2x_3 = 44$

$2x_1 + 10x_2 + x_3 = 51$

$x_1 + 2x_2 + 10x_3 = 61$

Q_3 :- Solve the System of Linear algebraic equation G.E.M.

$x - y + z - 2w = -1$

$-2x + 2y - z + 2w = 3$

$3x - 3y + 2z - 4w = -4$

$-4x + 4y - 3z + 6w = 5$

Q_4 :- Solve the following system of equation by Gauss - Seidel iteration method working to $e = 0.0001$

$10.27\,A_1 - 1.23\,A_2 + 0.67\,A_3 = 4.27$

$2.39\,A_1 - 12.65\,A_2 + 1.13\,A_3 = 1.26$

$1.79\,A_1 + 3.61\,A_2 + 15.11\,A_3 = 12.71$

Ans. $A_1 = 0.3693$

$A_2 = 0.0405$

$A_3 = 0.7878$

Numerical Differentiation and Integration :-

1- Numerical differentiation :-

A- Newton-Forward ($h = const.$)

$\because \; y = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$

and $\; u = \frac{x - x_0}{h}$ Then $\Delta u = \frac{\Delta x}{h} \Rightarrow \frac{du}{dx} = \frac{1}{h}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \xrightarrow{\text{Putting}} \frac{du}{dx} = \frac{1}{h} \quad \text{Then}$$

$$\frac{dy}{dx} = \frac{1}{h} \cdot \frac{dy}{du} \quad \text{--- (1)}$$

$+ \frac{u^4 - 6u^3 + 11u^2 - 6u}{\ldots}$

$$y = y_0 + u\,\Delta y_0 + \frac{u^2 - u}{2!}\Delta^2 y_0 + \frac{(u^3 - 3u^2 + 2u)}{3!}\Delta^3 y_0 + \cdots$$

$$\therefore \frac{dy}{du} = 0 + \Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!}\Delta^3 y_0 + \cdots \quad (2)$$

$+$

Sub 2 in 1

$$y'_{(x)} = \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!}\Delta^3 y_0\right] \quad \text{--- (3)}$$

and $\; \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{dy}{dx}\right) \cdot \frac{du}{dx}$

$$y''_{(x)} = \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \cdots\right] \quad \text{--- (4)}$$

Special cases : when $x = x_0$ Then $u = \frac{x_0 - x_0}{h} = 0$

$$y'(x_0) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2!}\Delta^2 y_0 + \frac{2}{3!}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right] \quad \text{--- (3)'}$$

$$y''(x_0) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right] \quad \text{--- (4)'}$$

Ex. Find the value of the derivative for the following function at $x = 2.3$; $x = x_0$

$y = f(x) = x^4 - \ln x$ when $x = 1\,(0.5)\,3.5$

Sol.

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
1.0	1.0			
		3.657		
1.5	4.657		6.993	
		10.650		5.196
2.0	15.307		12.189	
		22.839		6.727
2.5	38.146		18.916	
		41.755		8.238
3.0	79.901		27.154	
		68.909		
3.5	148.810			

2.3

$$u = \frac{x - x_0}{h} = \frac{2.3 - 1}{0.5} = 2.6$$

$$y'(2.3) = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!}\Delta^3 y_0\right]$$

$$= \frac{1}{0.5}\left[3.657 + \frac{2(2.6-1)}{2\times 1}(6.993) + \frac{3(2.6)^2 - 6(2.6) + 2}{3\times 2\times 1} 5.19\right]$$

$$\therefore y'(2.3) = 48.255$$

at $x = x_0 \Rightarrow u = \frac{x_0 - x_0}{h} = 0$

$$y'(x_0) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2!}\Delta^2 y_0 + \frac{2}{3!}\Delta^3 y_0\right]$$

$$y'(x_0) = \frac{1}{0.5}\left[3.657 - \frac{1}{2\times 1}(6.993) + \frac{2}{3!}(5.196)\right] = 3.785$$

B- Newton - Backward :- $(h = c)$

$$\therefore y = y_n + u\,\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n$$

and $u = \frac{x - x_n}{h} \Rightarrow \frac{du}{dx} = \frac{1}{h}$

Putting

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{h}\cdot\frac{dy}{du} \quad \text{—— (C)}$$

$$y = y_n + u\nabla y_n + \frac{u^2+u}{2!}\nabla^2 y_n + \frac{u^3+3u^2+2u}{3!}\nabla^3 y_n$$

$$\frac{dy}{du} = 0 + \nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2+6u+2}{3!}\nabla^3 y_n \quad \text{--- (2)}$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2+6u+2}{3!}\nabla^3 y_n\right] \quad \text{--- (3)}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{dy}{dx}\right)\cdot\frac{du}{dx} \quad \text{--- (4)}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (u+1)\nabla^3 y_n + \frac{6u^2+18u+11}{12}\nabla^4 y_n + \right]$$

Special cases: when $x = x_n$ then $u = \frac{x_n - x_n}{h} = 0$

$$y'(x_n) = \frac{1}{h}\left[\nabla y_n + \frac{1}{2!}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right] \quad \text{--- (3)}'$$

$$y''(x_n) = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right] \quad \text{--- (4)}'$$

Ex. By using Newton BackWard formula find the value of $x = 2.5$ use the table

x	0	1	2	3	4
y	-8	-7	0	19	56

Sol.

x_i	y_i	∇y_n	$\nabla^2 y_n$	$\nabla^3 y_n$	$\nabla^4 y_n$
0	-8				
		1			
1	-7		6		
		7		6	
2	0		12		0
		19		6	
3	19		18		
		37			
4	56				

$$u = \frac{x - x_n}{h} = \frac{2.5 - 4}{1} = -1.5 \qquad ; h = x_{i+1} - x_i$$

$$\therefore y'(x) = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2}\nabla^2 y_n + \frac{3u^2+6u+2}{6}\nabla^3 y_n\right]$$

$$= 18.75$$

H.W. Find (y'', y''').

c-($h \neq c$) → Numerical differentiation - Lagrang formula:-

$$y(x) = L_0 y_0 + L_1 y_1 + L_2 y_2 + \cdots + L_n y_n$$

$$L_0 = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{x^2 - x_1 x - x_2 x - x_1 x_2}{(x_0-x_1)(x_0-x_2)}$$

$$\frac{dL_0}{dx} = \frac{2x - x_1 - x_2}{(x_0-x_1)(x_0-x_2)}.$$

$$L_1 = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{x^2 - x_2 x - x_0 x - x_0 x_2}{(x_1-x_0)(x_1-x_2)}$$

$$\frac{dL_1}{dx} = \frac{2x - x_2 - x_0}{(x_1-x_0)(x_1-x_2)}$$

$$\therefore \frac{dy}{dx} = \frac{dL_0}{dx}y_0 + \frac{dL_1}{dx}y_1 + \frac{dL_2}{dx}y_2 + \cdots + \frac{dL_n}{dx}y_n$$

Ex. Find the first derivative of the function tabulated below at the point (1.3)

x	1.2	1.5	1.7
y	0.1823	0.4055	0.5306

Sol. $\frac{dy}{dx} = \frac{dL_0}{dx} y_0 + \frac{dL_1}{dx} y_1 + \frac{dL_2}{dx} y_2$

$$\frac{dL_0}{dx} = \frac{2X - X_1 - X_2}{(X_0 - X_1)(X_0 - X_2)} = \frac{2(1.3) - 1.5 - 1.7}{(-0.3)(-0.5)}$$

$$\frac{dL_1}{dx} = \frac{2X - X_2 - X_0}{(X_1 - X_0)(X_1 - X_2)} = \frac{2(1.3) - 1.7 - 1.2}{(0.3)(-0.2)}$$

$$\frac{dL_2}{dx} = \frac{2X - X_0 - X_1}{(X_2 - X_0)(X_2 - X_1)} = \frac{2(1.3) - 1.2 - 1.5}{(0.5)(0.2)}$$

$$\therefore \frac{dy}{dx} =$$

Ex. Find the exact and approximate value of the derivative for the following function at $X = 2.3$ and $X = X_0$; $y = f(x) = X^4 - \ln X$; $X = 1(0.5)3.5$

Sol.

X	1.0	1.5	2.0	2.5	3.0	3.5
f(x)	1.0	4.657	15.307	38.146	79.901	148.810

$\swarrow 2.3$

X	f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.0	1.0				
		3.657			
1.5	4.657		6.993		
		10.650		5.196	
2.0	15.307		12.189		1.531
		22.839		6.727	
2.5	38.146		18.916		1.511
		41.755		8.238	
3.0	79.901		27.15		
		68.909			
3.5	148.810				

$$U_{2.3} = \frac{2.3 - 1}{0.5} = 2.6$$

$$y' = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0\right]$$

$$y'_{(2.3)} = \frac{1}{0.5}\left[3.657 + \frac{6.993}{2}(2*2.6) + \frac{5.196}{6}(3*(2.6)^2 - 6*2.6 + 2)\right]$$

$$= \frac{1}{0.5}\left[3.657 + 14.685 + 5.785\right]$$

$$= 48.255 \quad \text{(approximate value)}$$

and $y' = 4x^3 - \frac{1}{x}$ when $x = 2.3$ then

$$y' = 4(2.3)^3 - \frac{1}{2.3} = 48.233 \quad \text{exact value.}$$

Special cases: $x = x_0$

$$y'(x_0) = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4}\right]$$

$$y'(1) = \frac{1}{0.5}\left[3.657 - \frac{6.993}{2} + \frac{5.196}{3}\right]$$

$$= 3.785 \Rightarrow$$ لثلاث حدود

$$3.785 - \frac{1.531}{4} = 0.382$$

$$\begin{array}{r} 3.785 \\ -\ 0.382 \\ \hline 3.403 \end{array}$$

$$y'_{(1)} = 4 - 1 \simeq 3.403$$ لأربع حدود

$$\text{true relative error} = \frac{48.233 - 48.255}{48.233} * 100\%$$

ملاحظة: هنالك تكون لحسابات تسمى بالخطأ النسبي الحقيقي او الخطأ النسبي الصحيح وهو ان

الخطأ النسبي الحقيقي (الصحيح) = $\frac{\text{القيمة الحقيقية} - \text{القيمة التقريبية}}{\text{القيمة الحقيقية}} * 100\%$

$$\text{true relative error} = \frac{\text{Exact value} - \text{approximate value}}{\text{Exact value}} * 100\%$$

2- Numerical Integration

A- Trapezoidal Rule ($h=c$)

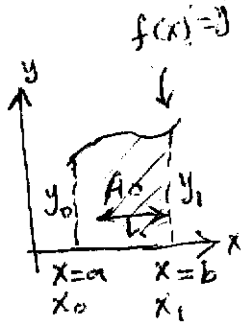


Fig.(1)

$$\text{Integ.} = I = \int_a^b f(x)\,dx \quad \text{and} \quad \underset{\text{step size}}{h} = \frac{b-a \rightarrow \text{Interval}}{N \rightarrow \text{number of Areas (Panels)}}$$

$$A_0 = \frac{1}{2} h \,(y_0 + y_1) \quad \text{for Fig. 1}$$

$$\int_a^b f(x)\,dx = \text{Total Area} = A_0 + A_1 + A_2 + \dots + A_n \quad (\text{Fig-2})$$

$$= \frac{1}{2} h (y_0 + y_1) + \frac{1}{2} h (y_1 + y_2) + \dots + \frac{1}{2} h (y_{n-1} + y_n)$$

$$\therefore \int_a^b f(x)\,dx = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n) = \boxed{\frac{h}{2}\{y_0 + 2(y_1 + \dots + y_{n-1}) + y_n\}}$$

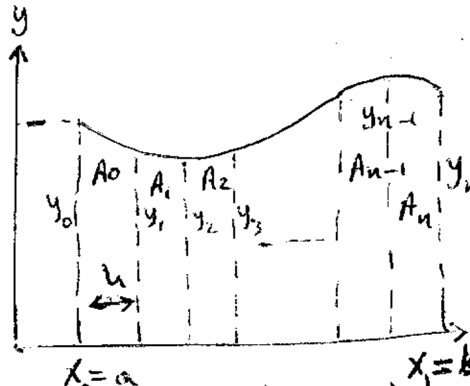
Ex. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ to (4D) by trapezoidal rul where the interval ($0 \rightarrow 1$) is sub-divided into (6) equal parts.

<u>Sol.</u>

$$\int_a^b f(x)\,dx = \frac{h}{2}[y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n]$$

$\because N = 6$

$$\therefore h = \frac{b-a}{N} = \frac{b-a}{N} = \frac{1-0}{6} = \frac{1}{6} \text{ step size}$$



(Fig-2)

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1.0
$f(x) = \frac{1}{1+x^2}$	1	0.9729	0.9	0.8	0.6923	0.5901	0.5
	y_0	y_1	y_2	—	—	—	

$$\therefore \int_0^1 \frac{dx}{1+x^2} = \frac{1/6}{2}\left[1 + 2\{0.9729 + 0.9 + 0.8 + 0.6923 + 0.5901\} + 0.5\right]$$

$$= 0.7842$$

D- Trapezoidal Rule ($h \neq c$)

$$I_{total} = I_T = h_1 \frac{f(x_1)+f(x_0)}{2} + h_2 \frac{f(x_2)+f(x_1)}{2} + \cdots + h_n \frac{f(x_n)+f(x_{n-1})}{2}$$

Ex. Use trapezoidal rule to determine the integral of data in the following table:-

	h_1	h_2	h_3								
x	0	0.12	0.22	0.32	0.36	0.4	0.44	0.54	0.64	0.7	0.8
$f(x)$	0.2	1.310	1.305	1.443	1.575	2.156	2.243	2.507	2.482	2.363	2.232

Sol.

$$I_T = 0.12 * \frac{1.310+0.2}{2} + 0.1 * \frac{1.305+1.310}{2} + 0.1 * \frac{1.4433+1.305}{2}$$
$$+ 0.04 * \frac{1.575+1.443}{2} + 0.04 * \frac{2.156+1.575}{2} + 0.04 * \frac{2.243+2.156}{2}$$
$$+ 0.1 * \frac{2.507+2.243}{2} + 0.1 * \frac{2.482+2.507}{2} + 0.06 * \frac{2.363+2.482}{2}$$
$$+ 0.1 * \frac{2.232+2.363}{2}$$
$$= 0.0905 + 0.1308 + 0.1374 + 0.0604 + 0.0746 + 0.088$$
$$+ 0.2375 + 0.2495 + 0.1454 + 0.2298$$
$$= 1.4438.$$

E- Multiple Integrals:-

$$\iint_A f(x,y)\,dA = \int_a^b \left\{ \int_c^d f(x,y)\,dy \right\} dx = \int_c^d \left\{ \int_a^b f(x,y)\,dx \right\} dy$$

$h_x = \frac{b-a}{N}$ step size for x ; $x \rightarrow (a-b)$;

$h_y = \frac{d-c}{N}$ step size for y : $y \rightarrow (c \rightarrow d)$;

EX. Evaluate the $\int_0^1 \int_0^2 (x^2+2xy+y^2)\,dy\,dx$ when step size of $x = 0.2$ and step size of $y = 0.5$?

Ex. Evaluate the double Integral $\int_{0.2}^{0.6}\int_{1.5}^{3.0} f(x,y)\,dx\,dy$
Use trapezoidal rule in x-direction and simpson rule in y-direction, and f(x,y) as a given in the following table :-

X \ y	0.1	0.2	0.3	0.4	0.5	0.6
0.5	0.165	0.428	0.687	0.942	1.190	1.431
1.0	0.271	0.640	1.0030	1.359	1.703	2.035
1.5	0.447	0.990	1.524	2.045	2.549	3.031
2.0	0.738	1.568	2.384	3.177	3.943	4.672
2.5	1.216	2.520	3.800	5.004	6.241	7.379
3.0	2.005	4.090	6.139	8.122	10.030	11.841
3.5	3.306	6.679	9.986	13.196	16.277	19.198

Sol. Interval $X = (1.5 \to 3.0)$ and $y = (0.2 \to 0.6)$

If $y = 0.2$ = constant Then $\therefore h_x = x_{i+1} - x_i = 2 - 1.5 = 0.5$

$$I_x = \int_{1.5}^{3.0} f(x,y)\,dx = \int_{1.5}^{3.0} f(x,0.2)\,dx = \frac{h_x}{2}[y_1 + 2(y_2 + y_3) + y_4]$$

and :-

$$\therefore I_{x_0}(y=0.2) = \frac{0.5}{2}[0.99 + 2(1.568 + 2.52) + 4.090] = 3.3140$$

$$I_{x_1}(y=0.3) = \frac{0.5}{2}[1.524 + 2(2.384 + 3.8) + 6.136] = 5.0070$$

$I_{x_2}(y=0.4) = 6.6522$; $I_{x_3}(y=0.5) = 8.2368$

$I_{x_4}(y=0.6) = 9.7435$

Then $I_{x_0} = y_0^* = 3.314$; $I_{x_1} = y_1^* = 5.007$; $I_{x_2} = y_2^* = 6.6522$

$I_{x_3} = y_3^* = 8.2368$; $I_{x_4} = y_4^* = 9.7435$

$\because h_y = y_{i+1} - y_i = 0.3 - 0.2 = 0.1$

$$\therefore \int_{0.2}^{0.6} f(x,y)\,dy = \frac{h_y}{3}[y_0^* + 4(y_1^* + y_3^*) + 2(y_2^*) + y_4^*] \rightarrow \text{when } X = \text{const}$$

$$= \frac{0.1}{3}[3.3140 + 4(5.007 + 8.2368) + 2(6.6522) + 9.7435]$$

$$= 2.6446$$

ملاحظة: تكون y = cons اذا اخذ hx وتطلع القيمة بالسطر

sheet

1- By using Simpson's $\frac{1}{3}$ rule solve the following $\int_0^{\pi} \sin x\, dx$; N = 6 and (2D). Then find the relative error
Ans. ($I \simeq 2.00$; t.r.e = 0.00%).

2- Evaluate the value of the next integration by using the trapezoidal rule ; Then find the absolute relative error $\int_0^{3\pi/20} [\sin(5x+1)]\,dx$; N=4 ; 4D
Ans. [$I \simeq 0.295$; $I = 0.303$; a.r.e = 2.640%].

3. By using Newton forward formula, evaluate the value of the derivative of the following equation at x = 6.6, then find the true relative error:
$f(x) = -46 + 45.4x - 13.8x^2 + 1.71x^3 - 0.0729x^4$
$x = 2(2)10$; 3D.
Ans. : $y'_{(6.6)} = 1.730$; $y''_{(6.6)} =$; t.r.e = 37.182 %

4. Evaluat the following using Simpson's $\frac{1}{3}$ rule ; h = 1
A- $\int_0^3 x^2 dx$; Ans.(8.667) B- $\int_1^5 \frac{x}{\sin x} dx$; Ans. (8.7159).

5. Find the integral value of the function below by using trapezoidal rule ; $y = x^3 + x^2 - 5$ (a=1, b=5 ; N=8).

6- Find $[y'_{(0.7)}]$ by using Newton backward formula and the table below

x	0.0	0.2	0.4	0.6	0.80	1.0
y	0.0	0.12	0.48	1.16	2.00	3.20

7- A rod is rotating in a plane about on of its ends if the following table given the angle (A) radians through which the rod has turend for different values of time (t) seconds, find its angular accelaration when (t=0.7 sec)

t sec	0.0	0.2	0.4	0.6	0.8	1.0
A rad	0.0	0.12	0.48	1.1	2.0	3.2

53

Numerical integration

53.1 Introduction

Even with advanced methods of integration there are many mathematical functions which cannot be integrated by analytical methods and thus approximate methods have then to be used. Approximate methods of definite integrals may be determined by what is termed **numerical integration**.

It may be shown that determining the value of a definite integral is, in fact, finding the area between a curve, the horizontal axis and the specified ordinates. Three methods of finding approximate areas under curves are the trapezoidal rule, the mid-ordinate rule and Simpson's rule, and these rules are used as a basis for numerical integration.

53.2 The trapezoidal rule

Let a required definite integral be denoted by $\int_a^b y\,dx$ and be represented by the area under the graph of $y = f(x)$ between the limits $x = a$ and $x = b$ as shown in Fig. 53.1.

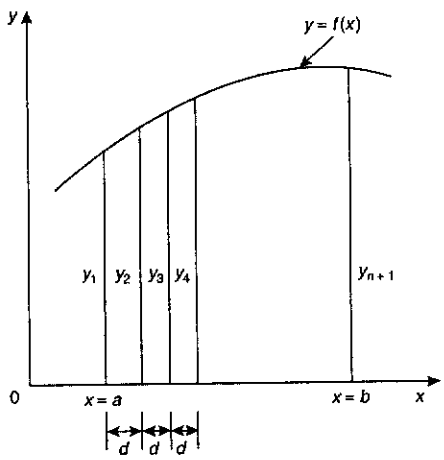


Figure 53.1

Let the range of integration be divided into n equal intervals each of width d, such that $nd = b-a$, i.e. $d = \dfrac{b-a}{n}$

The ordinates are labelled $y_1, y_2, y_3, \ldots y_{n+1}$ as shown.

An approximation to the area under the curve may be determined by joining the tops of the ordinates by straight lines. Each interval is thus a trapezium, and since the area of a trapezium is given by:

area $= \dfrac{1}{2}$(sum of parallel sides) (perpendicular distance between them) then

$$\int_a^b y\,dx \approx \frac{1}{2}(y_1+y_2)d + \frac{1}{2}(y_2+y_3)d + \frac{1}{2}(y_3+y_4)d + \cdots \frac{1}{2}(y_n+y_{n+1})d$$

$$\approx d\left[\frac{1}{2}y_1 + y_2 + y_3 + y_4 + \cdots + y_n + \frac{1}{2}y_{n+1}\right]$$

i.e. **the trapezoidal rule states:**

$$\boxed{\int_a^b y\,dx \approx \begin{pmatrix}\text{width of}\\ \text{interval}\end{pmatrix}\left\{\frac{1}{2}\begin{pmatrix}\text{first + last}\\ \text{ordinate}\end{pmatrix} + \begin{pmatrix}\text{sum of}\\ \text{remaining}\\ \text{ordinates}\end{pmatrix}\right\}} \quad (1)$$

> Problem 1. (a) Use integration to evaluate, correct to 3 decimal places, $\displaystyle\int_1^3 \frac{2}{\sqrt{x}}\,dx$
> (b) Use the trapezoidal rule with 4 intervals to evaluate the integral in part (a), correct to 3 decimal places

(a) $\displaystyle\int_1^3 \frac{2}{\sqrt{x}}\,dx = \int_1^3 2x^{-\frac{1}{2}}\,dx$

$$= \left[\frac{2x^{\left(\frac{-1}{2}\right)+1}}{-\frac{1}{2}+1}\right]_1^3 = \left[4x^{\frac{1}{2}}\right]_1^3$$

$$= 4\left[\sqrt{x}\right]_1^3 = 4\left[\sqrt{3}-\sqrt{1}\right]$$

$= \mathbf{2.928}$, correct to 3 decimal places.

