Chapter Two
Boundary Layer Theory

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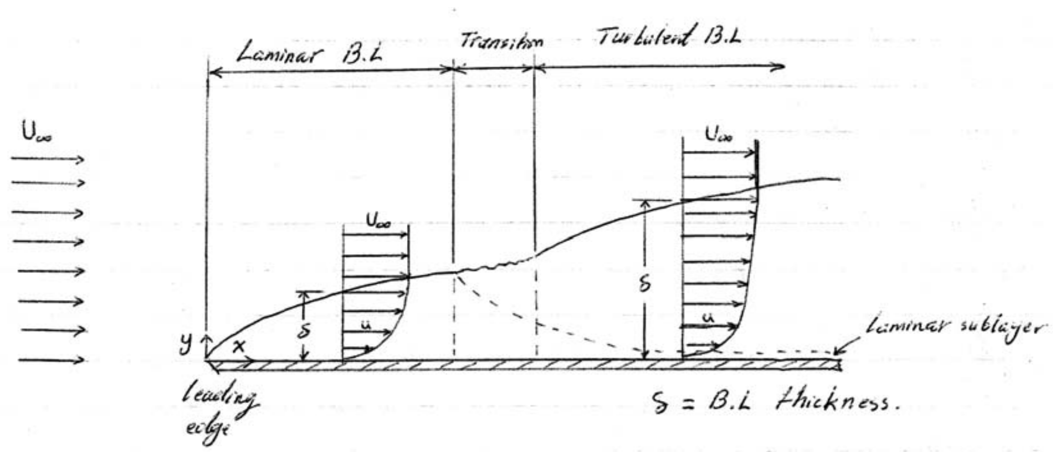
1- Introduction:

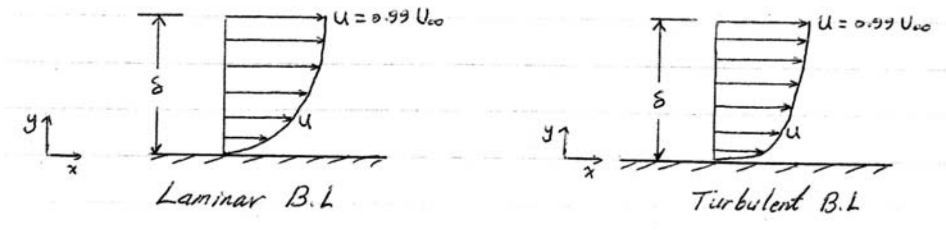
Development of boundary layer on a flat plate

The flow of a viscous fluid on a solid surface represents a region in which velocity increases from zero at the surface and approaches the velocity of the main stream. This region is known as *the boundary layer*.



The figure shows the development of a boundary layer on one side of a long flat plate held parallel to the flow direction.

Velocity distribution in boundary layer



The velocity gradient will give rise to a large shear stress at the wall τ_o (or τ_w).

$$\tau_o = \mu\left(\frac{du}{dy}\right)_{y=0} \qquad \tau_o \equiv \tau_w$$

As shown in figure:
the velocity gradient in the turbulent boundary layer is larger than that in the laminar boundary layer.

$$\left(\frac{du}{dy}\right)_{y=0} \text{(in turbulent BL)} \rangle \left(\frac{du}{dy}\right)_{y=0} \text{(in laminar BL)}$$

$$\therefore \quad \tau_o \ \text{(in turbulent BL)} \ \rangle \tau_o \ \text{(in laminar BL)}$$

The shear stress for a turbulent boundary layer is greater than the shear stress for a laminar boundary layer.

Boundary layer thickness (δ)

Boundary layer thickness is the distance from the solid surface to the point in the flow where $u = 0.99U_\infty$.

Displacement thickness (δ^*)

Displacement thickness represents the outward displacement of the streamlines caused by the viscous effects on the solid surface.