(b) The range of integration is the difference between the upper and lower limits, i.e. $3-1 = 2$. Using the trapezoidal rule with 4 intervals gives an interval width $d = \frac{3-1}{4} = 0.5$ and ordinates situated at 1.0, 1.5, 2.0, 2.5 and 3.0. Corresponding values of $\frac{2}{\sqrt{x}}$ are shown in the table below, each correct to 4 decimal places (which is one more decimal place than required in the problem).

x	$\frac{2}{\sqrt{x}}$
1.0	2.0000
1.5	1.6330
2.0	1.4142
2.5	1.2649
3.0	1.1547

From equation (1):

$$\int_1^3 \frac{2}{\sqrt{x}}\,dx \approx (0.5)\left\{\frac{1}{2}(2.0000+1.1547)\right.$$
$$\left. + 1.6330 + 1.4142 + 1.2649\right\}$$
$= \mathbf{2.945}$, correct to 3 decimal places.

This problem demonstrates that even with just 4 intervals a close approximation to the true value of 2.928 (correct to 3 decimal places) is obtained using the trapezoidal rule.

Problem 2. Use the trapezoidal rule with 8 intervals to evaluate $\int_1^3 \frac{2}{\sqrt{x}}\,dx$, correct to 3 decimal places

With 8 intervals, the width of each is $\frac{3-1}{8}$ i.e. 0.25 giving ordinates at 1.00, 1.25, 1.50, 1.75, 2.00, 2.25, 2.50, 2.75 and 3.00. Corresponding values of $\frac{2}{\sqrt{x}}$ are shown in the table below:

x	$\frac{2}{\sqrt{x}}$
1.00	2.0000
1.25	1.7889
1.50	1.6330
1.75	1.5119
2.00	1.4142
2.25	1.3333
2.50	1.2649
2.75	1.2060
3.00	1.1547

From equation (1):

$$\int_1^3 \frac{2}{\sqrt{x}}\,dx \approx (0.25)\left\{\frac{1}{2}(2.000+1.1547)+1.7889\right.$$
$$+ 1.6330 + 1.5119 + 1.4142$$
$$\left. + 1.3333 + 1.2649 + 1.2060\right\}$$
$= \mathbf{2.932}$, correct to 3 decimal places

This problem demonstrates that the greater the number of intervals chosen (i.e. the smaller the interval width) the more accurate will be the value of the definite integral. The exact value is found when the number of intervals is infinite, which is what the process of integration is based upon.

Problem 3. Use the trapezoidal rule to evaluate $\int_0^{\pi/2} \frac{1}{1+\sin x}\,dx$ using 6 intervals. Give the answer correct to 4 significant figures

With 6 intervals, each will have a width of $\frac{\frac{\pi}{2}-0}{6}$, i.e. $\frac{\pi}{12}$ rad (or 15°) and the ordinates occur at 0, $\frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{12}$ and $\frac{\pi}{2}$. Corresponding values of

$\dfrac{1}{1+\sin x}$ are shown in the table below:

x	$\dfrac{1}{1+\sin x}$
0	1.0000
$\dfrac{\pi}{12}$ (or 15°)	0.79440
$\dfrac{\pi}{6}$ (or 30°)	0.66667
$\dfrac{\pi}{4}$ (or 45°)	0.58579
$\dfrac{\pi}{3}$ (or 60°)	0.53590
$\dfrac{5\pi}{12}$ (or 75°)	0.50867
$\dfrac{\pi}{2}$ (or 90°)	0.50000

From equation (1):

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx \approx \left(\frac{\pi}{12}\right)\left\{\frac{1}{2}(1.00000 + 0.50000) + 0.79440 + 0.66667 + 0.58579 + 0.53590 + 0.50867\right\}$$

$$= \mathbf{1.006}, \text{ correct to 4 significant figures}$$

Now try the following exercise

Exercise 180 Further problems on the trapezoidal rule

Evaluate the following definite integrals using the **trapezoidal rule**, giving the answers correct to 3 decimal places:

1. $\displaystyle\int_0^1 \frac{2}{1+x^2}\,dx$ (Use 8 intervals) [1.569]

2. $\displaystyle\int_1^3 2\ln 3x\,dx$ (Use 8 intervals) [6.979]

3. $\displaystyle\int_0^{\pi/3} \sqrt{\sin\theta}\,d\theta$ (Use 6 intervals) [0.672]

4. $\displaystyle\int_0^{1.4} e^{-x^2}\,dx$ (Use 7 intervals) [0.843]

53.3 The mid-ordinate rule

Let a required definite integral be denoted again by $\int_a^b y\,dx$ and represented by the area under the graph of $y = f(x)$ between the limits $x = a$ and $x = b$, as shown in Fig. 53.2.

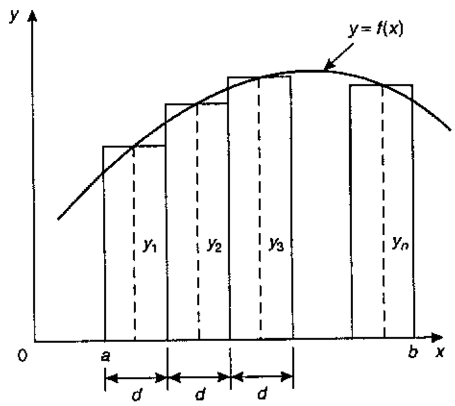


Figure 53.2

With the mid-ordinate rule each interval of width d is assumed to be replaced by a rectangle of height equal to the ordinate at the middle point of each interval, shown as $y_1, y_2, y_3, \ldots y_n$ in Fig. 53.2.

$$\text{Thus} \quad \int_a^b y\,dx \approx d\,y_1 + d\,y_2 + d\,y_3 + \cdots + d\,y_n$$

$$\approx d(y_1 + y_2 + y_3 + \cdots + y_n)$$

i.e. **the mid-ordinate rule states:**

$$\boxed{\int_a^b y\,dx \approx \begin{pmatrix}\textbf{width of}\\ \textbf{interval}\end{pmatrix}\begin{pmatrix}\textbf{sum of}\\ \textbf{mid-ordinates}\end{pmatrix}} \qquad (2)$$

Problem 4. Use the mid-ordinate rule with (a) 4 intervals, (b) 8 intervals, to evaluate $\int_1^3 \frac{2}{\sqrt{x}}\,dx$, correct to 3 decimal places

(a) With 4 intervals, each will have a width of $\frac{3-1}{4}$, i.e. 0.5 and the ordinates will occur at 1.0, 1.5, 2.0, 2.5 and 3.0. Hence the mid-ordinates y_1, y_2, y_3 and y_4 occur at 1.25, 1.75, 2.25 and 2.75

Corresponding values of $\frac{2}{\sqrt{x}}$ are shown in the following table:

x	$\frac{2}{\sqrt{x}}$
1.25	1.7889
1.75	1.5119
2.25	1.3333
2.75	1.2060

From equation (2):

$$\int_1^3 \frac{2}{\sqrt{x}}\,dx \approx (0.5)[1.7889 + 1.5119 + 1.3333 + 1.2060]$$

$$= \mathbf{2.920}, \text{ correct to 3 decimal places}$$

(b) With 8 intervals, each will have a width of 0.25 and the ordinates will occur at 1.00, 1.25, 1.50, 1.75, ... and thus mid-ordinates at 1.125, 1.375, 1.625, 1.875 .... Corresponding values of $\frac{2}{\sqrt{x}}$ are shown in the following table:

x	$\frac{2}{\sqrt{x}}$
1.125	1.8856
1.375	1.7056
1.625	1.5689
1.875	1.4606
2.125	1.3720
2.375	1.2978
2.625	1.2344
2.875	1.1795

From equation (2):

$$\int_1^3 \frac{2}{\sqrt{x}}\,dx \approx (0.25)[1.8856 + 1.7056 + 1.5689 + 1.4606 + 1.3720 + 1.2978 + 1.2344 + 1.1795]$$

$$= \mathbf{2.926}, \text{ correct to 3 decimal places}$$

As previously, the greater the number of intervals the nearer the result is to the true value of 2.928, correct to 3 decimal places.

Problem 5. Evaluate $\int_0^{2.4} e^{-x^2/3}\,dx$, correct to 4 significant figures, using the mid-ordinate rule with 6 intervals

With 6 intervals each will have a width of $\frac{2.4-0}{6}$, i.e. 0.40 and the ordinates will occur at 0, 0.40, 0.80, 1.20, 1.60, 2.00 and 2.40 and thus mid-ordinates at 0.20, 0.60, 1.00, 1.40, 1.80 and 2.20.

Corresponding values of $e^{-x^2/3}$ are shown in the following table:

x	$e^{\frac{-x^2}{3}}$
0.20	0.98676
0.60	0.88692
1.00	0.71653
1.40	0.52031
1.80	0.33960
2.20	0.19922

From equation (2):

$$\int_0^{2.4} e^{\frac{-x^2}{3}}\,dx \approx (0.40)[0.98676 + 0.88692 + 0.71653 + 0.52031 + 0.33960 + 0.19922]$$

$$= \mathbf{1.460}, \text{ correct to 4 significant figures.}$$

Now try the following exercise

Exercise 181 Further problems on the mid-ordinate rule

Evaluate the following definite integrals using the **mid-ordinate rule**, giving the answers correct to 3 decimal places.

1. $\displaystyle\int_0^2 \frac{3}{1+t^2}\,dt$ (Use 8 intervals) [3.323]
2. $\displaystyle\int_0^{\pi/2} \frac{1}{1+\sin\theta}$ (Use 6 intervals) [0.997]
3. $\displaystyle\int_1^3 \frac{\ln x}{x}\,dx$ (Use 10 intervals) [0.605]
4. $\displaystyle\int_0^{\pi/3} \sqrt{\cos^3 x}\,dx$ (Use 6 intervals) [0.799]

53.4 Simpson's rule

The approximation made with the trapezoidal rule is to join the top of two successive ordinates by a straight line, i.e. by using a linear approximation of the form $a + bx$. With Simpson's rule, the approximation made is to join the tops of three successive ordinates by a parabola, i.e. by using a quadratic approximation of the form $a + bx + cx^2$.

Figure 53.3 shows a parabola $y = a + bx + cx^2$ with ordinates y_1, y_2 and y_3 at $x = -d$, $x = 0$ and $x = d$ respectively.

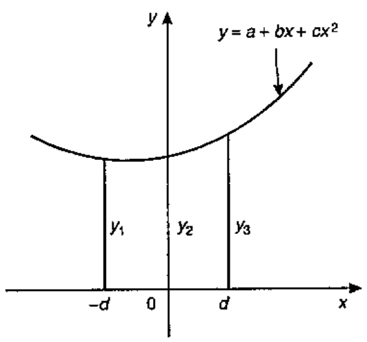


Figure 53.3

Thus the width of each of the two intervals is d. The area enclosed by the parabola, the x-axis and ordinates $x = -d$ and $x = d$ is given by:

$$\int_{-d}^{d} (a + bx + cx^2)\,dx = \left[ax + \frac{bx^2}{2} + \frac{cx^3}{3}\right]_{-d}^{d}$$

$$= \left(ad + \frac{bd^2}{2} + \frac{cd^3}{3}\right) - \left(-ad + \frac{bd^2}{2} - \frac{cd^3}{3}\right)$$

$$= 2ad + \frac{2}{3}cd^3$$

$$\text{or } \frac{1}{3}d(6a + 2cd^2) \qquad (3)$$

Since $y = a + bx + cx^2$,

at $x = -d$, $y_1 = a - bd + cd^2$

at $x = 0$, $y_2 = a$

and at $x = d$, $y_3 = a + bd + cd^2$

Hence $y_1 + y_3 = 2a + 2cd^2$

And $y_1 + 4y_2 + y_3 = 6a + 2cd^2$ (4)

Thus the area under the parabola between $x = -d$ and $x = d$ in Fig. 53.3 may be expressed as $\frac{1}{3}d(y_1 + 4y_2 + y_3)$, from equations (3) and (4), and the result is seen to be independent of the position of the origin.

Let a definite integral be denoted by $\int_a^b y\,dx$ and represented by the area under the graph of $y = f(x)$ between the limits $x = a$ and $x = b$, as shown in Fig. 53.4. The range of integration, $b - a$, is divided into an **even** number of intervals, say $2n$, each of width d.

Since an even number of intervals is specified, an odd number of ordinates, $2n + 1$, exists. Let an approximation to the curve over the first two intervals be a parabola of the form $y = a + bx + cx^2$ which passes through the tops of the three ordinates y_1, y_2 and y_3. Similarly, let an approximation to the curve over the next two intervals be the parabola which passes through the tops of the ordinates y_3, y_4 and y_5, and so on. Then

$$\int_a^b y\,dx \approx \frac{1}{3}d(y_1 + 4y_2 + y_3) + \frac{1}{3}d(y_3 + 4y_4 + y_5) + \frac{1}{3}d(y_{2n-1} + 4y_{2n} + y_{2n+1})$$

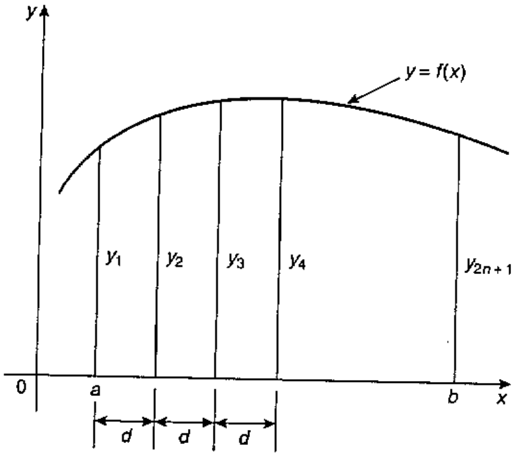


Figure 53.4

$$\approx \frac{1}{3}d[(y_1 + y_{2n+1}) + 4(y_2 + y_4 + \cdots + y_{2n}) + 2(y_3 + y_5 + \cdots + y_{2n-1})]$$

i.e. **Simpson's rule states:**

$$\int_a^b y\,dx \approx \frac{1}{3}\begin{pmatrix}\text{width of}\\ \text{interval}\end{pmatrix}\left\{\begin{pmatrix}\text{first + last}\\ \text{ordinate}\end{pmatrix} + 4\begin{pmatrix}\text{sum of even}\\ \text{ordinates}\end{pmatrix} + 2\begin{pmatrix}\text{sum of remaining}\\ \text{ordinates}\end{pmatrix}\right\} \quad (5)$$

Note that Simpson's rule can only be applied when an even number of intervals is chosen, i.e. an odd number of ordinates.

Problem 6. Use Simpson's rule with (a) 4 intervals, (b) 8 intervals, to evaluate $\int_1^3 \frac{2}{\sqrt{x}}\,dx$, correct to 3 decimal places

(a) With 4 intervals, each will have a width of $\frac{3-1}{4}$, i.e. 0.5 and the ordinates will occur at 1.0, 1.5, 2.0, 2.5 and 3.0.

The values of the ordinates are as shown in the table of Problem 1(b), page 440.

Thus, from equation (5):

$$\int_1^3 \frac{2}{\sqrt{x}}\,dx \approx \frac{1}{3}(0.5)\,[(2.0000 + 1.1547) + 4(1.6330 + 1.2649) + 2(1.4142)]$$

$$= \frac{1}{3}(0.5)[3.1547 + 11.5916 + 2.8284]$$

$= \mathbf{2.929}$, correct to 3 decimal places.

(b) With 8 intervals, each will have a width of $\frac{3-1}{8}$, i.e. 0.25 and the ordinates occur at 1.00, 1.25, 1.50, 1.75, ..., 3.0.

The values of the ordinates are as shown in the table in Problem 2, page 440.

Thus, from equation (5):

$$\int_1^3 \frac{2}{\sqrt{x}}\,dx \approx \frac{1}{3}(0.25)\,[(2.0000 + 1.1547) + 4(1.7889 + 1.5119 + 1.3333 + 1.2060) + 2(1.6330 + 1.4142 + 1.2649)]$$

$$= \frac{1}{3}(0.25)[3.1547 + 23.3604 + 8.6242]$$

$= \mathbf{2.928}$, correct to 3 decimal places.

It is noted that the latter answer is exactly the same as that obtained by integration. In general, Simpson's rule is regarded as the most accurate of the three approximate methods used in numerical integration.

Problem 7. Evaluate $\int_0^{\pi/3} \sqrt{1 - \frac{1}{3}\sin^2\theta}\,d\theta$, correct to 3 decimal places, using Simpson's rule with 6 intervals

With 6 intervals, each will have a width of $\frac{\frac{\pi}{3} - 0}{6}$, i.e. $\frac{\pi}{18}$ rad (or 10°), and the ordinates will occur at

$0, \frac{\pi}{18}, \frac{\pi}{9}, \frac{\pi}{6}, \frac{2\pi}{9}, \frac{5\pi}{18}$ and $\frac{\pi}{3}$

Corresponding values of $\sqrt{1-\frac{1}{3}\sin^2\theta}$ are shown in the table below:

θ	0	$\frac{\pi}{18}$ (or 10°)	$\frac{\pi}{9}$ (or 20°)	$\frac{\pi}{6}$ (or 30°)
$\sqrt{1-\frac{1}{3}\sin^2\theta}$	1.0000	0.9950	0.9803	0.9574

θ	$\frac{2\pi}{9}$ (or 40°)	$\frac{5\pi}{18}$ (or 50°)	$\frac{\pi}{3}$ (or 60°)
$\sqrt{1-\frac{1}{3}\sin^2\theta}$	0.9286	0.8969	0.8660

From equation (5):

$$\int_0^{\frac{\pi}{3}} \sqrt{1-\frac{1}{3}\sin^2\theta}\, d\theta$$

$$\approx \frac{1}{3}\left(\frac{\pi}{18}\right)[(1.0000+0.8660)+4(0.9950 + 0.9574+0.8969) + 2(0.9803+0.9286)]$$

$$= \frac{1}{3}\left(\frac{\pi}{18}\right)[1.8660+11.3972+3.8178]$$

$= \mathbf{0.994}$, correct to 3 decimal places.

Problem 8. An alternating current i has the following values at equal intervals of 2.0 milliseconds:

Time (ms)	0	2.0	4.0	6.0	8.0	10.0	12.0
Current i (A)	0	3.5	8.2	10.0	7.3	2.0	0

Charge, q, in millicoulombs, is given by $q = \int_0^{12.0} i\,dt$. Use Simpson's rule to determine the approximate charge in the 12 ms period

From equation (5):

$$\text{Charge, } q = \int_0^{12.0} i\,dt$$

$$\approx \frac{1}{3}(2.0)[(0+0)+4(3.5+10.0 + 2.0)+2(8.2+7.3)]$$

$$= \mathbf{62\,mC}$$

Now try the following exercise

Exercise 182 Further problems on Simpson's rule

In Problems 1 to 5, evaluate the definite integrals using **Simpson's rule**, giving the answers correct to 3 decimal places.

1. $\int_0^{\pi/2} \sqrt{\sin x}\,dx$ (Use 6 intervals) [1.187]

2. $\int_0^{1.6} \frac{1}{1+\theta^4}\,d\theta$ (Use 8 intervals) [1.034]

3. $\int_{0.2}^{1.0} \frac{\sin\theta}{\theta}\,d\theta$ (Use 8 intervals) [0.747]

4. $\int_0^{\pi/2} x\cos x\,dx$ (Use 6 intervals) [0.571]

5. $\int_0^{\pi/3} e^{x^2}\sin 2x\,dx$ (Use 10 intervals) [1.260]

In Problems 6 and 7 evaluate the definite integrals using (a) integration, (b) the trapezoidal rule, (c) the mid-ordinate rule, (d) Simpson's rule. Give answers correct to 3 decimal places.

6. $\int_1^4 \frac{4}{x^3}\,dx$ (Use 6 intervals)
[(a) 1.875 (b) 2.107 (c) 1.765 (d) 1.916]

7. $\int_2^6 \frac{1}{\sqrt{2x-1}}\,dx$ (Use 8 intervals)
[(a) 1.585 (b) 1.588 (c) 1.583 (d) 1.585]

In Problems 8 and 9 evaluate the definite integrals using (a) the trapezoidal rule, (b) the mid-ordinate rule, (c) Simpson's rule. Use 6 intervals in each case and give answers correct to 3 decimal places.

8. $\int_0^3 \sqrt{1+x^4}\,dx$

[(a) 10.194 (b) 10.007 (c) 10.070]

9. $\int_{0.1}^{0.7} \frac{1}{\sqrt{1-y^2}}\,dy$

[(a) 0.677 (b) 0.674 (c) 0.675]

10. A vehicle starts from rest and its velocity is measured every second for 8 seconds, with values as follows:

time t (s)	velocity v (ms^{-1})
0	0
1.0	0.4
2.0	1.0
3.0	1.7
4.0	2.9
5.0	4.1
6.0	6.2
7.0	8.0
8.0	9.4

The distance travelled in 8.0 seconds is given by $\int_0^{8.0} v\,dt$.

Estimate this distance using Simpson's rule, giving the answer correct to 3 significant figures. [28.8 m]

11. A pin moves along a straight guide so that its velocity v (m/s) when it is a distance x (m) from the beginning of the guide at time t (s) is given in the table below:

t (s)	v (m/s)
0	0
0.5	0.052
1.0	0.082
1.5	0.125
2.0	0.162
2.5	0.175
3.0	0.186
3.5	0.160
4.0	0

Use Simpson's rule with 8 intervals to determine the approximate total distance travelled by the pin in the 4.0 second period.

[0.485 m]

Assignment 14

This assignment covers the material in Chapters 50 to 53. *The marks for each question are shown in brackets at the end of each question.*

1. Determine: (a) $\int \frac{x-11}{x^2-x-2}\,dx$
(b) $\int \frac{3-x}{(x^2+3)(x+3)}\,dx$ (21)

2. Evaluate: $\int_1^2 \frac{3}{x^2(x+2)}\,dx$ correct to 4 significant figures. (12)

3. Determine: $\int \frac{dx}{2\sin x+\cos x}$ (5)

4. Determine the following integrals:

(a) $\int 5xe^{2x}\,dx$ (b) $\int t^2\sin 2t\,dt$ (12)

5. Evaluate correct to 3 decimal places:

$\int_1^4 \sqrt{x}\ln x\,dx$ (10)

6. Evaluate: $\int_1^3 \frac{5}{x^2}\,dx$ using

(a) integration

(b) the trapezoidal rule

(c) the mid-ordinate rule

(d) Simpson's rule.

In each of the approximate methods use 8 intervals and give the answers correct to 3 decimal places. (16)

7. An alternating current i has the following values at equal intervals of 5 ms:

Time t(ms)	0	5	10	15	20	25	30
Current i(A)	0	4.8	9.1	12.7	8.8	3.5	0

Charge q, in coulombs, is given by

$$q=\int_0^{30\times10^{-3}} i\,dt.$$

Use Simpson's rule to determine the approximate charge in the 30 ms period. (4)

Chapter: 6 - Interpolation

Mathematical function are often described in (tabular form), that is for prescribed values $X_1, X_2, ---, X_n$ of independent variable x, corresponding function values $f(x_1), f(x_2), ---, f(x_n)$ are given. The Logarithmic and trigonometric function are examples of functions which are presented in tabular form. The process of passing acurve through the given points.
In order to « determine functinal values of x not explicity shown in the table is called Interpolation».

1- Difference Table

A- Forward Differenes(Δ): Suppose that a table relating a dependent variable $f(x)$ to an independent variable x is given by

	X_i	$f(x_i)$
independ	X_0	$f(x_0)$
	X_1	$f(x_1)$
	X_2	$f(x_2)$
	⋮	⋮
	$f(x_n)$	$f(x_n)$

or

X_i	f_i
X_0	f_0
X_1	f_1
X_2	f_2
⋮	⋮
X_n	f_n

Ex.

Time	
5	
	86

and that $h = X_{i+1} - X_i$ and $X_0 < X_1 < X_2 < --- < X_n$

∴ Forward differences for the various known functional values can be established as ⇒ $\Delta f_0 = f_1 - f_0$

$\Delta f_1 = f_2 - f_1$

$\Delta f_K = f_{K+1} - f_K$

(degree of Polynomial = Number of points − 1)

$P(x) = aX^3 + bX^2 + cX + d$ (for 4 points)

In General $\Delta f_i = f_{i+1} - f_i \qquad i = 0, 1, 2, \dots, n$

Δ — first forward differences

- Further more higher difference expression can be readily defind by using equations :-

$$\Delta^n f_i = \Delta^{n-1}(f_{i+1} - f_i) \qquad i = 0, 1, 2, \dots, n$$

<u>Ex</u>: $\Delta^2 f_1 = \Delta(\Delta f_1) = \Delta(f_2 - f_1)$.

$\Delta^2 f_0 = \Delta(\Delta f_0) = \Delta(f_1 - f_0)$

B - BackWard Differences (∇) : The first Backward diff.

$$\nabla f_i = f_i - f_{i-1} \qquad i = 1, 2, \dots, n$$

$$\nabla^n f_i = \nabla^{n-1}(f_i - f_{i-1}) \qquad i = 1, 2, \dots, n$$

<u>Differences Tab.</u>: If $i = 0$ Then diff. table (Δ, ∇)

x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\Delta^4 f_i$
x_0	f_0				
		Δf_0			
x_1	f_1		$\Delta^2 f_0$		
		Δf_1		$\Delta^3 f_0$	
x_2	f_2		$\Delta^2 f_1$		$\Delta^4 f_0$
		Δf_2		$\Delta^3 f_1$	
x_3	f_3		$\Delta^2 f_2$		
		Δf_3			
x_4	f_4				

→ BackWard (∇) Diff. (max. Values)

→ for ward (Δ) Diff. (min. Values)

Ex:- find $\Delta^4 f_i$

$\Delta f_i = f_{i+1} - f_i \quad \text{--- (1)} \qquad \boxed{\Delta y_n = y_{n+1} - y_n}$

$\Delta^2 f_i = \Delta(f_{i+1} - f_i) = \Delta f_{i+1} - \Delta f_i = (f_{i+2} - f_{i+1}) - (f_{i+1} - f_i)$

$\Delta^2 f_i = f_{i+2} - 2f_{i+1} + f_i \quad \text{--- (2)} \qquad \boxed{\Delta^2 y_n = y_{n+2} - 2y_{n+1} + y_n}$

$\Delta^3 f_i = \Delta^2 f_{i+1} - \Delta^2 f_i = (f_{i+3} - 2f_{i+2} + f_{i+1}) - (f_{i+2} - 2f_{i+1} + f_i)$

$\Delta^3 f_i = f_{i+3} - 3f_{i+2} + 3f_{i+1} - f_i \quad \text{--- (3)}$

$\Delta^4 f_i = \Delta^3 f_{i+1} - \Delta^3 f_i = (f_{i+4} - 3f_{i+3} + 3f_{i+2} - f_{i+1}) - (f_{i+3} - 3f_{i+2} + 3f_{i+1} - f_i)$

$\therefore \Delta^4 f_i = f_{i+4} - 4f_{i+3} + 6f_{i+2} - 4f_{i+1} + f_i \quad \text{--- (4)}$

• The Interpolation is :-

1- Linea interpolation ($h = c$)

2- Second Interpolation (Quadratic In.) ($h = c$)

3- Newton - Gregory In. ($h = c$)

4- Lagrang Int. formula. ($h \neq c$

$\therefore P_n(x) = \sum_{k=0}^{n} \frac{f^{k}(x_0)}{k!}(x-x_0)^k$

$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{n}(x_n)}{n!}(x-x_0)^n$

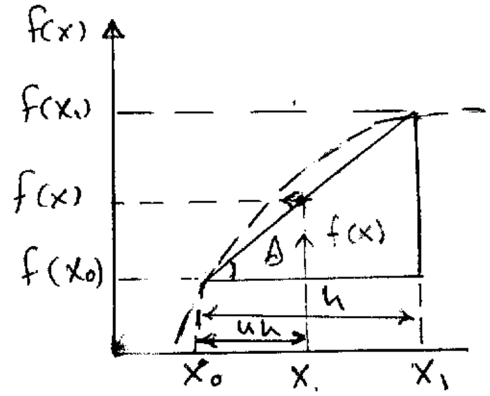
1- Linear interpolation : (L.I.) : Tylor's Series

If a table of value of $f(x)$ is given it is necessary to obtain values of $f(x)$ for values of [X] beteen [X] values :

$X = X_0 + rh$; h - step size

$r \overset{or}{=} u = \frac{X - X_0}{h}$; $h = X_{i+1} - X_i$

$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x) - f(x_0)}{x - x_0} = \tan\theta = \text{Slope}$



$\therefore f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \Rightarrow \boxed{P(x) = f_0 + u\,\Delta f_0}$

(3) $P(x) = a_1 + a_2(x-20) + a_3(x-20)(x-25)$ by Interpolatin parabala

Ex₁: Find the $\sin 22$ for the following table :

$0.5000 = 900a_1 + 30a_2 + a_3$

x	0	10	20	30	40
$f(x) = \sin x$	0	0.17365	0.34202	0.50000	0.69279

(2) Quadratic Int. $P(x) = a_1x^2 + a_2x + a_3 \Rightarrow 0.34202 = 400a_1 + 20a_2 + a_3$

solution :- $\sin 22^\circ = \sin 20 + \frac{\sin 30 - \sin 20}{30 - 20}(22 - 20) = f(x) = P(x)$

$\sin 22^\circ = 0.37861$; The actual value of $\sin 22^\circ = 0.37461$

Ex₂ : Given the following table find the value of (ln 9.2).

X	$y = \ln x = f(x)$
9.0	2.1970
9.5	2.2510
10.0	2.3026

Solution : $u = r = \frac{X - X_0}{X_1 - X_0} = \frac{9.2 - 9.0}{9.5 - 9.0} = 0.4$

$f(x) = P(x) = f_0 + u \cdot \Delta f_0$

$\therefore \ln 9.2 = \ln 9.0 + 0.4[\ln 9.5 - \ln 9.] = 2.219$

The differences are normally arranged as shown in the following table (forward)

x_i	$f(x_i)$	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\Delta^4 f_i$
x_i	f_i				
		Δf_i			
x_{i+1}	f_{i+1}		$\Delta^2 f_i$		
		Δf_{i+1}		$\Delta^3 f_i$	
x_{i+2}	f_{i+2}		$\Delta^2 f_{i+1}$		$\Delta^4 f_i$
		Δf_{i+2}		$\Delta^3 f_{i+1}$	
x_{i+3}	f_{i+3}		$\Delta^2 f_{i+2}$		
		Δf_{i+3}			
x_{i+4}	f_{i+4}				

$$f(x) = f_0 + \frac{u\Delta f_0}{1!} + \frac{u(u-1)}{2!}\Delta^2 f_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 f_0 + \cdots$$

Ex 2: (Backward Differences)

x_i	$f(x_i)$	∇f_i	$\nabla^2 f_i$	$\nabla^3 f_i$	$\nabla^4 f_i$
x_0	$f(x_0)$				
		∇f_1			
x_1	$f(x_1)$		$\nabla^2 f_2$		
		∇f_2		$\nabla^3 f_3$	
x_2	$f(x_2)$		$\nabla^2 f_3$		$\nabla^4 f_4$
		∇f_3		$\nabla^3 f_4$	
x_3	$f(x_3)$		$\nabla^2 f_4$		
		∇f_4			
x_4	$f(x_4)$				

$$f(x) = f_0 + u\nabla f_0 + \frac{u(u+1)}{2!}\nabla^2 f_0 + \frac{u(u+1)(u+2)}{3!}\nabla^3 f_0 + \cdots + \frac{u(u+1)(u+2)\cdots(u+n-1)}{n!}\nabla^n f_0$$

Ex 3: Tabulate the finite difference table for ($y = x^3$) for $x = 0(1)8$ when $h = 1$

Solution:

x_i	0	1	2	3	4	5	6	7	8
$y_i = x_i^3$	0	1	8	27	64	125	216	343	512

x_i	$y_i = x_i^3$	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$
0	0				
		1			
1	1		6		
		7		6	
2	8		12		0
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		0
		91		6	
6	216		36		6
		127		6	
7	343		42		
		169			
8	512				

2- Quadratic interpolation

$P(x) = a_1 x^2 + a_2 x + a_3$

$$\boxed{P(x) = f(x) = f_0 + \frac{r\,\Delta f_0}{1!} + \frac{r(r-1)}{2!}\,\Delta^2 f_0}$$

$0 \le r \le 2$

Ex. Find the value of (Int 9.2) in above Ex.

Solution :- $P(x_i) = f(x_i) = f(x_0) + \frac{u}{1!}\,\Delta f(x_0) + \frac{u(u-1)}{2!}\,\Delta^2 f(x_0)$.

$$\text{Int }9.2 = 2.1972 + 0.4\,(0.0541) + \frac{0.4\,(-0.6)}{2*1} * 0.0028$$

$$= 2.219$$

Ex. Given

x	2.1	2.4	2.5
y	0.61	2.09	2.00

use quadrati In. to evalute $y(2.127)$ and Linear Int. H.W.

Solution : $h_1 = 2.4 - 2.1 = 0.3$

$h_2 = 2.5 - 2.4 = 0.1$

3- A- Newton - Gregory forward diff. polynomial ; h = const.

$$P(x_i) = f(x_i) = f_0 + r\,\frac{\Delta f_0}{1!} + \frac{r(r-1)}{2!}\,\Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!}\,\Delta^3 f_0$$
$$+ \cdots + \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}\,\Delta^n f_0 \quad \cdots (1)$$

where $r = \frac{x - x_0}{h}$ $\quad 0 \le r \le h$

$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 = y_0 + u\,\Delta y_0 + \frac{1}{2}(u^2 - u)\Delta^2 y_0 + \frac{1}{6}(u^3 - 3u^2 + 2u)\Delta^3 y_0 +$

B- Newton - Gregory Backward diff. Interpolation Polynomial ; (N.-G. B. I.) - h = constant.

$$f(x) = P(r) = f_n + r\,\nabla f_1 + \frac{r(r+1)}{2!}\,\nabla^2 f_n + \frac{r(r+1)(r+2)}{3!}\,\nabla^3 f_n$$
$$+ \cdots + \frac{r(r+1)(r+2)\cdots(r+n-1)}{n!}\,\nabla^n f_n \quad \cdots (2)$$

$$u = r = \frac{x - x_n}{h}$$

Ex. construct difference table and find the Polynomial of minimum degree which fits the following data and compute 1- $f(10.6)$; 2- $f'(7.3)$

x	3	5	7	9	11
f(x)	6	24	58	108	174

X	f(x)	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
$3 = x_0$	6				
		18			
$5 = x_1$	24		16		
		34		0	
$7 = x_2$	58				
		50	16		0
$9 = x_3$ (10.3)	108			0	
		66	16		
$11 = x_4$	174				

$$f(x) = f_0 + \frac{r\Delta f_0}{1!} + \frac{r(r-1)}{2!}\Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 f_0 + \cdots$$

$$u = r = \frac{x - x_0}{h} = \frac{x-3}{2} = 0.5(x-3)$$

$$\therefore f(x) = 6 + \frac{r * 18}{1!} + \frac{r(r-1)}{2!} * 16 + 0$$

$$f(x) = 8r^2 + 10r + 6$$

$$= 8\left(\frac{x-3}{2}\right)^2 + 10\left(\frac{x-3}{2}\right) + 6 = 2x^2 - 7x + 9$$

$$\boxed{f(x) = 2x^2 - 7x + 9}$$

$$f(10.6) = 2(10.6)^2 - 7(10.6) + 9$$

$$f'(x) = 4x - 7$$

$$f''(7.3) = 4(7.3) - 7$$

$y = \log x$ find $\log 1044$

Ex/: Given the following function $y = \log_{10} x$ find $y = \log_{10} 1044$ for $(x = 1000\,(10)\,1050)$, by using interpolation: (h = constant)

X	$y = \log_{10} x$	∇^1	∇^2	∇^3
1000	3.00000	0.00432	+0.008698 / −0.000426	
1010	3.00423	0.004378	−0.000148 / −0.000418	0.0008 −0.00884
1020	3.00866	0.00423	−0.00003 / −0.000409	0.00009 +0.00118
1030	3.012830	0.004200		0.00009 − 0.00047
1040	3.017033	0.004156	−0.000505	
1050	3.021189			

$$u = \frac{x - x_n}{h} = \frac{1044 - 1050}{10} = -0.6$$

$\therefore$ (x) near the end of the table Then ($x = 1044$):

$$P_3(x) = \log_{10} 1044 = f_n + \frac{\nabla f_n}{1!}u + \frac{\nabla^2 f_n}{2!}u(u+1) + \frac{\nabla^3 f_n}{3!} * u(u+1)(u+2)$$

$$= 3.021189 + \frac{0.004156}{1!} * (-0.6) + \frac{(-0.000401)}{2!} * (-0.6) * (-0.6+1)$$

$$+ \frac{0.00009}{3!} * (-0.6)(-0.6+1) * (-0.6+2) = 3.01887005$$

4- Lagrange interpolation ($h \neq c$)

$$P_n(x) = \sum_{k=0}^{n} f(X_k)\, L_k = L_0 f(x_0) + L_1 f(x_1) + L_2 f(x_2) + \cdots + L_n f(x_n)$$

where:
$$L_k = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(X - X_i)}{(X_k - X_i)} \qquad \begin{array}{l} i = 0, 1, 2, 3, \ldots, n \\ k = 0, 1, 2, 3, \ldots, n \end{array}$$

$\prod$ - is the product of ---

L_k - polynomial Coefficients

for example $n = 1$ = first order

$$P_1(x) = \frac{X - X_1}{X_0 - X_1} f(X_0) + \frac{X - X_0}{X_1 - X_0} f(X_1)$$

and second order

$$P_2(X) = \frac{(X - X_1)(X - X_2)}{(X_0 - X_1)(X_0 - X_2)} f(X_0) + \frac{(X - X_0)(X - X_2)}{(X_1 - X_0)(X_1 - X_2)} f(X_1) +$$
$$+ \frac{(X - X_0)(X - X_1)}{(X_2 - X_0)(X_2 - X_1)} f(X_2)$$

Ex. Find the second degree interpolatio polynom,

Solution: ($n = 2$) ;

$i = 1 + 2 = 3 \qquad i = 0, 1, 2$

$K = 1 + 2 = 3 \qquad K = 0, 1, 2$

$$P_n(x) = \sum_{k=0}^{n} f(X_k)\, L_k$$

$$L_K = \prod_{\substack{i=0 \\ i \neq K}}^{n} \frac{(X - X_i)}{(X_K - X_i)} \quad ; K = 0, \quad i = 1, 2$$

$$L_0 = \left(\frac{X - X_1}{X_0 - X_1}\right)\left(\frac{X - X_2}{X_0 - X_2}\right)$$

$K=1 \quad , i=0,2$

$$L_1 = \left(\frac{X-X_0}{X_1-X_0}\right)\left(\frac{X-X_2}{X_1-X_2}\right)$$

$K=2 \quad , i=0,1$

$$L_2 = \left(\frac{X-X_0}{X_2-X_0}\right)\left(\frac{X-X_1}{X_2-X_1}\right)$$

$$\therefore P_2(X) = L_0 f(X_0) + f(X_1) L_1 + L_2 f(X_2)$$

Ex: Use $X_0=2$, $X_1=2.5$, $X_2=4$ find the second order Polynomial of the function $f(x)=\frac{1}{X}$

Solution:-

$n=2$

$$P_1(x) = \sum_{K=0}^{n=2} f(x_K) L_K$$

L_K = polynomial coefficients

$$L_K = \prod_{\substack{i=0 \\ i \neq K}}^{n} \left(\frac{X-X_i}{X_K-X_i}\right)$$

$$\therefore P_2(x) = f(x_0)L_0 + f(x_1)L_1 + f(x_2)L_2$$

$f(x) = \frac{1}{X}$; $X_0=2$, $X_1=2.5$, $X_2=4$

$f(X_0) = f(2) = \frac{1}{2} = 0.5$

$f(X_1) = f(2.5) = \frac{1}{2.5} = 0.4$

$f(X_2) = f(4) = \frac{1}{4} = 0.25$

$$L_0 = \left(\frac{X-X_1}{X_0-X_1}\right)\left(\frac{X-X_2}{X_0-X_2}\right) = \left(\frac{X-2.5}{2-2.5}\right)\left(\frac{X-4}{2-4}\right) = X^2-6.5X+10$$

$$L_1 = \frac{(X-X_0)(X-X_2)}{(X_1-X_0)(X_1-X_2)} = \frac{(X-2)(X-4)}{(2.5-2)(2.5-4)} = \frac{X^2-6X+8}{0-0.75}$$

$$L_2 = \frac{(X-X_0)(X-X_1)}{(X_2-X_0)(X_2-X_1)} = \frac{(X-2)(X-2.5)}{(4-2)(4-2.5)} = \frac{X^2-4.5X+5}{3}$$

$$\therefore P_2(X) = 0.5(X^2-6.5X+10) + 0.4\left(\frac{X^2-6X+8}{-0.78}\right) + \frac{0.25(X^2-4.5X+5)}{3}$$

$$\therefore P_2(X) = 0.05X^2 - 0.425X + 1.15$$ If X = any value

Sheet No.

1- Use Lagrange I.P to find the polynomial which fit the following dat and find $P(1.2)$

X	1	1.5	2.5	3	4
f(x)	0	8.625	43.878	72	153

2- If $y(1) = 12$, $y(2) = 15$, $y(5) = 25$ and $y(6) = 30$ find the four points Lagrange interpolation poly. that takes same value of the function (y) at the given points and estimate the value of $y(4)$?

3- Find the Velocity of the rocket by using Newton Int. Polynomial at $(t = 150)$ seconds:

t(s)	0	60	120	180	240	300	0
V(mile/sec)	0	0.0824	0.2747	0.6502	1.3851	3.2224	0

4- Use Newton Backward Polynomial to find $f(0.73)$

X	0.4	0.6	0.8	1
f(x)	0.423	0.684	1.03	1.557

5- The following table was obtained from a polynomial function

X	0	1	2	3	4	5
y	0	0	14	78	252	620

Determin the order of Polynomial and find the Value of y at $X = 3.15$.

6- Use Newton Gregory forward Interpolation Polynomial to estimate the minimum degree poly. to fit the following data and find $f(0.158)$ and $f(0.636)$

X	0.125	0.25	0.375	0.500	0.625	0.750
$f(X_i)$	0.79168	0.7733	0.7437	0.7041	0.65632	0.60228

2- Solution of differential Equation by Power Series method :-

Introduction : Solving diff. eq. by the Sol. called power series method which yields solutions in the form of power series. This is very efficient standard procedure in connection withe linear differential equations whose coefficients are variable. There are Infinite series function :-

or
$$a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$b(x) = b_0 + b_1 x + b_2 x^2 + \cdots$$
} Infinite funct.

apower series in power of $(x-a)$ is an infinite series of the form

$$y = \sum_{m=0}^{\infty} a_m (x-a)^m = a_0 + a_1 (x-a) + a_2 (x-a)^2 + \cdots \quad (1)$$

$a_0, a_1, \cdots \rightarrow$ are constant called the coefficient of series ([illegible])

$x \rightarrow$ Variable ; {m , r , [illegible]} power of variable.