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U_\infty}\right) dy$$

Or

$$\delta^* = \delta\int_0^1 (1 - f(\eta))\, d\eta \qquad \text{Where } \eta = \frac{y}{\delta} \quad \text{and} \quad f(\eta) = \frac{u}{U_\infty}$$

Momentum thickness (θ)

Momentum thickness, is defined as the thickness of a layer of fluid , with velocity U_∞ , for which the momentum flux is equal to the deficit of momentum flux through the boundary layer.

$$\theta = \int_0^\delta \frac{u}{U_\infty}\left(1-\frac{u}{U_\infty}\right)dy$$

Or

$$\theta = \delta\int_0^1 f(\eta)(1-f(\eta))d\eta$$

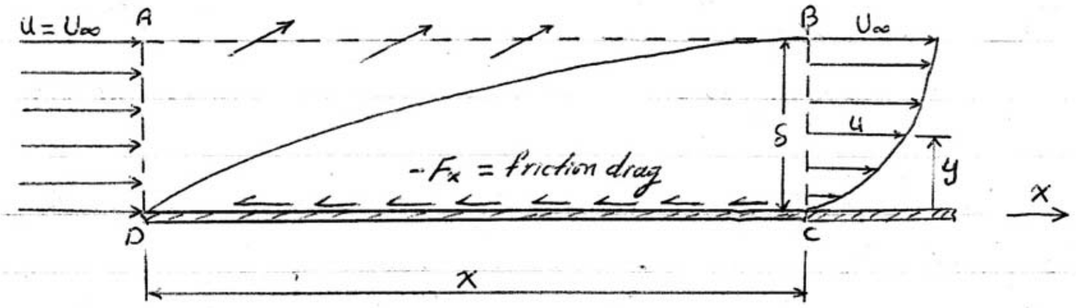
Shape factor (H)

H is a velocity profile shape factor.

$$H = \frac{\delta^*}{\theta}$$

2- Momentum equation for boundary layer:

Consider the control volume for flow over one side of a flat plate of width b.



$$-F_x = \rho b\int_0^\delta u^2 dy + \rho\left(U_\infty b\delta - b\int_0^\delta u dy\right)U_\infty - \rho(U_\infty b\delta)U_\infty$$

$$F_x = \rho b\int_0^\delta u(U_\infty - u)dy \qquad \text{------(*)}$$

F_x is the total friction drag on the plate from the leading edge up to x.

Assuming that the velocity profiles at various distances along the plate are similar to each other.

$$\frac{u}{U_\infty} = f\left(\frac{y}{\delta}\right) = f(\eta)$$

where

$$\eta = \frac{y}{\delta}$$

Equation (*) may be written as:

$$F_x = \rho b U_\infty^2 \delta \alpha \qquad ------(1)$$

where

$$\alpha = \int_0^1 f(\eta)\left(1 - f(\eta)\right) d\eta$$

The local wall shear stress is :

$$\tau_o = \rho U_\infty^2 \alpha \frac{d\delta}{dx} \qquad ------(2)$$

Equations (1) and (2) are valid for either laminar or turbulent flow in the boundary layer.

3- Laminar boundary layer:

The wall shear stress:

$$\tau_o = \mu \left(\frac{du}{dy}\right)_{y=0}$$

Let $\beta = \left[\frac{df(\eta)}{d\eta}\right]_{\eta=0}$

$$\Rightarrow \quad \tau_o = \frac{\mu U_\infty \beta}{\delta}$$

Another expression for shear stress:

$$\tau_o = \frac{1}{2} \rho U_\infty^2 c_f$$

Where c_f = local friction coefficient.

The total friction drag is:

$$F_f = b\int_0^L \tau_o dx$$

also

$$F_f = \frac{1}{2}\rho U_\infty^2 (b.L) C_f$$

Where C_f = total friction coefficient

The boundary layer thickness:

$$\delta = \sqrt{\frac{2\beta}{\alpha}} \frac{x}{\sqrt{\mathrm{Re}_x}}$$

Where Re_x = local Reynolds number

$$\mathrm{Re}_x = \frac{\rho U_\infty x}{\mu}$$

Blasius solution:

(α = 0.135 ; β = 1.63)

$$\frac{\delta}{x} = \frac{4.91}{\sqrt{\mathrm{Re}_x}}$$

$$\frac{\delta^*}{x} = \frac{1.721}{\sqrt{\mathrm{Re}_x}}$$

$$c_f = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$$

$$C_f = \frac{1.328}{\sqrt{\mathrm{Re}_L}}$$

Where Re_L = total Reynolds number

$$\mathrm{Re}_L = \frac{\rho U_\infty L}{\mu}$$

* The laminar boundary layer will remain laminar up to a value of Re_x = 500000

4- Turbulent boundary layer:

Velocity profile in turbulent boundary layer:

$$\frac{u}{U_\infty} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}} = \eta^{\frac{1}{7}}$$

$$\therefore \ \alpha = \int_0^1 f(\eta)(1 - f(\eta))d\eta = \int_0^1 \eta^{\frac{1}{7}}\left(1 - \eta^{\frac{1}{7}}\right)d\eta = \frac{7}{72}$$