1- Taylor series :

$$f(x) = f(a) + f'(a)(x-a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^m}{m!} f^m(a).$$

If $a = 0$

2- Maclaurin Series :

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f(0) + \quad + \frac{x^m}{m!} f^m(0).$$

Then we assume a solution in the form of apower series :

$$y = \sum a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_m x^m \quad (2)$$

note :

(a-) $a_m = 0$ for all $m < 0$

(b-) $a_0 \neq 0$

Examples of power series are the Maclaurin series.:

1- $\dfrac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots$

2- $e^x = \sum_{m=0}^{\infty} \dfrac{x^m}{m!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$

3- $\cos x = \sum_{m=0}^{\infty} \dfrac{(-1)^m x^{2m}}{(2m)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \cdots$

4- $\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \cdots$

5- $\sin x = \dfrac{x}{1!} - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots = \sum_{m=0}^{\infty} \dfrac{(-1)^m x^{2m+1}}{(2m+1)!}$

6- $\sinh x = \dfrac{x}{1!} + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!}$

7- $\ln(1-x) = -\left(x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!}\right)$

8- $\ln(1+x) = x - \dfrac{x^2}{2!} + \dfrac{x^3}{3!} - \dfrac{x^4}{4!} + \cdots = \sum_{m=0}^{\infty} (-1)^m \dfrac{x^{m+1}}{m+1}$

9- $\dfrac{1}{1+x} = \sum_{m=0}^{\infty} (-1)^m x^m = 1 - x + x^2 - x^3 + \cdots$

Power Series method :-

Solving the ordinary differential equation (1^{st}, 2^{nd}) order (homogenous, Linear) function withe Varible coefficient. In general two form of diff. eq. :

1- $P_0(x) \cdot \frac{dy}{dx} + P_1(x) \cdot y = \varphi(x)$

2- $P_0(x) \cdot \frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x) * y = 0$

or $\frac{d^2y}{dx^2} + \frac{P_1(x)}{P_0(x)} \cdot \frac{dy}{dx} + \frac{P_2(x)}{P_0(x)} \cdot y = 0$

To finde the solving of eq. (1, 2) by series method we have :- $\{ X = X_0 = 0 \}$

Solution :-

1- Near ordinary point - ($P_0(X_0) \neq 0$) :-

$y(x) = y_{(x)} = \sum_{m=0}^{\infty} C_m X^m \Rightarrow y'' - xy = 0 ; x = 0 \Rightarrow y'' = 0$

2- Near singular Point - ($P_0(X_0) = 0$) {Frobnius Method} :-

$y(x) = y_{(x)} = \sum_{m=0}^{\infty} C_m X^{m+r} = X^r \sum_{m=0}^{\infty} C_m X^m$ $\{ r = +n, -n, \frac{n}{1,2,3}, \dots \}$ to find (r) using the Indicial equation.

Exs. $X = 0$

1- $y'' - 3xy' + 3y \Rightarrow$ (Hermite eq.) near ordinary point

2- $2x^2y'' + 3xy' - (1+x)y \Rightarrow$ near Singular point ($x = 0$)

3- $x^2y'' - 2xy' - xy = 0 \Rightarrow$ " " " ($x = 0$)

Then we assume a solution in the form of a power series say:

$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots = \sum_{m=0}^{\infty} C_m X^m$ --- (3)

4 -- $\begin{cases} a- \ y' = C_1 + 2C_2 + 3C_3 x^2 + \dots = \sum_{m=1}^{\infty} m C_m X^{m-1} \\ b- \ y'' = 2C_2 + 3*2C_3 + 4*3C_4 + \dots = \sum_{m=2}^{\infty} m(m-1) C_m X^{m-2} \end{cases}$

ملاحظة: عندما يكون الحل القريب من النقطة الاعتيادية (O.P.) فان الحل [illegible]
يجب ان تتحقق والا فان الحل يكون بالقرب من نقطة شاذة (S.P.) وان
الحل [illegible] يكون حول (X=0)

EX 1: Solve the following differential eq. by Power Series ~~about~~ $x=0$; $y''+y=0$ --- (1)

note. $a_j = 0$ for all $j < 0$, $x^j \neq 0$

Solution:-

$$\left[y=\sum a_j x^j \;;\; y'=\sum j a_j x^{j-1} \;;\; y''=\sum j(j-1) a_j x^{j-2} \right] \text{--- (2)}$$

subsit eq. (2) in eq. (1)

$$\sum a_j \cdot j \cdot (j-1)\, x^{j-2} + \sum a_j x^j = 0 \quad \text{--- (3)}$$

replace each j by $j+2$ in 1st term eq (3) becomes

$$\sum a_{j+2}(j+2)(j+1)\, x^j + \sum a_j\, x^j = 0$$

$$\sum \left[a_{j+2}(j+2)(j+1) + a_j \right] x^j = 0 \qquad x^j \neq 0$$

$$\therefore a_{j+2}(j+2)(j+1) + a_j = 0$$

صيغة ابتداء

$$a_{j+2} = \frac{-a_j}{(j+2)(j+1)} \qquad \text{Recurrence formula.}$$

الأعلى

Putting $j = 0, 1, 2, 3, \ldots$

$$a_2 = \frac{-a_0}{2.1} = \frac{-a_0}{2!} \qquad a_4 = \frac{-a_2}{4.3} = \frac{a_0}{4!}$$

$$a_3 = \frac{-a_1}{3.2} = \frac{-a_1}{3!} \qquad a_5 = \frac{a_1}{5!}$$

$$\therefore y = \sum_{j=0}^{\infty} a_j x^j = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

note $a_0 \neq 0$; $a_1, a_2, a_3, \ldots$ are coefficient ((constant))

$$y = a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 - \cdots$$

$$y = a_0 - \frac{a_0}{2!} x^2 + \frac{a_0}{4!} x^4 - \cdots + a_1 x - \frac{a_1}{3!} x^3 + \frac{a_1}{5!} x^5 - \cdots$$

$$y = a_0 \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \right] + a_1 \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right]$$

$$\therefore y = a_0 \cos x + a_1 \sin x = A \cos x + B \sin x$$

Method of Frobenius :-

Frobenius Method use to solve the following type of diff. eq.:-

$P(x)\, y'' + q(x)\, y' + r(x)\, y = 0$; where $P(x), q(x)$ and $r(x)$ are Polynomial

دوال متعددة حدود

assume for solution $y = \sum_{j=0}^{\infty} a_j (x-a)^{j+c}$

if $a = 0$ $\quad y = \sum_{j=0}^{\infty} a_j\, x^{j+c}$ or zero

$\therefore y = x^c [a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots]$ (N)

The Cases 1- Distinct roots which do not differ by an integer $[c_1 - c_2 \neq 0]\ [c_1 - c_2 \neq n]\ [\mp\frac{1}{2}, \mp\frac{1}{4}]$

الجذور $(c_1 - c_2) \neq$ roots is not integr

2- Double root $[c_1 = c_2 = \alpha] \Rightarrow [c_1 - c_2 = 0]$

3- Distinct roots which differ by an integer $[c_1 - c_2 = n]\ [2, 4, 5, \ldots]$

Ex. 1 Case 1 :

Solve the diff. eq. by the method of Frobenius about $x = 0$; $\quad 4x \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

Solution :-

$y = \sum_{j=0}^{\infty} a_j x^{j+c}$; $y' = \sum a_j (j+c) x^{j+c-1}$; $y'' = \sum a_j (j+c)(j+c-1) x^{j+c-2}$

$\sum 4(j+c)(j+c-1)\, a_j\, x^{j+c-1} + \sum 2(j+c)\, a_j\, x^{j+c-1} + \sum a_j\, x^{j+c} = 0$ $\quad j \to j+1$

$\Rightarrow \sum [4(j+c+1)(j+c)\, a_{j+1} + 2(j+c+1)\, a_{j+1} + a_j]\, x^{j+c} = 0$

$\sum [(2j+2c+2)(2j+2c+1)\, a_{j+1} + a_j]\, x^{j+c} = 0$

$$(2j+2c+2)(2j+2c+1)\,a_{j+1} + a_j = 0 \quad ---- \circledast$$

Indicial Equation

Putting $j=-1$

$$(-2+2c+2)(-2+2c+1)\,a_0 + \cancel{a_{-1}}^{0} = 0 \qquad a_0 \neq 0$$

$$(2c)(2c-1) = 0 \Rightarrow c_1 = 0 \; ; \; c_2 = \frac{1}{2}$$

$\therefore c_1 - c_2 = 0 - \frac{1}{2} = -\frac{1}{2}$ not integer.

Case a :- $c_1 = 0$; $y = \sum a_j x^{j+c_1} = \sum a_j x^{j+0} = \sum a_j x^j$

$\therefore y_1 = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

Subsit $c_1 = 0$ in eq. $\circledast$ we get

$$(2j+2\cancel{c}^{0}+2)(2j+2\cancel{c}^{0}+1)\,a_{j+1} + a_j = 0$$

$$a_{j+1} = \frac{-a_j}{(2j+2)(2j+1)} \qquad \text{Recurrence formula (R.f. 1)}$$

Putting $j = 0, 1, 2, \ldots$ $\quad a_1 = \frac{-a_0}{2!}$, $a_2 = \frac{-a_1}{4.3} = \frac{a_0}{4!}$

$$a_3 = \frac{-a_2}{6.5} = \frac{-a_0}{6!}$$

$$\therefore y_1 = a_0 - \frac{a_0}{2!}x + \frac{a_0}{4!}x^2 - \frac{a_0}{6!}x^3 + \cdots$$

$$y_1 = a_0\left[1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots\right] = a_0 \cos\sqrt{x}$$

Case b :- $c_2 = \frac{1}{2}$

$$y_2 = \sum a_j x^{j+c_2} = \sum a_j x^{j+\frac{1}{2}} = x^{\frac{1}{2}}\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right]$$

Sub. $c_2 = \frac{1}{2}$ in eq. $\circledast$ we get

$$a_{j+1} = -\frac{a_j}{(2j+3)(2j+2)} \qquad \text{Recurrence formula (R.F}$$

$j=3 \quad a_3 = \frac{-a_1}{3!}, \quad a_4 = \frac{a_0}{4!}, \quad a_5 = \frac{-a_3}{5.4} = \frac{a_1}{5!}$

$a_6 = \frac{-a_0}{6!}, \quad a_7 = \frac{-a_1}{7!}$

$\therefore y_2 = \sum a_j \, x^{j-\frac{1}{2}} = \frac{1}{\sqrt{x}}[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots]$

$y = \frac{1}{\sqrt{x}}[a_0(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots) + a_1(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots)$

$\therefore y_2 = \frac{1}{\sqrt{x}}[a_0 \cos x + a_1 \sin x] = \frac{a_0 \cos x}{\sqrt{x}} + \frac{a_1 \sin x}{\sqrt{x}}$

The General solution is:

$y_{G.s} = y_1 + y_2$

$= \frac{a_0 \sin x}{\sqrt{x}} + \frac{a_0 \cos x}{\sqrt{x}} + \frac{a_1 \sin x}{\sqrt{x}}$

Case 3: when the case Double roots: $c_1 = c_2 = \alpha$

We can obtain directly one solution and other solution can be obtained as in the following:

Ex 3: Solve the following diff. Eq. by Frobenius method about $x=0$.

$x(x-1)y'' + (3x-1)y' + y = 0 \quad \cdots \quad * \quad \cdots \quad (1)$

Solution: $y = \sum a_j x^{j+c}; \; y' = \sum (j+c) a_j x^{j+c-1}; \; y'' = \sum (j+c)(j+c-1) a_j x^{j+c-2}$ (2)

subsit eq (2) in eq (1)

$x^2 \sum a_j (j+c)(j+c-1) x^{j+c-2} - x \sum a_j (j+c)(j+c-1) x^{j+c-2} + \cancel{3x \sum a_j (j+c)}$

$+ 3x \sum a_j (j+c) x^{j+c-1} - \sum a_j (j+c) x^{j+c-1} + \sum a_j x^{j+c} = 0$

$\sum a_j (j+c)(j+c-1) x^{j+c} - \sum a_j (j+c)(j+c-1) x^{j+c-1} + 3 \sum a_j (j+c) x^{j+c}$

$- \sum a_j (j+c) x^{j+c-1} + \sum a_j x^{j+c} = 0.$

$$\sum a_j (j+c)(j+c-1)\, x^{j+c} - \sum a_{j+1} (j+c+1)(j+c)\, x^{j+c}$$
$$+ 3\sum a_j (j+c)\, x^{j+c} - \sum a_{j+c} (j+c+1)\, x^{j+c} + \sum a_j \cdot x^{j+c} = 0$$
$$\sum\Big([(j+c)(j+c-1) + 3*(j+c)+1]a_j - [(j+c+1)(j+c)+(j+c+1)]a_{j+1}\Big)x^{j+c}=0$$

Butting $j=-1$ we get :-

$$(\quad)\, a_{-1}^{\nearrow 0} - [c(c-1)+c]a_0 = 0 \qquad a_0 \neq 0$$
$$-c^2+c-c=0 \Rightarrow -c^2=0 \Rightarrow c^2=0$$
$$c_1 = 0 \quad , c_2 = 0 \quad \text{-- roots}$$

$\therefore$ Indicial Eq. is :

$$[(j+c)(j+c-1)+3(j+c)+1]a_j - [(j+c+1)(j+c)+(j+c+1)]a_{j+1}=0 \quad (3)$$

<u>Case a</u> If $c_1=0$ then subsit. in eq. (3)

$$a_{j+1} = \frac{a_j[(j+c)(j+c-1)+3(j+c)+1]}{(j+c+1)(j+c)+(j+c+1)} = \frac{j(j-1)+3j+1}{j(j+1)+(j+1)}a_j$$

$$\frac{(j^2+2j+1)a_j}{(j+1)(j+1)} = \frac{(j+1)(j+1)}{(j+1)(j+1)}a_j = 1*a_j$$

$\therefore \boxed{a_{j+1} = a_j} \Rightarrow$ Recurrence formula.

If

$j=0 \qquad a_1 = a_0 \quad , a_2 = a_1 = a_0 , \quad a_3 = a_2 = a_0 , \text{---}$

$$\therefore y_1 = \sum a_j x^{j+c_1} = \sum a_j x^j = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
$$y_1 = a_0 [1 + x + x^2 + x^3 + \cdots] \Rightarrow y_1 = \sum_{n=0}^{\infty} a_0 x^n$$
$$\therefore y_1 = a_0 \sum x^n$$

But we have from series : $\frac{1}{1-x} = \sum x^n = 1 + x + x^2 + \cdots$

$$\boxed{\therefore y_1 = \frac{a_0}{1-x}}$$

To find a Solution of y_2 we must solve as follows:

assume $y_2 = \phi y_1$

$$\phi = \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx = \int \frac{e^{-\int \frac{3x-1}{x(x-1)}dx}}{y_1^2}\,dx \Rightarrow \text{But}$$

$$\int \frac{3x-1}{x(x-1)}\,dx = \int \frac{3x}{x(x-1)}\,dx - \int \frac{1}{x(x-1)}\,dx$$

$$= 3\ln(x-1) - \int \frac{-1}{x}\,dx + \int \frac{1}{x-1}\,dx$$

$$= 3\ln(x-1) + \ln x - \ln(x-1) = 2\ln(x-1) + \ln x$$

$$= \ln x(x-1)^2$$

$$\therefore \phi = \int \frac{e^{-\ln x(x-1)^2}}{\frac{a_0^2}{(1-x)^2}}\,dx = \int \frac{+(1-x)^2}{x(x-1)^2 a_0^2}\,dx$$

$$\phi = \int \frac{-(x-1)^2}{x(x-1)^2 a_0^2}\,dx = \frac{-1}{a_0^2}\int \frac{dx}{x} = -\frac{1}{a_0^2}\ln x = A\ln x$$

$$\therefore \phi = A\ln x \qquad \therefore y_2 = \phi y_1 = A\ln x\,\frac{a_0}{1-x} = \frac{A_0}{1-x}\ln x$$

$$\boxed{\therefore y_{G.S.} = y_1 + y_2 = \frac{A_0}{1-x}(1+\ln x)}$$

Sheet NO: 2 Solve the following diff. eq. by power series about $x=0$ to find the General Solution:-

1- $x^2y'' + 4xy' + (x^2+2)y = 0$

2- $xy'' + 3y' + 4x^3y = 0$

3- $xy'' + y' - xy = 0$

4- $xy'' + 2y' - xy = 0$

5- $(2x^2+3x)y'' - (4x+3)y' + (4+\frac{3}{x})y = 0$

6- $(x^2-1)x^2y'' - x(x^2+1)y' + (x^2+1)y = 0$

7- $x(x-1)y'' + xy' + y = 0$ about $x = 1$.

8- $4x^2y'' + 2y' + 2y = 0$

9- $(x-x^2)y'' + (1-5x)y' - 4y = 0$

10- $x(1-x)y'' - 3xy' - y = 0$

EX. 4: Solve the following differential Equation by Power Series method (Frobenius method) about $x=0$

$$x^2y'' + 5xy' + (x+4)y = 0 \quad \text{----} (1)$$

Solution: here $x^2 = 0 \Rightarrow x = 0$ is a singular Point, then

$$y = \sum a_j x^{j+c} \; ; \; y' = \sum a_j (j+c) x^{j+c-1} \; ; \; y'' = \sum a_j (j+c)(j+c-1) x^{j+c-2} \quad \text{----} (2)$$

Subsit eq. (2) into eq. (1) we get:

$$\sum a_j (j+c)(j+c-1) x^2 x^{j+c-2} + \sum 5 a_j (j+c) x x^{j+c-1} + \sum a_j x^{j+c+1} + \sum 4 a_j x^{j+c} = 0$$

$$\sum a_j (j+c)(j+c-1) x^{j+c} + \sum 5 a_j (j+c) x^{j+c} + \sum a_j x^{j+c+1} + \sum 4 a_j x^{j+c} = 0$$

$$\sum [((j+c)(j+c-1) + 5(j+c) + 4) a_j + a_{j-1}] x^{j+c} = 0 \; ; \; x^{j+c} \neq 0$$

$$\boxed{\therefore [(j+c)(j+c-1) + 5(j+c) + 4] a_j + a_{j-1} = 0} \quad \text{put } j=0$$

$$\therefore ((c)(c-1) + 5c + 4) a_0 + \cancel{a_{-1}}^{\,0} = 0 \quad , \; a_0 \neq 0$$

$$c^2 - c + 5c + 4 = 0 \Rightarrow c^2 + 4c + 4 = 0 \Rightarrow (c+2)(c+2) = 0$$

$$\boxed{c_1 = c_2 = -2} \; ; \; \boxed{a_j = \frac{-a_{j-1}}{(j+c)(j+c-1) + 5(j+c) + 4}} \quad \text{----} (*)$$

Case (a): $c_1 = -2 \Rightarrow a_j = \dfrac{-a_{j-1}}{(j-2)(j-3) + 5(j-2) + 4}$

$$a_j = \frac{-a_{j-1}}{(j-2)(j-3+5) + 4}$$

$$a_j = \frac{-a_{j-1}}{(j-2)(j+2) + 4} = \frac{-a_{j-1}}{j^2} \qquad \boxed{\therefore a_j = \frac{-a_{j-1}}{j^2}} \text{ R.F.1}$$

Putting $j_1 = 1, 2, 3, 4, \dots \Rightarrow a_1 = -a_0 \; , \; a_2 = \dfrac{-a_1}{4} = \dfrac{a_0}{4} \; ; \; a_3 = \dfrac{-a_2}{9}$

$$a_3 = \frac{-a_0}{36} \; ; \; a_4 = \frac{-a_3}{16} = \frac{a_0}{576}$$

Thus $y_1 = \sum a_j x^{j+c_1} = x^{-2} [\sum a_j x^j] = x^{-2} [a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots]$

$$\therefore y_1 = a_0 [x^{-2} - x^{-1} + \tfrac{1}{4} - \tfrac{1}{36} x + \tfrac{1}{576} x^2 + \dots]$$

$$y_1 = a_0 [\frac{1}{x^2} - \frac{1}{x} + \frac{1}{4} - \frac{1}{36} x + \frac{1}{576} x^2 + \dots]$$

Case (b): For find $y_2 = y_1(x)\ln x + \sum_{n=1}^{\infty} b_j x^{j+c_2}$

$$b_j = \frac{\partial}{\partial c}(a_j) = \frac{\partial}{\partial c}\left[\frac{-a_{j-1}}{(j+c)(j+c-1)+5(j+c)+4}\right]$$

When $j=1$

$$b_1 = \frac{\partial a_1}{\partial c} = \frac{\partial}{\partial c}\left[\frac{-a_0}{(c+1)c+5(c+1)+4}\right] = \frac{\partial}{\partial c}\left[\frac{-a_0}{(c+1)(c+5)+4}\right]$$

$$\therefore b_1 = \frac{a_0[(c+1)+(c+5)]}{[(c+1)(c+5)+4]^2} \quad \text{If } c_2=-2 \text{ then } b_1 = 2a_0$$

When $j=2$

$$b_2 = \frac{\partial a_2}{\partial c} = \frac{\partial}{\partial c}\left[\frac{-a_1}{(c+2)(c+1)+5(c+2)+4}\right] = \frac{\partial}{\partial c}\left[\frac{-a_1}{(c+2)(c+6)+4}\right]$$

$$b_2 = \frac{\partial}{\partial c}\left[\frac{a_0}{[(c+2)(c+6)+4][(c+1)(c+5)+4]}\right]$$

$$b_2 = \frac{-a_0\left[[(c+2)(c+6)+4][(c+1)+(c+5)] + [(c+1)(c+5)+4][(c+2)+(c+6)]\right]}{[(c+2)(c+6)+4]^2[(c+1)(c+5)+4]^2}$$

Sub $c=-2$ $\therefore b_2 = \frac{-3}{4}a_0$

When $j=3$

$$b_3 = \frac{\partial a_3}{\partial c} = \frac{\partial}{\partial c}\left[\frac{-a_2}{(c+3)(c+2)+5(c+3)+4}\right]$$

$$b_3 = \frac{\partial}{\partial c}\left[\frac{-a_0}{[(c+3)(c+7)+4][(c+2)(c+6)+4][(c+1)(c+5)+4]}\right]$$

$$b_3 = \frac{a_0\left[[(c+3)(c+7)+4][(c+2)(c+6)+4][(c+1)+(c+5)] + [(c+3)(c+7)+4] * [(c+1)(c+5)+4]\right.}{[(c+3)(c+7)+4]^2[(c+2)(c+6)+4]^2[(c+1)(c+5)+4]^2}$$

$$+ \frac{[(c+2)+(c+6)] + [(c+3)+(c+7)][(c+2)(c+6)+4][(c+7)(c+5)+4]}{[(c+3)(c+7)+4]^2[(c+2)(c+6)+4]^2[(c+1)(c+5)+4]^2}$$

$$b_3 = \frac{11a_0}{108}$$

$$\boxed{\therefore y_2 = y_1\ln x + a_0\left[\frac{2}{x} - \frac{3}{4} + \frac{11}{108}x + \cdots\right]}$$ The required solution

Ch. 5 : Solution of Simultaneaus linear algebraic Equation

The following linear system of equation

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

Where a_{ij} $\quad i = 1, 2, \cdots, m \quad j = 1, 2, \cdots, n$ are the coefficient of n

and $x_1, x_2, \cdots, x_n$ Variable ;

$b_1, b_2, \cdots, b_m$ are Constant

The above system can be written in the form :-

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \Rightarrow A \cdot X = B$$

To solve the above system we have two to types of method :-

1- The direct methods :

A- The matrix Inversion method.

B- The Gauss Elimination method.

C- The Gauss-Jordan Elimination Method.

2- The indirect Methods :-

A- Jacobi's Method.

B- Gaus-Seidel Method.

C- Relaxation Method.

A- The matrix Inversion Method :-

$\because A \cdot X = B \quad \therefore X = A^{-1} \cdot B$

The inverse of $(A) \Rightarrow \quad A^{-1} = \dfrac{Adj(A)}{|A|} \; ; |A| \neq 0$

EX. Use the matrix inversion method to solve the following Linear equation :-

$$2X_1 + 4X_2 - 8X_3 = 6$$
$$-X_1 - 3X_2 + 6X_3 = 4$$
$$5X_1 + 7X_2 - 2X_3 = 24$$

Solution :

$$\begin{bmatrix} 2 & 4 & -8 \\ -1 & -3 & 6 \\ 5 & 7 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 24 \end{bmatrix}$$

$$|A| = 2\begin{vmatrix} -3 & 6 \\ 7 & -2 \end{vmatrix} - 4\begin{vmatrix} -1 & 6 \\ 5 & -2 \end{vmatrix} + (-8)\begin{vmatrix} -1 & -3 \\ 5 & 7 \end{vmatrix} = -24 \neq 0$$

To find A^{-1}, form the matrix $[A|I]$ and change it to $[I|B]$ as follows

$$\begin{bmatrix} 2 & 4 & -8 \\ -1 & -3 & 6 \\ 5 & 7 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \longrightarrow New\, R_1 = \frac{R_1}{2}$$

$$\begin{bmatrix} 1 & 2 & -4 \\ -1 & -3 & 6 \\ 5 & 7 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[NR_3 = R_3 + R_1(-5)]{NR_2 = R_2 + R_1}$$

$$\begin{bmatrix} 1 & 2 & -4 \\ 0 & -1 & 2 \\ 0 & -3 & 18 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & +1 & 0 \\ -5/2 & 0 & 1 \end{bmatrix} \longrightarrow NR_2 = \frac{R_2}{-1}$$

$$\begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & -2 \\ 0 & -3 & 18 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ -\frac{1}{2} & -1 & 0 \\ -5/2 & 0 & 0 \end{bmatrix} \xrightarrow[NR_3 = R_3 + R_2(3)]{NR_1 = R_1 + R_2(-2)}$$

$$\left[\begin{array}{ccc}1 & 0 & 0\\ 0 & 1 & -2\\ 0 & 0 & 12\end{array}\right]\left[\begin{array}{ccc}\frac{3}{2} & 2 & 0\\ -12 & -1 & 0\\ -4 & -3 & 1\end{array}\right] \longrightarrow NR_3 = \frac{R_3}{12}$$

$$\left[\begin{array}{ccc}1 & 0 & 0\\ 0 & 1 & -2\\ 0 & 0 & 1\end{array}\right]\left[\begin{array}{ccc}\frac{3}{2} & 2 & 0\\ -\frac{1}{2} & -1 & 0\\ -\frac{1}{3} & -\frac{1}{4} & \frac{1}{12}\end{array}\right] \longrightarrow R_2 = R_2 + R_3(2) \Rightarrow I|A^{-1}$$

$$\left[\begin{array}{ccc}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right]\left[\begin{array}{ccc}\frac{3}{2} & 2 & 0\\ -\frac{7}{6} & -\frac{3}{2} & \frac{1}{6}\\ -\frac{1}{3} & -\frac{1}{4} & \frac{1}{12}\end{array}\right]$$

Now, we have the inverse matrix of A

Thus $X = A^{-1} * B$

$$\begin{bmatrix}X_1\\ X_2\\ X_3\end{bmatrix} = \begin{bmatrix}3/2 & 2 & 0\\ -7/6 & -3/2 & 1/6\\ -1/3 & -1/4 & 1/12\end{bmatrix} * \begin{bmatrix}6\\ 4\\ 24\end{bmatrix}$$

$\therefore X_1 = \frac{3}{2} * 6 + 2 * 4 + 0 * 24 = 9 + 8 + 0 = 17$

$\therefore X_1 = 17$, $X_2 = -9$, $X_3 = -1$

B_ Gauss Elimination Method :-

- Form the matrix $[a_{ij} | b_i]$ $\quad i = 1, 2, \ldots, n$ $\quad j = 1, 2, \ldots, n$
- We will get an upper-triangular matrix.

EX. Find the solution of the following set of simultaneous equtions, using the Gauss Elimintion method work 4D

$2.37\,X_1 + 3.06\,X_2 - 4.28\,X_3 = 1.76$

$1.46\,X_1 - 0.78\,X_2 + 3.75\,X_3 = 4.69$

$-3.6\,X_1 + 5.13\,X_2 - 1.06\,X_3 = 5.74$

Solution:
$$\left[\begin{array}{ccc|c}2.37 & 3.06 & -4.28 & 1.76\\ 1.46 & -0.78 & 3.75 & 4.69\\ -3.69 & 5.13 & -1.06 & 5.74\end{array}\right] \Rightarrow \begin{array}{l}New\,R_2 = R_2 - R_1\frac{a_{21}}{a_{11}}\\ New\,R_3 = R_3 - R_1\frac{a_{31}}{a_{11}}\end{array}$$

$$\left[\begin{array}{ccc|c} 2.37 & 3.06 & -4.28 & 1.76 \\ 0 & -2.6650 & 6.3865 & 3.6058 \\ 0 & 9.8944 & -5.604 & 8.4803 \end{array}\right] \Rightarrow NR_3 = R_3 - R_2 * \frac{a_{32}}{a_{22}}$$

$$\left[\begin{array}{ccc|c} 2.37 & 3.06 & -4.28 & 1.76 \\ 0 & -2.665 & 6.3865 & 3.6058 \\ 0 & 0 & 18.1072 & 21.8676 \end{array}\right]$$

$\therefore X_3 = \dfrac{21.8676}{18.1072} = 1.2077$; $X_2 = (3.6058 - 6.3865 * 1.2077)/-2.66$

$\therefore X_2 = 1.5412$

$\therefore X_1 = (1.76 - 3.06 * 1.5412 - (-4.28) * 1.2077)/2.37 = 0.9337$

C- Gauss - Jordan Elimination Method :-

- Form the matrix [A|B], and by same elimination steps chang the matrix to [I|B].

EX. Solve the following linear equations using Gauss-Jordan method.

$$2X_1 + 3X_2 - X_3 = 1$$
$$4X_1 + 4X_2 - 3X_3 = 17$$
$$-2X_1 + 3X_2 - X_3 = -1$$

Solution

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 11 \\ 4 & 4 & -3 & 17 \\ -2 & 3 & -1 & -1 \end{array}\right] \longrightarrow New\ R_1 = \frac{R_1}{a_{11}} = \frac{R_1}{2}$$

$$\left[\begin{array}{ccc|c} 1 & 1.5 & 0.5 & 5.5 \\ 4 & 4 & -3 & 17 \\ -2 & 3 & -1 & -1 \end{array}\right] \Longrightarrow \begin{array}{l} NR_2 = R_2 - a_{21}R_1 = R_2 - 4R_1 \\ NR_3 = R_3 - a_{31}R_1 = R_3 + 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1.5 & -0.5 & 5.5 \\ 0 & -2 & -1 & -5 \\ 0 & 6 & -2 & 10 \end{array}\right] \qquad NR_2 = \frac{R_2}{a_{22}} = \frac{R_2}{-2}$$

$$\left[\begin{array}{ccc|c} 1 & 1.5 & -0.5 & 5.5 \\ 0 & 1 & 0.5 & 2.5 \\ 0 & 6 & -2 & 10 \end{array}\right] \qquad \begin{array}{l} NR_1 = R_1 - a_{12} R_2 = R_1 - 1.5 R_2 \\ NR_3 = R_3 - a_{32} R_2 = R_3 - 6 R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1.25 & 1.75 \\ 0 & 1 & 0.5 & 2.5 \\ 0 & 0 & -5 & -5.0 \end{array}\right] \qquad NR_3 = \frac{R_3}{a_{33}} = \frac{R_3}{-5}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1.25 & 1.75 \\ 0 & 1 & 0.5 & 2.5 \\ 0 & 0 & 1 & 1 \end{array}\right] \qquad \begin{array}{l} NR_1 = R_1 - a_{13} R_3 = R_1 + 1.25 R_3 \\ NR_2 = R_2 - a_{23} R_3 = R_2 - 0.5 R_3 \end{array}$$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \qquad \therefore X_1 = 3 \ ; \ X_2 = 2 \ ; \ X_3 = 1$$

2- The Indirect Methods :-

In this method we have a sufficient condition for a solution to be found which is :-

$$|a_{ij}| > \sum_{\substack{i=1 \\ j \neq i}}^{n} |a_{ij}| \quad , \quad i = 1, 2, \ldots, n$$

A- Jacobi's Method :-

Ex. Solve the following set of linear equation using the Jacobi's method.

$$5X_1 - 2X_2 + X_3 = 4$$
$$X_1 + 4X_2 - 2X_3 = 3$$
$$X_1 + 2X_2 + 4X_3 = 17$$

Solution :-

$$|5| > |-2| + |1| \Rightarrow 5 > 3$$
$$|4| > |1| + |-2| \Rightarrow 4 > 3$$
$$|4| > |1| + |2| \Rightarrow 4 > 3$$

So we have

$$X_1^{k+1} = \left(\frac{4}{5} + \frac{2}{5} X_2^{k} - \frac{1}{5} X_3^{k}\right) \quad \text{--- (1)}$$
$$X_2^{k+1} = \left(\frac{3}{4} - \frac{1}{4} X_1^{k} + \frac{1}{2} X_3^{k}\right) \quad \text{--- (2)}$$
$$X_3^{k+1} = \left(\frac{17}{4} - \frac{1}{4} X_1^{k} - \frac{1}{2} X_2^{k}\right) \quad \text{--- (3)}$$

assume $X_1^{0} = 0$; $X_2^{0} = 0$; $X_3^{0} = 0$ and substuting this values in the Last three equation then we will have $X_1^{(1)}$; $X_2^{(1)}$; $X_3^{(1)}$ and So on X_1^{k} ; X_2^{k} ; $X_3^{(k)}$.

i	1	2	3	4	5	6	7	8	9	10
X_1	0.8	0.25	1.14	1.24	1.02	0.92	0.98	1.02	1.01	0.99
X_2	0.75	2.68	2.53	1.89	1.79	1.99	2.07	2.02	1.98	1.99
X_3	4.25	3.68	2.85	2.70	2.99	3.10	3.02	2.97	2.98	3.01

accuracy : we must satisfied the accuracy condition

$$|X_i^{k+1} - X_i^{k}| < e \quad ; \quad i = 1, 2, 3, \ldots$$

B- Gauss - Seidel Method :-

EX. Solve the following set of Linear equation using the Gauss - Seidel method.

$$5X_1 - 2X_2 + X_3 = 4$$
$$X_1 + 4X_2 - 2X_3 = 3$$
$$X_1 + 2X_2 + 4X_3 = 17$$

If $\lambda = 1$ It is Gauss - seidel
If $0 < \lambda < 1$ It is called under relaxation
If $1 < \lambda < 2$ It is called over relaxation

Ex.
Solve the following set of Linear equations using over relaxation with $\lambda = 1.1$

$$10X_1 + X_2 + X_3 = 12$$
$$X_1 + 10X_2 + X_3 = 12$$
$$X_1 + X_2 + 10X_3 = 12$$

we begin our solution by first checking the diagonal coefficients:

$$|10| > |1| + |1| \Rightarrow 10 > 2$$
$$|10| > |1| + |1| \Rightarrow 10 > 2$$
$$|10| > |1| + |1| \Rightarrow 10 > 2$$

So we have $X_1^{(k+1)} = 1.2 - 0.1 X_2^{(k)} - 0.1 X_3^{(k)}$

$$X_1^{(k+1)*} = \lambda X_1^{(k+1)} + (1-\lambda) X_1^{(k)}$$

and $X_2^{k+1} = 1.2 - (0.1) X_1^{(k+1)*} - (0.1) X_3^{(k)}$

$$X_2^{(k+1)*} = \lambda X_2^{(k+1)} + (1-\lambda) X_2^{k}$$

also $X_3^{(k+1)} = 1.2 - (0.1) X_1^{(k+1)*} - (0.1) X_2^{(k+1)*}$

$$X_3^{(k+1)*} = \lambda X_3^{(k+1)} + (1-\lambda) X_3^{(k)}$$

Now assuming an initial value of $X_2 = X_3 = 0$

so $X_1^{(1)} = 1.2$; $X_1^{(1)*} = \lambda X_1^{(1)} + (1-\lambda) X_1^{(0)}$

$$X_1^{(1)*} = (1.1)(1.2) + (1-1.1)(0) = 1.32$$

Solution : We begin our solution by checking

$|5| > |-2| + |1| \Rightarrow 5 > 3$

$|4| > |+1| + |-2| \Rightarrow 4 > 3$

$|4| > |1| + |2| \Rightarrow 4 > 3$

$$X_1^{k+1} = \frac{4}{5} + \frac{2}{5} X_2^{k} - \frac{1}{5} X_3^{k} \quad \text{--- (1)}$$

$$X_2^{k+1} = \frac{3}{4} - \frac{1}{4} X_1^{k+1} + \frac{1}{2} X^{k} \quad \text{--- (2)}$$

$$X_3^{k+1} = \frac{17}{4} - \frac{1}{4} X_1^{k+1} - \frac{1}{2} X_2^{k+1} \quad \text{--- (3)}$$

assume $X_2^{(0)} = 0$ and $X_3^{(0)} = 0$ and Sub. into eq. (1,2,3)

$X_1^{(1)} = 0.8 \Rightarrow$ subst. in eq. (2)

$X_2^{(1)} = 0.75 - 0.2\,(0.8) + 0 = 0.55$ Sub. in eq. (3)

$X_3^{(1)} = 4.25 - 0.25\,(0.8) - 0.5\,(0.55) = 3.775$

and go on until $|X_i^{k+1} - X_i^{k}| < e$

So we will have the following values :-

i	1	2	3	4	5	6	7
X_1	0.8	0.265	1.249	0.956	1.002	1.001	0.999
X_2	0.55	2.571	1.887	2.008	2.003	1.999	2.000
X_3	3.775	2.898	2.994	3.007	3.007	3.000	3.000

C- Relaxation Method :-

After each new value of (x) is computed using Gauss-Seidel method that value is modified by

$$X_i^{(new)*} = \lambda X_i^{new} + (1-\lambda) X_i^{old}$$

where (λ) is corrected term its value $0 < \lambda < 2$

Now $X_2^{(1)} = 1.2 - (0.1)(1.32) - (0.1)(0) = 1.068$

$$X_2^{(1)*} = \lambda X_2^{(1)} + (1-\lambda) X_2^{(0)}$$
$$= (1.1)(1.068) + (1-1.1)(0) = 1.1748$$

Now $X_3^{(1)} = 1.2 - (0.1)(1.1748) - (0.1)(1.32) = 0.95052$

$$X_3^{(1)*} = \lambda X_3^{(1)} - (1-\lambda) X_3^{(0)}$$
$$= (1.1)(0.95052) + (1-1.1)(0) = 1.04572$$

Thus we get $X_1^{(1)*} = 1.32$

$X_2^{(1)*} = 1.1748$

$X_3^{(1)*} = 1.0457$

i Iteration	1	2	3	4	5
X_1	1.32	0.955	1.005	0.996	1.000
X_2	1.1748	0.9931	1.000	1.001	1.000
X_3	1.0456	1.001	0.9993	1.001	1.000

Q_1 :- Solve the following system of linear equation by Gauss elimintion method :-

$$X_1 - X_2 + 3X_3 = 10$$
$$2X_1 + 3X_2 + X_3 = 15$$
$$4X_1 + 2X_2 - X_3 = 6$$

Ans. $X_1 = 1$; $X_2 = 3$; $X_3 = 4$

Q_2 :- Solve the following system ~~of~~ linear eq. by :-

1- Gauss-Seidel $(\lambda = 1)$ 2- Relaxation method $(\lambda = 1.2)$

3- Relaxation method $(\lambda = 1.7)$.

①
$$x - 3y + 2z = 1$$
$$2x - 2y = k^2$$
$$3x - 5y + z = 0$$
$$-2x + 8y + 4z = 49$$

$k = \mp 2$

②
$$10x_1 + x_2 + 2x_3 = 44$$
$$2x_1 + 10x_2 + x_3 = 51$$
$$x_1 + 2x_2 + 10x_3 = 61$$

Q3 :- Solve the System of linear algebraic equation G.E.M.

$$x - y + z - 2w = -1$$
$$-2x + 2y - z + 2w = 3$$
$$3x - 3y + 2z - 4w = -4$$
$$-4x + 4y - 3z + 6w = 5$$

Q4 :- Solve the following system of equation by Gauss - Seidel iteration method working to $e = 0.0001$

$$10.27\,\delta_1 - 1.23\,\delta_2 + 0.67\,\delta_3 = 4.27$$
$$2.39\,\delta_1 - 12.65\,\delta_2 + 1.13\,\delta_3 = 1.26$$
$$1.79\,\delta_1 + 3.61\,\delta_2 + 15.11\,\delta_3 = 12.71$$

Ans. $\delta_1 = 0.3693$

$\delta_2 = 0.0405$

$\delta_3 = 0.7878$

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CRANFIELD INSTITUTE OF TECHNOLOGY

DEPARTMENT OF MATHEMATICS

Supplementary Mathematics Notes

Matrix Algebra and the Theory of Linear Equations

1. Elementary Matrix Algebra

Matrix algebra is directly applicable to the solution of simultaneous linear equations and has many other applications including the analysis of vibrating mechanical systems, electric network theory, geometry, statistics.

1.1 Basic Definitions:

An $m \times n$ matrix is a rectangular array of mn elements (usually numbers) arranged in m rows and n columns.

Conventionally a matrix is denoted by a capital letter [e.g. A] its elements being denoted by the corresponding small letter together with a pair of suffices to indicate the position of the element in the matrix, the first of these suffices referring to the row, the second to the column [e.g. a_{23} denotes the element is the 2nd row and 3rd column of matrix A].

A 'typical' element is usually denoted a_{ij} signifying a element in the ith row and jth column of A where i, j are arbitrary.

Example:

$$A = \begin{pmatrix} 2 & 1 & 2 & 3 \\ 3 & -1 & 2 & -5 \\ 4 & 1 & 0 & 2 \end{pmatrix}$$

is a 3×4 matrix with $a_{13} = 2$, $a_{31} = 4$, etc.

Two matrices are said to be equal if they are of the same size and if all corresponding pairs of elements are equal. More precisely this can be stated: $A = B$ if A and B are of the same size and if $a_{ij} = b_{ij}$ for all i,j. Matrices of different sizes are not comparable.

1.2 Addition of Matrices

If A and B are both $m \times n$ matrices we define a matrix $C = A + B$ by the equations $c_{ij} = a_{ij} + b_{ij}$, for all $i = 1 \dots m$, all $j = 1 \dots n$

Example: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 2 \end{pmatrix}$ $\quad B = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \end{pmatrix}$

$$A + B = \begin{pmatrix} 3 & 2 & 4 \\ 3 & 7 & 5 \end{pmatrix}$$

n.b. Addition is only defined between matrices of the same size.

From the definition the algebraic properties listed below can be quickly proved.

(1) $A + B = B + A$

(2) $A + (B + C) = (A + B) + C$

(3) $A + 0 = A$ where 0 is the $m \times n$ matrix having every element zero.

(4) Given any $m \times n$ matrix A there is a matrix $-A$ such that $A + (-A) = 0$.

Specimen Proof:

To prove any of the above properties it is necessary to establish 3 things (a) that each side of the equation does in fact exist (b) that the matrices obtained on each side are of the same size and hence comparable (c) that the corresponding pairs of elements in these matrices are equal.

To Prove (1):

Let $A + B = C$, $B + A = D$.

Since A, B are both $m \times n$ matrices $A + B$ is defined and is an $m \times n$ matrix.

Similarly $B + A$ is a well defined $m \times n$ matrix. Hence C and D both exist and are comparable.

$$c_{ij} = a_{ij} + b_{ij} = b_{ij} + a_{ij} = d_{ij} \qquad \text{for all } i, j$$

hence finally $C = D$.

1.3 Multiplication by a Scalar

Let A be an $m \times n$ matrix and λ be a scalar (or number) an $m \times n$ matrix $B = \lambda A$ is then defined by the equations $b_{ij} = \lambda a_{ij}$ for all i, j.