The wall shear stress for the turbulent boundary layer on smooth plate is:

$$\tau_o = 0.023\rho U_\infty^2\left(\frac{\mu}{\rho U_\infty \delta}\right)^{\frac{1}{4}} \quad \text{--------(**)}$$

$$\frac{\delta}{x} = \frac{0.377}{(\mathrm{Re}_x)^{\frac{1}{5}}}$$

$$c_f = \frac{0.0587}{(\mathrm{Re}_x)^{\frac{1}{5}}}$$

The total friction coefficient is calculated form the following relations:

$$C_f = \frac{0.0735}{(\mathrm{Re}_L)^{\frac{1}{5}}} \qquad \text{for } \mathrm{Re}_L \langle 10^7$$

$$C_f = \frac{0.455}{(\log_{10}\mathrm{Re}_L)^{2.58}} \qquad \text{for } \mathrm{Re}_L \rangle 10^7$$

and

$$F_f = \frac{1}{2}\rho U_\infty^2 (b.L)C_f$$

Note: equation (**) was obtained from the following pipe equations:

$$\tau_o = \frac{1}{8}\rho f U_{av}^2 \qquad (f = 4c_f)$$

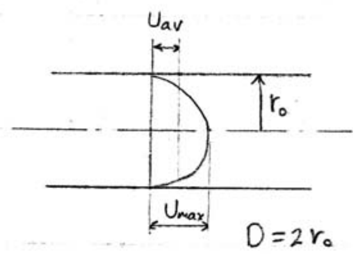
$$f = \frac{0.316}{(\mathrm{Re})^{\frac{1}{4}}} \qquad ; \qquad \mathrm{Re} = \frac{\rho D U_{av}}{\mu}$$

$$U_{av} = \frac{U_{max}}{1.235}$$

To transfer to the flat plate :

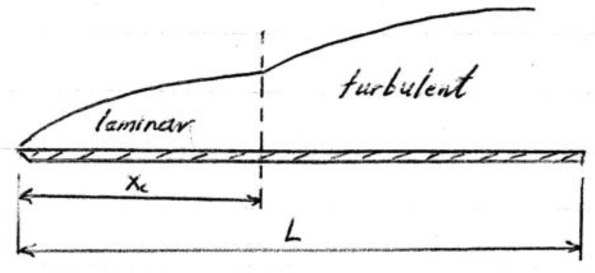
$$r_o \approx \delta \qquad , \qquad U_{max} \approx U_\infty$$

$$\therefore \quad \tau_o = \frac{1}{8}\rho \frac{0.316}{\left[\dfrac{\rho(2\delta)\dfrac{U_\infty}{1.235}}{\mu}\right]^{\frac{1}{4}}}\left(\frac{U_\infty}{1.235}\right)^2$$



$$\Rightarrow \quad \tau_o = 0.023\rho U_\infty^2\left(\frac{\mu}{\rho U_\infty \delta}\right)^{\frac{1}{4}}$$

5- Friction drag in transition region:

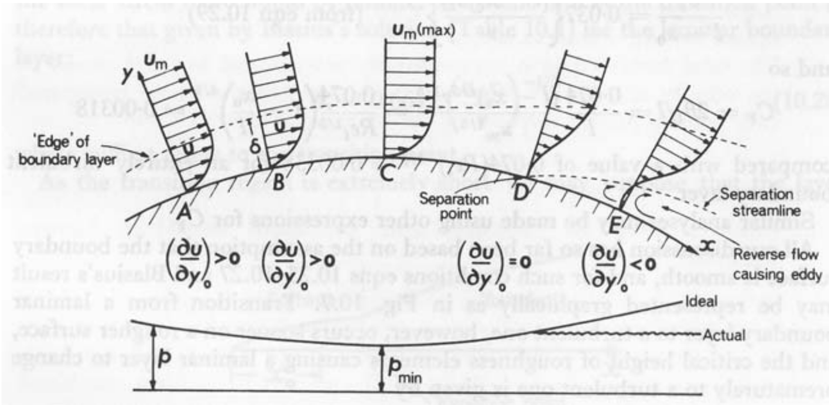


$F_f = F_{laminar} + F_{turbulent}$

$$F_f = \frac{1}{2}\rho U_\infty^2 (b.L) C_f$$

$$C_f = \left[1.328\frac{\sqrt{\mathrm{Re}_{xc}}}{\mathrm{Re}_L} + \frac{0.455}{(\log_{10}\mathrm{Re}_L)^{2.58}} - 0.0735\frac{\mathrm{Re}_{xc}^{\frac{4}{5}}}{\mathrm{Re}_L}\right]$$

6- Effect of pressure gradient:



Consider the flow over a curved surface as shown in figure. As the fluid is deflected round the surface it is accelerated over the left-hand section (points A and B) until at point C, the velocity just outside the boundary layer is a maximum and the pressure is a minimum.