Example: $A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 3 & 7 \end{pmatrix}$ $\quad \lambda A = \begin{pmatrix} 2\lambda & 3\lambda & \lambda \\ \lambda & 3\lambda & 7\lambda \end{pmatrix}$

In particular taking $\lambda = 2$ we have $2A = \begin{pmatrix} 4 & 6 & 2 \\ 2 & 6 & 14 \end{pmatrix}$

From the definition the following algebraic properties can be easily verified:

(1) $\lambda(A + B) = \lambda A + \lambda B$

(2) $(\lambda + \mu)A = \lambda A + \mu A$

(3) $0.A = 0$

(4) $\lambda(\mu A) = (\lambda\mu)A.$

{it is assumed throughout that A and B are $m \times n$ matrices λ, μ are scalars}.

1.4 The Product of Two Matrices

The definitions given for equality, addition and scalar multiplication of matrices will appear to be elementary and obvious. It might seem natural to define the product of two matrices in a similar way [i.e. by multiplying together corresponding pairs of elements]. Unfortunately, although this would offer the simplest definition it would be of very limited practical application and a definition suited to the practical application of matrices is preferred.

Suppose we have variables x_1, x_2; y_1, y_2, y_3; z_1, z_2 related by the following linear equations:

$$x_1 = a_{11}y_1 + a_{12}y_2 + a_{13}y_3 \qquad y_1 = b_{11}z_1 + b_{12}z_2$$

$$x_2 = a_{21}y_1 + a_{22}y_2 + a_{23}y_3 \qquad y_2 = b_{21}z_1 + b_{22}z_2$$

$$y_3 = b_{31}z_1 + b_{32}z_2$$

where a_{ij}, b_{ij} are constants for all i,j.
Then clearly the relationship between the variables $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

is determined by the 2×3 matrix A.

The relationship between Y and the variables $Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ is

determined by the 3×2 matrix B.

This could be expressed more briefly by writing:

$$X = AY, \quad Y = BZ.$$

There is an implicit relationship between the variables x_1, x_2 and the variables z_1, z_2; suppose this is expressed in terms of a matrix C as $X = CZ$.

It would be convenient if the matrix product was defined in such a way that $C = AB$.

Carrying out the necessary linear substitutions to express X in terms of Z suggests the method of defining a matrix product.

Substituting for y_1, y_2, y_3 into the expressions for x_1 , x_2 gives:

$$x_1 = a_{11}(b_{11}z_1 + b_{12}z_2) + a_{12}(b_{21}z_1 + b_{22}z_2) + a_{13}(b_{31}z_1 + b_{12}z_2)$$

$$x_2 = a_{21}(b_{11}z_1 + b_{12}z_2) + a_{22}(b_{21}z_1 + b_{22}z_2) + a_{23}(b_{31}z_1 + b_{32}z_2)$$

these can be simplified and re-arranged to give:

$$x_1 = (a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31})z_1 + (a_{21}b_{12} + a_{12}b_{22} + a_{13}b_{32})z_2$$

$$x_2 = (a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31})z_1 + (a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32})z_2$$

These equations should be compared with:

$$x_1 = c_{11}z_1 + c_{12}z_2$$

$$x_2 = c_{21}z_1 + c_{22}z_2$$

giving for example:

$$c_{12} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}$$

$$c_{22} = a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}$$

From this we obtain the more general relationship:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} \qquad \text{for} \quad \begin{matrix} i = 1,2 \\ j = 1,2. \end{matrix}$$

In this way we define the product of the 2×3 matrix A and the 3×2 matrix B to give the 2×2 matrix $C = AB$.

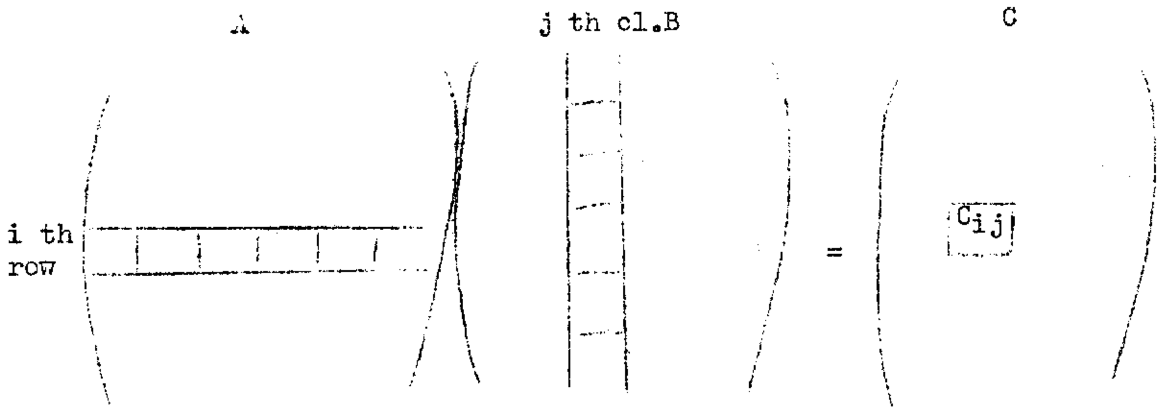
Formal Definition of Matrix Product

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix the product $C = AB$ is then an $m \times p$ matrix defined by the equations:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

$$\text{for } i = 1 \dots m, \; j = 1 \dots p.$$

Note. The element c_{ij} in the i th row and j th column of the product AB is obtained by multiplying together in pairs and adding the elements of the i th row of A and the elements of the j_{th} column of B. c_{ij} can be considered as the 'product' of the i th row of A and the j th column of B.



The product AB is only defined if the number of columns of A is equal to the number of rows of B. If this condition is satisfied AB has the same number of rows as A and the same number of columns as B.

<u>Example</u>:

(i) $A = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 0 \\ 2 & 3 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 1 \\ -3 & 2 \\ 0 & -5 \end{pmatrix}$ $AB = \begin{pmatrix} -5 & -12 \\ -4 & 5 \\ -5 & 3 \end{pmatrix}$

(ii) $C = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ $D = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$ $CD = \begin{pmatrix} 11 & 5 \\ 9 & 5 \end{pmatrix}$

$DC = \begin{pmatrix} 2 & 9 \\ 2 & 14 \end{pmatrix}$

(iii) $E = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 4 & 6 \end{pmatrix}$ $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $EI = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 4 & 6 \end{pmatrix}$

1.5 <u>Properties of Matrix Products</u>

(1) $(A\,B)C = A(B\,C)$, where A is any $m \times n$ matrix, B any $n \times p$ matrix and C any $p \times q$ matrix.

(2) $A(B + C) = AB + AC$ where A is any $m \times n$ matrix, B and C are $n \times p$ matrices.

(3) $(D + E)A = DA + EA$ where D, E are $1 \times m$ matrices and A is an $m \times n$ matrix.

(4) $I_m A = A$, $AI_n = A$, where A is an $m \times n$ matrix, I_m is the $m \times m$ identity matrix

$$I_m = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

and I_n is the $n \times n$ identity matrix.

(5) $A.0 = 0$ where A is an $m \times n$ matrix the zero on the left is the $n \times p$ zero matrix and the zero on the right is the $m \times p$ zero matrix.

(6) $A(\lambda B) = \lambda(AB) = (\lambda A)B$ for any scalar λ.

Specimen Proofs.

These properties can be proved directly from the definition of the product of two matrices. In each case the existence and the equality of the two sides must be separately established.

(2) Suppose A is an $m \times n$ matrix, B and C are $n \times p$ matrices.

Then $B + C = D$ is a well defined $n \times p$ matrix and $A(B + C) = AD$ is a well defined $m \times p$ matrix.

Similarly $AB = F$, $AC = G$ are well defined $m \times p$ matrices and $H = F + G$ is also an $m \times p$ matrix.

Hence the two sides of the equation are comparable.

Let $A(B + C) = E = AD$.

Then $e_{ij} = \sum_{k=1}^{n} a_{ik}d_{kj} = \sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj})$

$= \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj} = f_{ij} + g_{ij}$ {from AB, AC}

$= h_{ij}$ {since $H = F + G$}.

Hence $E = H$ or $A(B + C) = AB + AC$.

(4) To prove $I_m A = A$.

I_m is properly defined as $I_m = \{\delta_{ij}\}$, the $m \times m$ matrix whose typical element is δ_{ij}. δ_{ij} [the Kronecker Delta] being defined as:

$$\delta_{ij} = 1 \quad \text{if } i = j$$

$$\delta_{ij} = 0 \quad \text{if } i \neq j.$$

Since I_m is an $m \times m$ matrix and A is an $m \times n$ matrix $I_m A$ is a well defined $m \times n$ matrix.

Let $I_m A = B$

Then $b_{ij} = \sum_{k=1}^{m} \delta_{ik} a_{kj} = a_{ij}$ {since $\delta_{ik} = 0$ except for $k = i$ when $\delta_{ik} = 1$}.

Thus $B = A$ or $I_m A = A$.

Similarly $AI_n = A$.

It is usual to drop the suffices m, n to give the equations $IA = A$, $AI = A$ it being understood that the appropriate sized identity matrix is chosen in each case.

Exceptional Properties of Matrices

(a) $AB \neq BA$

This is to say that in general AB will not be equal to BA. Unless A,B are both square matrices of the same size AB and BA will not be comparable {they need not even both exist, see example (i)}.

In the special case of A and B both being $n \times n$ matrices it is still not generally true that $AB = BA$. A counter example to this is provided by example (ii).

If in fact $AB = BA$, this is a special property of the pair of matrices which is expressed by saying that A and B commute.

In particular all square matrices commute with the identity matrix and with the $n \times n$ zero matrix.

Since matrices do not in general commute a great deal of care must be taken when manipulating matrix products to keep the matrices in the correct order.

e.g. $(A + B)^2 = A^2 + AB + BA + B^2 \neq A^2 + 2AB + B^2$

$$ABA \neq A^2B$$

(b) $AB = 0 \not\Rightarrow A = 0$ or $B = 0$

Real numbers have the elementary property that if a, b are numbers such that $a.b = 0$ then either $a = 0$ and/or $b = 0$. This is expressed symbolically as $ab = 0 \Rightarrow a = 0$ or $b = 0$.

It is certainly true for matrices that $A.0 = 0$ and $0.B = 0$ but as the example below shows it is also possible to have $AB = 0$ with A and B both non zero.

<u>Example:</u>

Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ -3 & -6 & -3 \end{pmatrix}$ $\quad B = \begin{pmatrix} -1 & 2 & 1 \\ 1 & -2 & -1 \\ -1 & 2 & 1 \end{pmatrix}$

Then $AB = 0$. {but note that $BA \neq 0$ in this case}.

As a consequence of this property it is not always possible to conclude that if $AB = AC$ then $B = C$.

{$AB = AC \Rightarrow A(B - C) = 0$ but this may be possible with $B - C \neq 0$}.

1.6 <u>Non Singular Matrices</u>

A square n $\times$ n matrix A for which it is possible to find an <u>inverse matrix</u> A^{-1} with the properties $AA^{-1} = I$, $A^{-1}A = I$ is said to be <u>non-singular</u>.

If A is a non singular matrix the cancellation law $AB = AC \Rightarrow B = C$ is valid since we have:

$$\begin{aligned} & AB = AC \\ \Rightarrow\ & A^{-1}(AB) = A^{-1}(AC) \\ \Rightarrow\ & (A^{-1}A)B = (A^{-1}A)C && \text{\{by property (1)\}} \\ \Rightarrow\ & IB = IC \\ \Rightarrow\ & B = C && \text{\{by property (4)\}.} \end{aligned}$$

<u>Example:</u>

Let $A = \begin{pmatrix} 2 & 4 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{pmatrix}$ $\quad B = \begin{pmatrix} -2 & -1 & 3 \\ 1 & 0 & -1 \\ 1 & 2 & -2 \end{pmatrix}$

Then $AB = I$ and $BA = I$.

Hence both A and B are non singular with $B = A^{-1}$ and $A = B^{-1}$.

By no means all square matrices have inverses for example

$C = \begin{matrix} 1 & 2 \\ 2 & 4 \end{matrix}$ can have no inverse,

since $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

and the existence of an inverse for C would imply that

$$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which is clearly false.

There are a number of methods for determining whether a given matrix is non singular and if so finding the inverse but those will be considered after the theory of linear equations.

1.7 The Transpose of a Matrix

Defn. Let A be an $m \times n$ matrix the transpose of A, denoted A', is an $n \times m$ matrix obtained from A by interchanging the rows and columns. More precisely if α_{ij} is the element in the i th row and j th column of A' then $\alpha_{ij} = a_{ji}$ for all i, j.

A square matrix A such that $A' = A$ is said to be symmetric.

A square matrix B such that $B' = -B$ is said to be skew-symmetric.

Examples

$$A = \begin{pmatrix} 5 & 1 & 2 \\ 2 & 3 & 4 \end{pmatrix} \qquad A' = \begin{pmatrix} 5 & 2 \\ 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 2 & 5 \\ -1 & 5 & 4 \end{pmatrix} \text{ is symmetric} \qquad C = \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{pmatrix} \text{ is skew-symmetric}$$

Properties of the Transpose

(1) $(A + B)' = A' + B'$

(2) $(AB)' = B'A'$

Exercises on Elementary Matrix Algebra

(1) $A = \begin{pmatrix} 2 & -1 & 2 \\ 3 & 4 & 7 \\ 2 & 1 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 5 & 1 \\ 4 & 3 \\ -3 & -1 \end{pmatrix}$ $C = \begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}$

Calculate AB and BC and verify that $(AB)C = A(BC)$.

(2) By assuming the existence of an inverse of the form $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$

where x,y,z,t are to be determined, find the inverse of the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

Hence solve the equations $2a + b = 7$, $3a + 2b = 4$,
and find a matrix X such that $XA = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

(3) Expand the following expressions where A and B are $n \times n$ matrices which do not commute.

(i) $(A + B)(A - B)$

(ii) $(A - B)^3$

(iii) $(I + A)^3$

(4) If A is any $m \times n$ matrix and B is any $n \times p$ matrix prove that $(AB)' = B'A'$.

(5) A matrix P is said to be orthogonal if it satisfies the equation $PP' = I$. Show that the matrix

$$P = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$ is orthogonal.

(6) $$A = \begin{pmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{pmatrix}$$

find the values of A^2 and A^3.

Deduce the values of A^{17} and A^{123}

2. Theory of Linear Equations

The system:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = c_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = c_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = c_n$$

of m linear equations in n unknowns can be written in matrix form as $AX = C$ where A is the $m \times n$ matrix of coefficients X is the $n \times 1$ matrix of column vector $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and C is the constant column vector

$$C = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$$

The solutions of the above system are closely linked to the related homogeneous system $AX = 0$. This relationship is established in the first two theorems below.

2.1 Theorem 1

If $X = Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ is any solution to the homogeneous set of the equations $AX = 0$, then $X = \lambda Y$ is also a solution for any value of the constant λ.

If $X = Z$ is a further solution so is $X = Y + Z$.

Proof

Since $X = Y$ is a solution we have $AY = 0$.

Substituting $X = \lambda Y$ into the equation gives

$$AX = A\lambda Y = \lambda AY = \lambda . 0 = 0$$

showing that λY is a solution for all values of λ.

Since $X = Y$ and $X = Z$ are both solutions we have $AY = 0$ and $AZ = 0$ substituting $X = Y + Z$ gives:

$$AX = A(Y + Z) = AY + AZ = 0 + 0 = 0$$

hence $X = Y + Z$ is a solution.

Note The equation $AX = 0$ is always soluble since $X = 0$ is a solution. This is called the trivial solution, any solution other than $X = 0$ is called a non trivial solution. An immediate consequence of the above theorem is that if the homogeneous equations $AX = 0$ have a non trivial solution then they have an infinite number of distinct solutions.

2.2 Theorem 2

If $X = Y$ is any particular solution of the non homogeneous equations $AX = C$ and if $X = Z$ is a non trivial solution of the related homogeneous equations $AX = 0$ then a more general solution of the non homogeneous equations is $X = Y + \lambda Z$ for any value of λ.

Proof

Since $X = Y$ is a solution of $AX = C$ then $AY = C$

Since $X = Z$ is a solution of $AX = 0$ then by Th 1 so is $X = \lambda Z$.

Substituting $X = Y + \lambda Z$ into the equations gives $AX = A(Y + \lambda Z)$

$$= AY + A(\lambda Z) = C + 0 = C.$$

Hence $Y + \lambda Z$ is a more general solution.

Converse to Theorem 2.

If the equations $AX = C$ have two distinct solutions then corresponding homogeneous equations have a non trivial solution.

Proof

Let $X = Y$ and $X = Z$ be two distinct solutions to $AX = C$.

Let $X = Y - Z$ then since these solutions are distinct $Y - Z \neq 0$.

$AX = A(Y - Z) = AY - AZ = 0 - 0 = 0$, verifying that $X = Y - Z$ is the required non-trivial solution.

Deductions from Theorems 1 and 2.

Consider the equation $AX = C$ where A is an $m \times n$ matrix. As a consequence of theorems 1 and 2 there are 3 mutually distinct possibilities for the existance of solutions to these equations:

(i) $AX = C$ has a unique solution, in this case $AX = 0$ has only the trivial solution.

(ii) $AX = C$ has no solution or the equations are incompatible

(iii) $AX = C$ has an infinite number of solutions and $AX = 0$ has a non-trivial solution.

From this it can be seen that it is of critical importance to know whether or not $AX = 0$ has a non trivial solution.

2.3 Theorem 3

The set of m homogeneous equations in n unknowns $AX = 0$ always has a non trivial solution if $m < n$.

Proof

The proof is by induction on m.

When $m = 1$ the set of equations is:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$

in this case we can assume all coefficients are non zero (otherwise the corresponding variable would not appear).

Let $x_1 = a_{12}$, $x_2 = -a_{11}$, $x_3 = \dots = x_n = 0$.

This gives a non trivial solution.

Suppose the theorem is true for all sets of $m - 1$ equations in a greater number of unknowns.

Consider the equations: where $n > m$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned}$$

in at least one of these equations x_n must appear with a non zero coefficient, we may suppose $a_{mn} \neq 0$ {otherwise the order of the equations may be altered as necessary}.

$$\text{Let } x_n = -\frac{1}{a_{mn}}\left(a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn-1}x_{n-1}\right).$$

For any values of $x_1, x_2, \dots x_n$, the final equation will then be satisfied. Substitution into the preceding $m - 1$ equations gives:

$$\begin{aligned} &\left(a_{11} - a_{1n}\frac{a_{m1}}{a_{mn}}\right)x_1 + \left(a_{12} - \frac{a_{1n}a_{m2}}{a_{mn}}\right)x_2 + \dots + \left(a_{1n-1} - a_{1n}\frac{a_{mn-1}}{a_{mn}}\right)x_{n-1} = 0 \\ &\vdots \\ &\left(a_{m-11} - a_{m-1n}\frac{a_{m1}}{a_{mn}}\right)x_1 + \dots + \left(a_{m-1} - a_{m-1n}\frac{a_{mn-1}}{a_{mn}}\right)x_{n-1} = 0 \end{aligned}$$

a set of $m - 1$ equations in $n - 1$ unknowns.

But $m < n \Rightarrow m-1 < n-1$ and so by the induction hypothesis these equations have a non trivial solution.

Substitution of this into the expression for x_n gives a non trivial solution of the original m equations.

Hence by induction the theorem is proved.

Corollary

The set $AX = C$ of m non homogeneous equations in n unknowns can never have a unique solution if $m < n$.

2.4 Case of n equations in n unknowns

In this case the homogeneous equations $AX = 0$ may or may not have a non trivial solution depending upon the matrix of coefficients. If in particular the matrix A is non singular there is an inverse matrix A^{-1} and from $AX = 0$ we obtain $A^{-1}(AX) = (A^{-1}A)X = IX = 0$, giving $X = 0$ and showing that only the trivial solution is possible.

A simple example will suffice to show that on occasions n homogeneous equations in n unknowns can have a non trivial solution.

Examples

Equations $x_1 - x_2 = 0$, $x_1 + x_2 = 0$ have only the trivial solution

Equations $x_1 - 2x_2 = 0$, $-2x_1 + 4x_2 = 0$ have the non trivial solution $x_1 = 2$, $x_2 = 1$.

Considering related non homogeneous equations we have:

$$\begin{aligned} x_1 - x_2 &= c_1 \\ x_1 + x_2 &= c_2 \end{aligned} \qquad \text{Solution: } x_1 = \frac{c_1 + c_2}{2}, \quad c_2 = \frac{c_2 - c_1}{2}$$

which is unique for all values of c_1 and c_2.

$$\begin{aligned} x_1 - 2x_2 &= 5 \\ -2x_1 + 4x_2 &= 6 \end{aligned} \qquad \text{are insoluble}$$

$$\begin{aligned} x_1 - 2x_2 &= 2 \\ -2x_1 + 4x_2 &= -4 \end{aligned} \qquad \text{have solutions: } \begin{aligned} x_1 &= 4 + 2\lambda \\ x_2 &= 1 + \lambda \end{aligned} \quad \text{for any } \lambda.$$

Hence in this case the distinction between no solutions and an infinite number depends upon the constants on the right hand side of the equation. [In this example the coefficients in the 2nd equation are precisely $-2 \times$ coefficients in the first equation unless the constants are similarly related the equations will be incompatible].

In the case of 2 equations in 2 unknowns it is a relatively easy matter to spot incompatible equations but as the number of equations and unknowns increases this becomes very difficult unless a systematic method is used. Such a method of recognising incompatible equations and obtaining solutions of compatible equations is outlined below.

2.5 Use of Augmented Matrix and Row Operations to Solve Linear Equations.

The values of the solutions of the set $AX = C$ of m equations in n unknowns depends upon the coefficients and the constants $c_1, \ldots, c_m$ from the right hand side, the actual symbols used for the unknowns are irrelevant.

As a first step in the calculation the augmented matrix is introduced. This is the $m \times (n+1)$ matrix

$$A_1 = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & c_1 \\ a_{21} & a_{22} & & a_{2n} & c_2 \\ \vdots & & & & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} & c_m \end{array}\right)$$

This augmented matrix contains all the information necessary to solve the equations.

In elementary methods of solving linear equations it is usual to attempt to eliminate some of the unknowns by means of one or more of the following operations:

(i) Rearranging the order of the equations.

(ii) Multiplying (or dividing) any equation by a non zero constant.

(iii) Adding (or subtracting) any multiple of one equation from another.

In the augmented matrix the rows correspond to the equations, as a method of computation the following Elementary Row Operations are introduced:

(I) Interchange two rows.

(II) Multiply any row by a non zero constant.

(III) Add to any row a multiple of another.

If B_1 is obtained from A_1 by a sequence of elementary row operations we say B_1 is row-equivalent to A_1 or more simply B_1 is equivalent to A_1.

Since the row operations are reversible A_1 is also equivalent to B_1.

Equivalent augmented matrices will correspond to equivalent sets of equations having precisely the same set of solutions.

The basis of the method is to use row operations to reduce the augmented matrix A_1 to an equivalent matrix of a particularly simple form, called the row echelon form in which successive rows contain an increasing number of zeros in the leading positions. For an augmented matrix in echelon form it is an elementary matter to obtain the solutions or to determine that the equations are incompatible. The method is best illustrated by examples.

Examples:

(1) Solve, if possible, the equations:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ 2x_1 + 3x_2 - x_3 &= 2 \\ -x_1 + x_2 + 3x_3 &= -1. \end{aligned}$$

Corresponding augmented matrix is

$$A_1 = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 2 & 3 & -1 & 2 \\ -1 & 1 & 3 & -1 \end{array}\right)$$

The object is to reduce this matrix to row echelon form by the use of elementary row operations. This is done systematically by first obtaining zeros in all positions lower than the first in the first column, then obtaining zeros in the requisite places in the second column, etc. At each stage of the calculation a note is made of the row operations used.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 2 & 3 & -1 & 2 \\ -1 & 1 & 3 & -1 \end{array}\right) \sim \begin{array}{c} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -1 & 1 & -6 \\ 0 & 3 & 2 & 3 \end{array}\right)$$

$$\sim \begin{array}{c} \\ \\ R_3+3R_2 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -1 & 1 & -6 \\ 0 & 0 & 5 & -15 \end{array}\right)$$

The final matrix obtained corresponds to the equations:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -x_2 + x_3 &= -6 \\ 5x_3 &= -15. \end{aligned}$$

From this the solution is: $x_3 = -3$, $x_2 = 3$, $x_1 = -5$

Note As far as possible the first row is used as an operator when introducing zeros into the first column, the second row for the second column, etc. If at any stage there is a zero in the critical position it is necessary to interchange the order of the rows before continuing. Operations using the same row may be carried out simultaneously.

(2) Solve, if possible, the equations

$$\begin{aligned} x_1 - x_2 + 2x_3 - x_4 &= 1 \\ 2x_1 - 2x_2 + x_3 + 2x_4 &= 3 \\ 3x_1 - 2x_2 - 3x_3 + 4x_4 &= -1. \end{aligned}$$

Augmented matrix is:

$$A_1 = \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 1 \\ 2 & -2 & 1 & 2 & 3 \\ 3 & -2 & -3 & 4 & -1 \end{array}\right) \sim \begin{array}{c} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 1 \\ 0 & 0 & -3 & 4 & 1 \\ 0 & 1 & -9 & 7 & -4 \end{array}\right)$$

$$\sim R_2 - R_3 \quad \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 1 \\ 0 & 1 & -9 & 7 & -4 \\ 0 & 0 & -3 & 4 & 1 \end{array}\right)$$

In this case the final row corresponds to the equation $-3x_3 + 4x_4 = 1$, this has no unique solution, letting $x_4 = \lambda$ (arbitrary constant) gives

$$x_3 = \frac{4}{3}\lambda - \frac{1}{3}$$ substituting into the equations

corresponding to the first and second rows gives:

$$x_2 = 9x_3 - 7x_4 - 4 = 5\lambda - 7$$

$$x_1 = x_2 - 2x_3 + x_4 + 1 = \frac{10}{3}\lambda - \frac{16}{3}.$$

(3) Show that the equations below are only compatible for one value of k. For this value of k find the solution.

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 3 \\ 3x_1 + 7x_2 + 5x_3 &= 5 \\ 2x_1 + 4x_2 + 3x_3 &= k \end{aligned}$$

Augmented matrix is :

$$A_1 = \left(\begin{array}{ccc|c} 1 & 3 & 2 & 3 \\ 3 & 7 & 5 & 5 \\ 2 & 4 & 3 & k \end{array}\right) \sim \begin{array}{c} R_2-3R_1 \\ R_3-2R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 3 \\ 0 & -2 & -1 & -4 \\ 0 & -2 & -1 & k-6 \end{array}\right)$$

$$\sim \begin{array}{c} \\ \\ R_3-R_2 \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 3 \\ 0 & -2 & -1 & -4 \\ 0 & 0 & 0 & k-2 \end{array}\right)$$

Since the last row of the echelon matrix corresponds to the equation $0x_1 + 0x_2 + 0x_3 = k - 2$ the equations are incompatible unless $k = 2$.

For $k = 2$ the solutions are: $x_3 = \lambda$, $x_2 = 2 - \frac{\lambda}{2}$, $x_1 = -3 - \frac{\lambda}{2}$.

Note Examples (1) and (3) illustrate all the possibilities for the solution of n equations in n unknowns. If the matrix of coefficients is row equivalent to a matrix of the form $\begin{pmatrix} b_{11} & \cdots & b_{ij} & \cdots \\ 0 & b_{22} & & \vdots \\ 0 & & \ddots & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & b_{nn} \end{pmatrix}$

where all the diagonal elements $b_{11}, b_{22}, \ldots, b_{nn}$ are non zero a unique solution to the equations $AX = C$ can be obtained for all values of c, [including $C = 0$]. If, on the other hand, A is row equivalent to an $n \times n$

matrix with zero final row the existence of a solution will depend upon the values of $c_1, c_2, \ldots, c_n$ and will in any case not be unique. $AX = 0$ will in this case have a non trivial solution since it is equivalent to a set of $n - 1$ homogeneous equations in n unknowns.

2.6 The Determinant of an n x n Matrix

It has previously been noted that for a system in n homogeneous equations in n unknowns the existence or otherwise of a non trivial solution depends upon the coefficient matrix. The determinant of an n x n matrix is a number so defined that it is zero if the corresponding homogeneous equations have a non trivial solution and is non zero otherwise.

1 x 1 Determinant

The equation $a_{11}x_1 = 0$ has a non trivial solution only if $a_{11} = 0$ hence define $|a_{11}| = a_{11}$.

2 x 2 Determinant

Consider the equations
$$a_{11}x_1 + a_{12}x_2 = 0$$
$$a_{21}x_1 + a_{22}x_2 = 0$$

For these to have a non trivial solution the matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

must be row equivalent to a matrix with zero final row

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \sim \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} - \dfrac{a_{12}\,a_{21}}{a_{11}} \end{pmatrix} \qquad \text{(provided } a_{11} \neq 0\text{)}$$

Hence in the case $a_{11} \neq 0$ $AX = 0$ has a non trivial solution only if $a_{11}\,a_{22} - a_{12}\,a_{21} = 0$.

If $a_{11} = 0$ A is row equivalent to a matrix with zero row only if $a_{12} = 0$ or $a_{21} = 0$ which once more leads to the condition $a_{11}a_{22} - a_{12}a_{21} = 0$
The 2 x 2 determinant $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is then defined as $a_{11}a_{22} - a_{12}a_{21}$.

Cramers Rule for Solution of 2 Simultaneous Equations

Consider
$$a_{11}x_1 + a_{12}x_2 = c_1$$
$$a_{21}x_1 + a_{22}x_2 = c_2 , \qquad \text{suppose } |A| \neq 0.$$

The augmented matrix of this set of equations is:

$$\left(\begin{array}{cc|c} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \end{array}\right) \sim \left(\begin{array}{cc|c} a_{11} & a_{12} & c_1 \\ 0 & a_{22} - \dfrac{a_{12}a_{21}}{a_{11}} & c_2 - \dfrac{a_{21}}{a_{11}}\,c_1 \end{array}\right)$$

Giving solution $x_2 = \dfrac{c_2 - \frac{a_{21}}{a_{11}}c_1}{a_{22} - \frac{a_{12}}{a_{11}}a_{21}} = \dfrac{a_{11}c_2 - a_{21}\ c_1}{a_{11}a_{22} - a_{12}\ a_{21}}$

or $x_2 = \dfrac{\begin{vmatrix} a_{11} & c_1 \\ a_{21} & c_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$ interchanging x_1 and x_2 enables the solution

$x_1 = \dfrac{\begin{vmatrix} c_1 & a_{21} \\ c_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$ to be written down.

These solutions are usually put in the form:

$$\frac{x_1}{\begin{vmatrix} c_1 & a_{12} \\ c_1 & a_{22} \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} a_{11} & c_1 \\ a_{21} & c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

this result being a special case of Cramers Rule.

<u>3 x 3 Determinant</u>

Suppose the equations
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= 0 \end{aligned}$$

have a non trivial solution.

The second and third equations can be re-written :

$$\begin{aligned} a_{21}x_1 + a_{22}x_2 &= -\ a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 &= -\ a_{33}x_3 \end{aligned}$$

Using Cramer's Rule the solution of these is :

$$\frac{x_1}{\begin{vmatrix} -a_{23} & a_{22} \\ -a_{33} & a_{32} \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} a_{21} & -a_{23} \\ a_{31} & -a_{33} \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}$$

or
$$\frac{x_1}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}$$

Substituting these values for x_1, x_2, x_3 in the first equation gives as a condition for the existence of a non trivial solution :

$$a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

This then gives the definition of the 3×3 determinant $|A|$.

Note The coefficient of a_{11} is the 2×2 determinant obtained by omitting the first row and first column from $|A|$. The coefficient of a_{12} is $-$ the 2×2 determinant obtained by omitting the first row and second column from $|A|$. The coefficient of a_{13} is the 2×2 determinant obtained by omitting the first row and third column from $|A|$.

A comparison with the expression for the 2×2 determinant shows that it could have been expressed in a similar way, the coefficients of a_{11} and a_{12} being 1×1 determinants.

n x n Determinants

The above definitions can be generalised to define an $n \times n$ determinant as :

$$\begin{vmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{n1} & \cdots\cdots & & a_{nn} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & \cdots\cdots & a_{2n} \\ a_{32} & & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ a_{n2} & & a_{nn} \end{vmatrix}$$

$$- a_{12}\begin{vmatrix} a_{21} & a_{23} & \cdots\cdots & a_{2n} \\ & & & \\ & & & \\ a_{n1} & & & a_{nn} \end{vmatrix} + a_{13}\begin{vmatrix} & \cdot & \cdot & \\ & & & \end{vmatrix} \cdots\cdots \quad \text{etc.}$$

This definition is in terms of $n(n-1) \times (n-1)$ determinants the coefficient of a_{1j} being $(-1)^{1+j}$ times the determinant obtained from $|A|$ by omitting the 1st row and j th column.

2.7 Properties of Determinants

From the definition of an $n \times n$ determinant the following properties can be established. Proofs of these are omitted but they can be found in any suitable Algebra text-book.

(1) $|A| = |A'|$ where A' denotes the transpose of A. {This result means that any property of the rows of a determinant will be immediately applicable to the columns}.

(2) If B is obtained from A by interchanging two rows (or columns) then $|B| = -|A|$.

(3) If A has two identical rows (or columns) then $|A| = 0$. {This is a direct consequence of (2)}}

(4) If B is obtained from A by multiplying all the elements in one row (or column) by a constant c then $|B| = c|A|$.

(5) If B is obtained from A by adding to one row (or column) k times another row (or column) then $|B| = |A|$.

(6) If A and B are any n x n matrices then $|AB| = |A|.|B|$.

2.8 Numerical Evaluation of Determinants

The n x n determinant $|A|$ has been defined in terms of the elements of the first row and certain smaller determinants; this is usually referred to as expanding the determinant by its first row. In practice this full expansion can be very laborious and the above properties offer some short-cuts in this process. An immediate consequence of property (1) is that the expansion could just as well have been defined in terms of the elements of the first column (or as will be seen later of any row or column). Properties (4) and (5) are particularly useful since they enable constant factors to be removed and the rows and columns to be manipulated to obtain zero elements in certain positions of the determinant. If this is done systematically the determinant can be rapidly evaluated.

Example:

Evaluate
$$\begin{vmatrix} 1 & 5 & 7 & -2 \\ 2 & 4 & 3 & 1 \\ -1 & 2 & 8 & 1 \\ 0 & 1 & 3 & 4 \end{vmatrix} = \begin{Bmatrix} R_2-2R_1 \\ R_3+R_1 \end{Bmatrix} \begin{vmatrix} 1 & 5 & 7 & -2 \\ 0 & -6 & -11 & 5 \\ 0 & 7 & 15 & -1 \\ 0 & 1 & 3 & 4 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 5 & 7 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 7 & 15 & -1 \\ 0 & -6 & -11 & 5 \end{vmatrix} = \begin{Bmatrix} R_3-7R_2 \\ R_4+6R_2 \end{Bmatrix} - \begin{vmatrix} 1 & 5 & 7 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -6 & -29 \\ 0 & 0 & 7 & 29 \end{vmatrix}$$

Expanding by the first column this gives :

$$- \begin{vmatrix} 1 & 3 & 4 \\ 0 & -6 & -29 \\ 0 & 7 & 29 \end{vmatrix} = - \begin{vmatrix} -6 & -29 \\ 7 & 29 \end{vmatrix} = -29$$

2.9 Minors and Cofactors

The n-1 x n-1 determinant obtained by removing the ith row and j th column from an n x n determinant $|A|$ is called the minor of a_{ij}, it is usually denoted M_{ij}

The coefficient of a_{ij} in the full expansion of the determinant is called the <u>cofactor</u> of a_{ij}, it is denoted A_{ij}.

The minors and cofactors are related by the rule $A_{ij} = (-1)^{i+j}M_{ij}$.

From the definition of the determinant this rule clearly holds for elements in the first row of A, it can be deduced for elements in the i th row, where $i \neq 1$, by repeated application of property (2) (interchange of rows). By a suitable interchange of rows any row can be brought into the first row position, expanding in terms of this row gives :

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in},$$

this is the formula for the expansion of $|A|$ by the i th row.

Application of this result to the j th row of A' gives the expansion by the j th column of (a) :

$$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + \dots + a_{nj}A_{nj}.$$

Consider next the expansion of an n × n determinant with equal i th and j th rows, expanding by the j th row gives :

$$0 = \begin{array}{r} \\ \\ i\text{ th} \rightarrow \\ \\ j\text{ th} \rightarrow \\ \\ \\ \end{array} \begin{vmatrix} a_{11} & a_{12} & \dots\dots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \dots\dots & a_{in} \\ \vdots & & & \\ a_{i1} & a_{i2} & \dots\dots & a_{in} \\ \vdots & & & \\ a_{n1} & \dots & \dots\dots & a_{nn} \end{vmatrix} = a_{i1}A_{j1} + a_{i2}A_{j2} + \dots + a_{in}A_{jn}$$

{The cofactors are equal to those in $|A|$ since apart from the elements of the j th row the two matrices are identical}.

Application of this to $|A'|$ gives a similar result for columns :

$$0 = a_{1i}A_{1j} + a_{2i}A_{2j} + \dots + a_{ni}A_{nj} \quad \text{where } i \neq j$$

These 4 properties of cofactors can be summarised by the equations:

$$\sum_{k=1}^{n} a_{ik}A_{jk} = \delta_{ij}|A| \qquad (1)$$

$$\sum_{k=1}^{n} a_{ki}A_{kj} = \delta_{ij}|A| \qquad (2)$$

Expressed in words this is 'The sum of the products of the elements of one row (or column with the corresponding cofactors of another row (or column) is zero. The sum of the products of the elements of one row with their own cofactors is equal to the determinant.'

2.10 Cramer's Rule

The more general version of Cramer's rule for the solution of n linear equations is n unknowns can be derived from the above properties of cofactors.

Consider the equations :

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= c_1 \\
a_{21}x_1 + a_{12}x_2 + \dots + a_{2n}x_n &= c_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= c_n
\end{aligned}
$$

Multiplying the first equation by A_{1i}, second by A_{2i} etc., and adding gives :

$$\left(\sum_{k=1}^{n} a_{k1}A_{ki}\right)x_1 + \dots + \left(\sum_{k=1}^{n} a_{ki}A_{ki}\right)x_i + \dots + \left(\sum_{k=1}^{n} a_{kn}A_{ki}\right)x_n$$

$$= c_1A_{1i} + c_2A_{2i} + \dots + c_nA_{ni}$$

Using properties of cofactors this reduces to :

$$|A|x_i = \begin{vmatrix} a_{11} & \dots & a_{1i-1} & c_1 & a_{1i+1} & \dots & a_{1n} \\ a_{21} & & & c_2 & a_{2i+1} & \dots & a_{2n} \\ \vdots & & & & & & \\ a_{n1} & & a_{ni-1} & c_n & a_{ni+1} & \dots & a_{nn} \end{vmatrix}$$

or (provided $|A| \neq 0$)

$$x_i = \frac{|B_i|}{|A|}$$

where B_i is the matrix obtained from A by replacing the i th column by $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$

This result is usually expressed as :

$$\frac{x_1}{|B_1|} = \frac{x_2}{|B_2|} = \dots = \frac{x_n}{|B_n|} = \frac{1}{|A|} .$$

Cramer's rule provides a convenient method of writing down immediately the solutions of a set of linear equations, but since these solutions include n x n determinants they are not in a very practical form.

3. The inverse of an n × n Matrix

An n × n matrix A is said to be non singular if there is a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$, A^{-1} is called the inverse of A. It will be shown that A is non singular if and only if $|A| \neq 0$.

3.1 Calculation of inverse using Adjoint Matrix

Let A be any n × n matrix, AdjA, the adjoint of A, is then defined as the transpose of the matrix whose elements are the cofactors of $|A|$. i.e. if AdjA = B, then $b_{ij} = A_{ji}$ for all ij = 1, ... n.

Consider A.AdjA

Let A.AdjA = C

Then by defn. of matrix product and of adjA

$$C_{ij} = \sum_{k=1}^{n} a_{ik}A_{jk} = \delta_{ij}|A| \quad \text{using (1)}$$

Hence $A.\text{Adj}A = |A|.I$ (3)

Similarly if $D = \text{Adj}A.A$

$$d_{ij} = \sum_{k=1}^{n} A_{ki}a_{kj} = \delta_{ij}|A| \quad \text{using (2)}$$

or $\text{Adj}A.A = |A|.I$ (4)

Equations (3) and (4) are valid whether or not $|A| = 0$.

If, moreover $|A| \neq 0$ the equations may be divided by the scalar $|A|$ to give:

$$\left(\frac{1}{|A|}\text{Adj}A\right)A = I = A\left(\frac{1}{|A|}\text{Adj}A\right)$$

showing that A is non singular if $|A| \neq 0$ with inverse $A^{-1} = \frac{1}{|A|}.\text{Adj}A$.

If, on the other hand, $|A| = 0$ then the equations $AX = 0$ have a non-trivial solution $X = Y \neq 0$, but if we assume the existence of an inverse matrix A^{-1} : $AY = 0 \Rightarrow A^{-1}(AY) = A^{-1}0$

$$\Rightarrow IY = 0$$

$$\Rightarrow Y = 0$$

which is a contradiction.

Hence when $|A| = 0$ A can have no inverse.

It will be appreciated that the formula $A^{-1} = \frac{1}{|A|}\,\mathrm{Adj}\,A$ will always anable A^{-1} to be calculated when A is non singular but in practice this calculation becomes very laborious when A is larger than 3×3. [For a 3×3 matrix the calculation involves 1 3×3 determinant and 9 2×2 determinants, for a 4×4 matrix the same calculation involves 1 4×4 determinant and 16 3×3 determinants].

3.2 Calculation of Inverse Using Elementary Row Operations

For the solution of n linear equations in n unknowns the use of elementary row operations on the augmented matrix provided an alternative to the calculation of the solution using Cramer's rule and determinants, the work involved being much less in the case of larger values of n. A similar method can be applied to the calculation of an inverse matrix. This method is demonstrated below, the theoretical justification being given later.

Example

Find the inverse of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 2 & -2 & 1 \end{pmatrix}$

The aim of the method is to begin by forming a 3×6 matrix consisting of two 3×3 matrices A and I. Elementary row operations are then performed on this matrix until A is reduced to the unit matrix I, the matrix obtained by performing the same operations on I is A^{-1}.

$$\begin{array}{ccc|ccc} & A & & & I & \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{array} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -2 & 1 & 0 \\ 0 & -4 & -1 & -2 & 0 & 1 \end{array}\right)$$

$$\begin{array}{l} R_1 + \frac{1}{2}R_2 \\ -\frac{1}{2}R_2 \\ R_3 - 2R_2 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -1 & 2 & -2 & 1 \end{array}\right)$$

$$\begin{array}{l} R_1 + R_3 \\ \\ -R_1 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -\frac{3}{2} & 1 \\ 0 & 1 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & 2 & -1 \end{array}\right)$$

$$\begin{array}{ccc|ccc} & I & & & A^{-1} & \end{array}$$

A check shows that

$$\begin{pmatrix} 2 & -\frac{3}{2} & 1 \\ 1 & -\frac{1}{2} & 0 \\ -2 & 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

verifying that the matrix obtained is indeed A^{-1}.

3.3 Theoretical Justification

To justify the above method it is necessary to introduce the so-called elementary matrices [or elementary row operation matrices].

An $n \times n$ elementary matrix E is a matrix obtained from the $n \times n$ identity matrix I by performing a single elementary row operation upon it. The elementary row operations and corresponding elementary matrices E are such that if A is any $n \times m$ matrix and E is an $n \times n$ elementary matrix EA is the matrix obtained by performing the corresponding operation on A. This is proved by considering separately the three types of elementary row operation and their corresponding elementary matrices.

Type I. Interchange the order of 2 rows.