Beyond C, the velocity outside the boundary layer decreases, resulting in an increase in pressure. The velocity of the layer close to the wall is reduced and finally brought to a stop at D. Now the increasing pressure calls for further retardation so the boundary layer separates from the wall. At E there is a backflow (reverse flow) next to the wall, driven in the direction of decreasing pressure.

Down stream from the separation point the flow is characterized by irregular turbulent eddies. This disturbed region is called the wake of the body. The pressure within the wake remains close to that at the separation point. The pressure is always less than the pressure at the forward stagnation point.

An additional drag force is resulted from differences of pressure. This force is known as *the pressure drag (or form drag)*

** The total drag on a body is the sum of the friction drag and the pressure drag.

$$F_D = F_f + F_P$$

$$F_D = \frac{1}{2}\rho U_\infty^2 A C_D$$

Where A = projected area of the body perpendicular to the oncoming flow.

C_D = total drag coefficient.

Values of C_D for two- and three-dimensional bodies are given in figures (1) and (2) respectively.

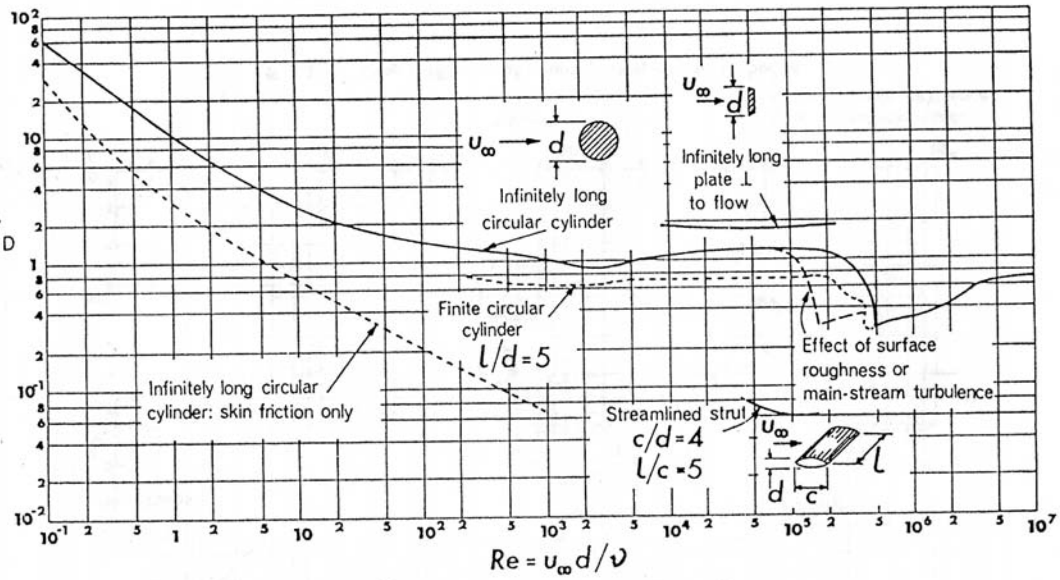


FIG. 1 Drag coefficient for two-dimensional bodies

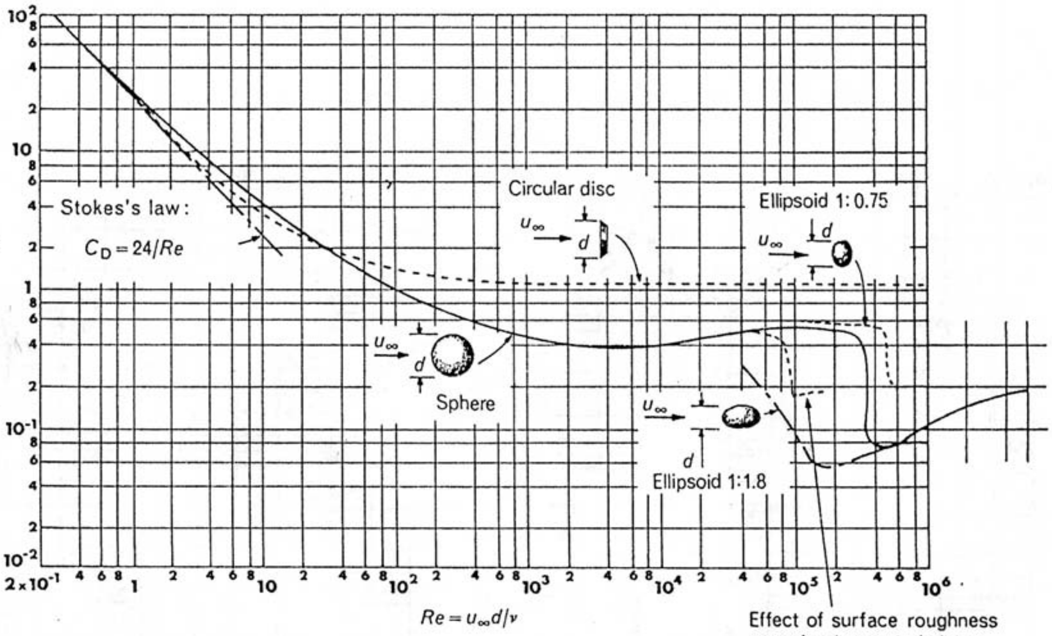


FIG. 2 Drag coefficients of smooth, axially-symmetric bodies

Typical drag coefficients for objects of interest are given in table (1).

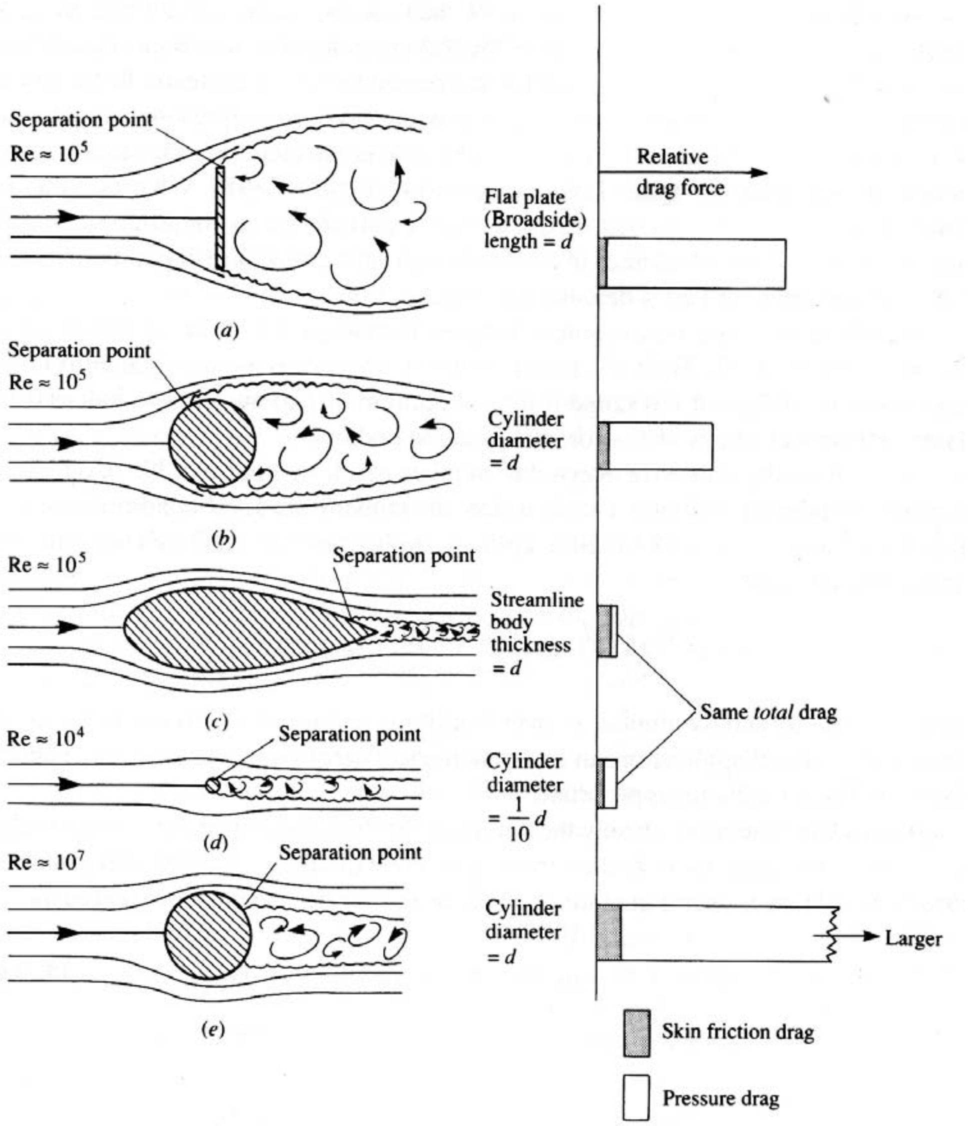
Shape	Reference area	Drag coefficient C_D
Parachute	Frontal area $A = \frac{\pi}{4}D^2$	1.4
Porous parabolic dish	Frontal area $A = \frac{\pi}{4}D^2$	Porosity: 0 / 0.2 / 0.5 → : 1.42 / 1.20 / 0.82 ← : 0.95 / 0.90 / 0.80 Porosity = open area/total area
Average person	Standing	$C_DA = 9\ ft^2$
	Sitting	$C_DA = 6\ ft^2$
	Crouching	$C_DA = 2.5\ ft^2$
Fluttering flag	$A = \ell D$	ℓ/D = 1: 0.07 ℓ/D = 2: 0.12 ℓ/D = 3: 0.15
Empire State Building	Frontal area	1.4
Six-car passenger train	Frontal area	1.8
Bikes		
Upright commuter	$A = 5.5\ ft^2$	1.1
Racing	$A = 3.9\ ft^2$	0.88
Drafting	$A = 3.9\ ft^2$	0.50
Streamlined	$A = 5.0\ ft^2$	0.12
Tractor-trailor tucks		
Standard	Frontal area	0.96
With fairing	Frontal area	0.76
With fairing and gap seal	Frontal area	0.70
Tree, $U = 10$ m/s	Frontal area	0.43
Tree, $U = 20$ m/s		0.26
Tree, $U = 30$ m/s		0.20
Dolphin	Wetted area	0.0036 at $Re = 6 \times 10^6$ (flat plate has $C_{Df} = 0.0031$)
Large birds	Frontal area	0.40