Suppose the r th and s th rows are interchanged.

Let the corresponding elementary matrix be E.

Then $e_{ij} = \delta_{ij}$ for $i \neq r$ or s.

$e_{rj} = \delta_{sj}$, $e_{sj} = \delta_{rj}$

Let EA = B

Then for $j = 1 \ldots m$ we have:

$$b_{ij} = \sum_{k=1}^{n} \delta_{ik} a_{kj} = a_{ij} \qquad \text{for } i \neq r \text{ or } s$$

$$b_{rj} = \sum_{k=1}^{n} \delta_{sk} a_{kj} = a_{sj}$$

$$b_{sj} = \sum_{k=1}^{n} \delta_{rk} a_{kj} = a_{rj}$$

Hence B is the matrix obtained by interchanging the r th and s th rows of A,

Type II Multiply any row by non-zero constant.

Suppose the r th row is multiplied by λ. Then if the corresponding elementary matrix is E and EA = B,

$$e_{ij} = \delta_{ij} \qquad \text{for } i \neq r,$$

$$e_{rj} = \lambda\delta_{rj} \qquad \text{and for } j = 1 \ldots m.$$

$$b_{ij} = \sum_{k=1}^{n} \delta_{ik}a_{kj} = a_{ij} \qquad \text{for } i \neq r.$$

$$b_{rj} = \sum_{k=1}^{n} \lambda\delta_{rk}a_{kj} = \lambda a_{rj}.$$

Hence B is the matrix obtained from A by multiplying the r th row by λ.

Type III Add to any row a constant multiple of another row.

Suppose μ times r th row is added to s th row.

Then if E is corresponding elementary matrix and EA = B

$$e_{ij} = \delta_{ij} \text{ for } i \neq s. \quad e_{sj} = \delta_{sj} + \mu\delta_{rj}$$

$$\text{for } j = 1 \ldots m:$$

$$b_{ij} = \sum_{k=1}^{n} \delta_{ik}a_{kj} = a_{ij} \qquad \text{for } i \neq s.$$

$$b_{sj} = \sum_{k=1}^{n} (\delta_{sk} + \mu\delta_{rk})a_{kj} = a_{sj} + \mu a_{rj}$$

Hence B is the matrix obtained by adding μ times the r th row of A to the s th row of A.

Suppose now that $E_1, E_2, \ldots E_k$ are the elementary matrices corresponding to row operations which taken in order will reduce an $n \times n$ matrix A to the identity matrix.

Since the application of these operations corresponds to pre-multiplying

A in turn by the elementary matrices we have: $I = E_k E_{k-1} \dots E_2 E_1 A$

$$\text{or} \quad (E_k E_{k-1} \dots E_1) = A^{-1}.$$

But the matrix $E_k E_{k-1} \dots E_1$ is precisely the matrix obtained by applying the same sequence of elementary row operations to I, in the previous numerical calculation this is the right hand half of the 3×6 matrix.

Notes (i) Properties similar to the above can be proved for elementary column operations but in this case the corresponding elementary matrices must post-multiply the matrix being operated upon [i.e. AE is matrix obtained from A by performing appropriate column operation].

(ii) The properties of the elementary matrices can be used to justify the method of solving linear equations using the augmented matrix.

3.4 Properties of Inverse Matrices

(1) The inverse of a non singular $n \times n$ matrix is unique, any left inverse will also be a right inverse and conversely.

Proof Suppose A is a non singular matrix.

Let A_L^{-1} be a left inverse of A such that $A_L^{-1}A = I$.

Let A_R^{-1} be a right inverse of A such that $AA_R^{-1} = I$

Then $A_L^{-1}AA_R^{-1} = (A_L^{-1}A)A_R^{-1} = I\,A_R^{-1} = A_R^{-1}$

and $A_L^{-1}AA_R^{-1} = A_L^{-1}(AA_R^{-1}) = A_L^{-1}I = A_L^{-1}$

hence $A_L^{-1} = A_R^{-1}$.

Showing that any left inverse is also a right inverse and conversely. Hence there is no necessity to distinguish between left and right inverses.

Now suppose A_1^{-1} and A_2^{-1} are two, if possible distinct, inverses for A.

Then $A_1^{-1} = A_1^{-1}I = A_1^{-1}(AA_2^{-1}) = (A_1^{-1}A)A_2^{-1} = A_2^{-1}$.

Here inverse is uniquely determined.

(2) If A and B are both non singular $n \times n$ matrices then AB is non singular and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof

Consider $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$.

Showing that AB is non singular with inverse $B^{-1}A^{-1}$.

(3) If A is a non singular matrix the matrix equation $AX = B$ has unique solution $X = A^{-1}B$ and the matrix equation $YA = C$ has unique solution $Y = CA^{-1}$.

Proof

Consider $AX = B$

Pre-multiplying by A^{-1} gives $A^{-1}(AX) = A^{-1}B = IX = A^{-1}B$

$= X = A^{-1}B.$

If $AX_1 = B$ and $AX_2 = B$.

Then $AX_1 = AX_2$ and pre-multiplying by A^{-1} gives $(A^{-1}A)X_1 = A^{-1}A)X_2$

$= X_1 = X_2$

Showing that the solution is unique.

Similarly post-multiplying by A^{-1} shows that $Y = CA^{-1}$ is the unique solution of $YA = C$ [note unless C and A^{-1} commute $Y = A^{-1}C$ is not a solution].

Note. The above property shows that the inverse matrix provides a further method of calculating the solution of a set of n linear equations in n unknowns when the matrix of coefficients is non-singular. This method although rather laborious for one set of equations is very useful for obtaining the solutions of a number of sets of related equations with the same coefficients. This situation is likely to arise when the matrix of coefficients represents a linear system and inputs have to be calculated for a number of different outputs.

Exercises

1. Solve the equations

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ 3x_1 + 6x_2 + x_3 &= 11 \\ 2x_1 + 3x_2 + 2x_3 &= 3 \end{aligned}$$

(a) By using Cramer's Rule

(b) By inverting the matrix of coefficients

(c) By performing row operations on the augmented matrix

2. Solve, if possible, the following sets of equations:

(a)
$$\begin{aligned} x_1 + y_1 + z_1 &= 1 \\ 2x_1 + 4y_1 - 3z_1 &= 9 \\ 3x_1 + 5y_1 - 2z_1 &= 11 \end{aligned}$$

(b)
$$\begin{aligned} 2x_2 - y_2 + z_2 &= 7 \\ 3x_2 + y_2 - 5z_2 &= 13 \\ x_2 + y_2 + z_2 &= 5 \end{aligned}$$

3. Find the values of λ for which the equations

$$\begin{aligned} x_1 + 2x_2 + x_3 &= \lambda x_1 \\ 2x_1 + x_2 + x_3 &= \lambda x_2 \\ x_1 + x_2 + 2x_3 &= \lambda x_3 \end{aligned}$$

have a non trivial solution. For one of these values of λ give the most general solution.

4. Find the most general form of solution of the equations:

$$\begin{aligned} x_1 - x_2 + 2x_3 - x_4 &= 1 \\ 2x_1 - x_2 + 3x_3 - 4x_4 &= 2 \\ -x_1 + 3x_2 - x_3 - x_4 &= -1 \end{aligned}$$

5. (a) Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a - b)(b - c)(c - a).$$

(b) Show that $x^2 + y^2 + z^2$ is a factor of

$$\begin{vmatrix} y^2 + z^2 & x^2 & yz \\ z^2 + x^2 & y^2 & zx \\ x^2 + y^2 & z^2 & xy \end{vmatrix}$$

and hence factorise this determinant completely.

6. Show that the equations:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 - x_2 + 2x_3 &= 0 \\ x_1 + 7x_2 - 11x_3 &= 0 \end{aligned}$$

have a non trivial solution. Find a solution which also satisfies

$$x_1^2 + x_2^2 + x_3^2 = 1.$$

7. Find the inverses of the matrices

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$$

Hence find a matrix X such that

$$BXA = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

8. Find the inverse of the matrix:

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & -1 & 1 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 4 & -1 & 5 \end{pmatrix}$$

hence solve:

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 2 \\ -x_1 - x_2 + x_3 &= 1 \\ 2x_1 + 3x_2 &= 5 \\ x_1 + 4x_2 - x_3 + 5x_4 &= 0 \end{aligned}$$

9. $$A = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

show that

$$P^{-1}AP = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Using this result find the values of A^7 and A^{-1}.

10 Find the inverses of the matrices

$$A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$

Find X if $ABX = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

11. Show that the matrix $\mathbf{A} = \begin{pmatrix} 2 & 2 & -2 \\ 1 & 3 & -2 \\ 1 & 2 & -1 \end{pmatrix}$

satisfies the matrix equation $A^2 - 3A + 2I = 0$. Use this result to find the value of A^{-1}.

Hence solve the equations

$$\begin{aligned} 2x_1 + 2x_2 - 2x_3 &= 4 \\ x_1 + 3x_2 - 2x_3 &= 1 \\ x_2 + 2x_2 - x_3 &= 3 \end{aligned}$$

and

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 1 \\ 2x_1 + 3x_2 + 2x_3 &= 2 \\ -2x_1 - 2x_2 - x_3 &= 3\,. \end{aligned}$$

Summary. Matrix Algebra

An $m \times n$ matrix A is a rectangular array of mn elements arranged in m rows and n columns. a_{ij} is the element in the i th row and j th column of A.

Sum of two matrices: If A and B are $m \times n$ matrices $C = A + B$ is an $m \times n$ matrix with $c_{ij} = a_{ij} + b_{ij}$.

Scalar Product: $kA = D$ is an $m \times n$ matrix with $d_{ij} = ka_{ij}$.

Product of Two Matrices: If A is an $m \times n$ matrix and B is an $n \times p$ matrix $C = AB$ is an $m \times p$ matrix with

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Properties: $A(BC) = (AB)C$, $A(B + C) = AB + AC$.

$AI = A$ where $I = \{\delta_{ij}\}$ $\delta_{ij} = 1$ if $i = j$

and $IA = A$ $\delta_{ij} = 0$ if $i \neq j$.

But in general $AB \neq BA$.

A is said to be non singular if A is an $n \times n$ matrix and there exists a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$. A^{-1} is called the inverse of A. A^{-1} is given as $\frac{1}{|A|}\,\text{Adj}A$ or by the application of elementary row operations to $(A|I)$ to give $(I|A^{-1})$.

Adj$A$ = transpose of matrix of cofactors of $|A|$.

Determinants

$|A| = a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n}$ or more generally,

$$|A| = \sum_{k=1}^{n} a_{ik}A_{ik} = \sum_{l=1}^{n} a_{lj}A_{lj}$$

Cofactors: A_{ij} is the cofactor of a_{ij} in $|A|$ it is given by

$$(-1)^{i+j}M_{ij}.$$

Minors: M_{ij}, the minor of a_{ij} is the $(n-1)\times(n-1)$ determinant obtained by omitting the i th row and j th column from $|A|$.

Theory of Linear Equations

The solutions of the set $AX = C$ of m linear equations in n unknowns are related to those of the corresponding homogeneous system $AX = 0$; if $X = Y$ is any solution of $AX = C$ and if $X = Z$ is a solution of the homogeneous equations $AX = 0$ then, for any k, $X = Y + kZ$ is also a solution of $AX = C$.

If the equations $AX = C$ are compatible {i.e. have a solution}. this solution is unique only if $AX = 0$ has only the trivial solution $X = 0$

If $m < n$ $AX = 0$ always has a non-trivial solution and consequently the solution of $AX = C$ can never be unique.

If $m = n$ {i.e. n equations in n unknowns}. $AX = 0$ has a non-trivial solution only if $|A| = 0$. If $|A| \neq 0$ $AX = C$ has a unique solution for any value of C, namely $X = A^{-1}C$. If $|A| = 0$ then, depending upon the values of C, $AX = C$ will either be insoluble or have an infinite number of solutions.

Numerical Solution of Ordinary Differential Equation « First-order » :-

We know that a differential equation of the first order is of the form $F(x, y, y') = 0$ and often it will be possible to write the equation in the explicit form $y' = f(x, y)$. An initial value problem consists of a differential equation and a condition which the solution must satisfy (or several conditions referring to the same value of (x) if the equation is of higher order). In this we shall consider initial value problems of the form:

$$y' = f(x, y) \quad , \quad y(x_0) = y_0$$

We shall discuss methods for computing numerical values of the solution. These methods are step by step methods, that is we start from $y_0 = y(x_0)$ and proceed stepwise. In the first step we compute an approximate value (y_1) of the solution (y) of (1) at $x = x_1 = x_0 + h$ in the second step we compute an approximate value (y_2) of the solution at $x = x_2 = x_0 + 2h$ etc. here (h) is a fixed number. In each step the computations are done by the same formula. Such formulas are suggested by the Taylor series :-

$$y(x+h) = y(x) + h y'(x) + \frac{h^2}{2!} y''(x) + \cdots + \frac{h^n}{n!} y^{n}(x).$$

EX. Find the numerical solution of the following by using Taylor series method :-

$$\frac{dy}{dx} = x + y \quad ; \quad y(x_0) = y_0 = 1 \quad ; \text{ for } x = 0(0.1)0.1$$

Solution :- We have $h = 0.1$, $x_n = 0.1$, $x_0 = 0$

$$y(x_1) = y(x_0) + y'(x_0) \cdot h + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \cdots$$

and :

$$y' = x + y \Rightarrow y'(x_0) = x_0 + y_0 = 0 + 1 = 1$$

$$y'' = 1 + y' \Rightarrow y''(x_0) = 1 + y'(x_0) = 1 + 1 = 2$$

$$y''' = 0 + y'' \Rightarrow y'''(x_0) = y''(x_0) = 2$$

$$y^{iv}(x_0) = y''' \Rightarrow y^{iv}(x_0) = y'''(x_0) = 2$$

Then $$y_{(0.1)} = 1 + \frac{1}{1} * 0.1 + \frac{\cancel{2}}{\cancel{2} * 1} * (0.1)^2 + \frac{2}{3*2*1} (0.1)^3$$

$$+ \frac{2}{4*3*2*1} * (0.1)^4 =$$

$$y_{(0.1)} = 1 + 0.1 + 0.01 + 0.00033 + 0.000008 = 1.110311$$

H.W.

Resolve the above Example take $h = 0.02$.

⊛ Euler's method :

$$\frac{dy}{dx} = y' = f(x_i, y_i)$$

The general formula of Euler's method is :-

$$y_{i+1} = y_i + h\, f(x_i, y_i) \quad \text{for each } i = 0, 1, 2, \ldots, n$$

x_i is defined on interval $a \le x \le b$;

$x_i = a + ih$ or $x_i = x_0 + ih$

$x_0 = \alpha$, $x_1 = x_0 + h$; $x_2 = x_1 + h$; $x_3 = x_2 + h$ or

$x_1 = x_0 + h$; $x_2 = x_0 + 2h$; $x_3 = x_0 + 3h$.

$y(a) = \alpha$; h is the step size.

$h = \frac{b-a}{n}$ or $h = \frac{x_n - x_0}{n}$

EX.1. Solve $\frac{dy}{dx} = x^2 + y$ using Euler's method from $x = 1$ to $x = 2$; $h = 0.1$; with I.V. $y(1) = 1$.

Solution :-

$$y_{i+1} = y_i + h\,(x_i^2 + y_i)$$

$y_1 = y_0 + h\,(x_0^2 + y_0)$; $x_0 = 1$; $y_0 = 1$ from the Question

$$y_1 = 1 + 0.1\,[(1)^2 + 1] = 1.2$$

$$y_2 = y_1 + 0.1\,(x_1^2 + y_1)$$
$$= 1.2 + 0.1\,[(1.1)^2 + 1.2] = 1.441$$
$$y_3 = 1.441 + 0.1\,[(1.2)^2 + 1.441] = 1.729$$

⋮

until

$$y_{i+1} = y_i + h\,(x_i^2 + y_i) \text{ for } x = 2.$$

EX.2. $\frac{dy}{dx} = y' = \frac{x-y}{x+y}$, $y(0) = 1$ $0 < x < 0.1$; $h = 0.02$

Sol.

$$y_1 = y_0 + h\,f(x_0, y_0) = 1 + 0.02\left(\frac{0-1}{0+1}\right) = 0.98$$

$$y_2 = 0.98 + 0.02\,\frac{x_1 - y_1}{x_1 + y_1} = 0.98 + 0.02\,\frac{0.02 - 0.98}{0.02 + 0.98} = 0.9608$$

Then

x	0	0.02	0.04	0.06	0.08	0.1
y	1	0.98	0.9608	0.9426	0.9249	0.9080

Euler's modified method

The errors introduced by the use of the straight forward Euler's method and the build up of these errors as one proceeds can by reduled by the use of the Euler's modified method. This method also called; The Trapezoidle method.

The method uses the Euler's method as apredictor:

$$y^{n}_{i+1} = y_i + h f(x_i; y_i) \quad --- ①$$

and Trapezoidal rule as a corrector:

$$y^{n+1}_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y^{n}_{i+1}) \right] -- ②$$

EX. Use Euler's method (modified) to approximate

$$y' = x^2 + y \quad \text{for } y(1) = 1 ; \; h = 0.1, \; 1 \leq x \leq 2$$

withe accuracy of $\epsilon = 10^{-4}$

Solution: $x_0 = 1$, $y_0 = 1$

$$y^{0}_{1} = y_0 + h(x_0^2 + y_0) = 1 + 0.1(1^2 + 1) = 1.2$$

Now use the formula of Euler modified (eq. 2))

$$x_1 = x_0 + h = 1 + 0.1 = 1.1$$

$$y^{1}_{1} = 1 + \frac{0.1}{2}\left[(1^2+1) + (1.1)^2 + 1.2\right] = 1.2205$$

$$y^{2}_{1} = 1 + \frac{0.1}{2}\left[(1^2+1) + (1.1)^2 + 1.2205)\right] = 1.2215$$

$$y^{3}_{1} = 1 + \frac{0.1}{2}\left[(1^2+1) + (1.1)^2 + 1.2215\right] = 1.2216$$

must be fixed or agree with ~~the~~ the considere accuracy.

Now repeat the iteration for anew x:

$x_1 = 1.1$, $y_1 = 1.2216$

$x_2 = x_1 + h$ or $x_0 + 2h = 1 + 0.2 = 1.2$

$y_2^0 = 1.2216 + 0.1[(1.1)^2 + 1.2216] = 1.46476$

$y_2^1 = 1.2216 + \frac{0.1}{2}[(1.1)^2 + 1.2216 + ((1.2)^2 + 1.46476)] = 1.487818$

$y_2^2 = 1.226 + \frac{0.1}{2}[(1.1)^2 + 1.2216 + ((1.2)^2 + 1.487818)] = 1.48957$ must be fixed

$y_2^3 = 1.2216 + \frac{0.1}{2}[(1.1)^2 + 1.2276 + ((1.2)^2 + 1.48957)] = 1.4897$

Now repeat the iteration until reach $(x = 2)$

$\epsilon = |y_3^3 - y_3^2|$ or $|y_n^3 - y_n^2|$ must be applied at each stage.

x	1	1.1	1.2	-----
y	1	1.2216	1.48957	-----

Complete the solution until reach $x = 2$

Runge-Kutta Method (4th order)

The equation used is :-

$$y_{i+1} = y_i + \frac{1}{6}[K_0 + 2K_1 + 2K_2 + K_3]$$

$$K_0 = h.f(x_i, y_i)$$
$$K_1 = h.f(x_i + \frac{h}{2}, y_i + \frac{K_0}{2})$$
$$K_2 = h.f(x_i + \frac{h}{2}, y_i + \frac{K_1}{2})$$
$$K_3 = h.f(x_i + h, y_i + K_2)$$

Ex. Apply 4th order Runge-Kutta to solve the following Value problem :

$y' = x^2 + y$ for $h = 0.1$, $y(1) = 1$ and $1 \le x \le 2$

Solution :-

for $x_0 = 1$, $y_0 = 1$ i.e $i = 0$

$K_0 = h f(x_i, y_i) = h f(x_0, y_0) = 0.1(1^2 + 1) = 0.2$

$K_1 = h.f(x_0 + \frac{h}{2}, y_0 + \frac{K_0}{2}) = 0.1[(1 + \frac{0.1}{2})^2 + (1 + \frac{0.2}{2})] = 0.22025$

$K_2 = h.f(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}) = 0.1[(1 + \frac{0.1}{2})^2 + (1 + \frac{0.22025}{2})] = 0.2212625$

$K_3 = h.f(x_0 + h, y_0 + K_2) = 0.1[(1 + 0.1)^2 + (1 + 0.2212625)] = 0.243126$

$\therefore y_{i+1} = y_i + \frac{1}{6}[k_0 + 2(k_1 + k_2) + k_3]$

$y_1 = y_0 + \frac{1}{6}[k_0 + 2(k_1 + k_2) + k_3]$

$y_1 = 1 + \frac{1}{6}[0.2 + 2(0.22025 + 0.2212625) + 0.2431262]$

$\therefore y_1 = 1.2210252$

For $x_1 = x_0 + h = 1 + 0.1 = 1.1$, $y_1 = 1.2210252$ Complete the Procedure

$k_0 = hf(x_1, y_1) = 0.1[(1.1)^2 + 1.2210252] = 0.2431025$

$k_1 = h.f(x_1 + \frac{h}{2}, y_1 + \frac{k_0}{2}) = 0.1[((1.1) + \frac{0.1}{2})^2 + (1.2210252 + \frac{0.2431025}{2})]$

$k_1 = 0.26650$

$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.1[(1.1 + \frac{0.1}{2})^2 + (1.2210252 + \frac{0.266507}{2})]$

$k_2 = 0.267677$

$k_3 = h.f(x_1 + h, y_1 + k_2) = 0.1[(1.1 + 0.1)^2 + (1.2210252 + 0.267677)]$

$k_3 = 0.2807677$

$\therefore y_2 = y_1 + \frac{1}{6}[k_0 + 2k_1 + 2k_2 + k_3]$

$\therefore y_2 = 1.4863982$

Continue unitl ($x = 2$)

x	1	1.1	1.2	1.3	1.4	---	2.0
y	1	1.2210252	1.48639		-----		----