Table (1)

- **Streamlined body:** the pressure drag on streamlined body is small and the friction drag is the major part of the total drag.

- **Bluff body:** the pressure drag on bluff body is much greater than the friction drag.

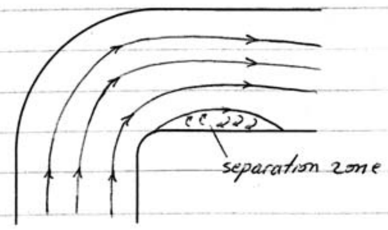
Relative comparison between skin friction drag and pressure drag for various aerodynamic shapes

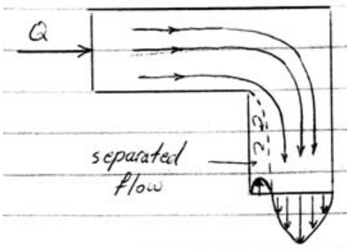


7- Separation of flow inside duct systems:

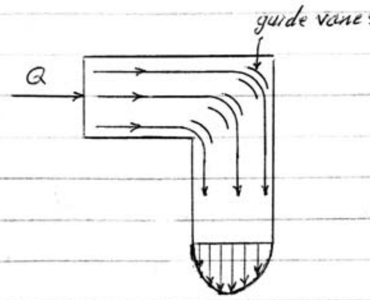
Most duct systems consist of more than straight duct. These additional components produce head losses (termed minor losses).
The minor losses are due to the separation region of flow inside duct as shown in the following figures:

Flow pattern in 90° bend.

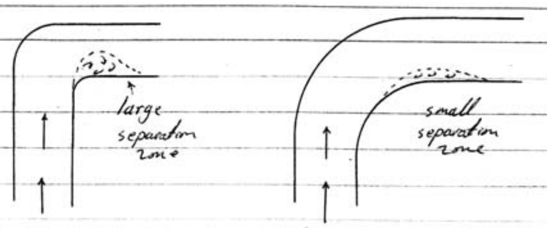




Flow in a 90° bend without guide vanes

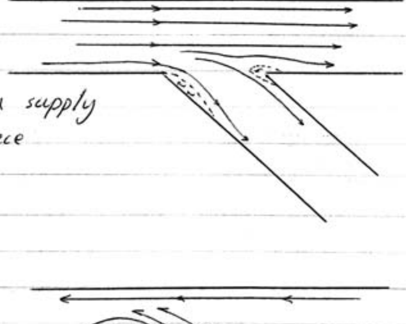


Flow in a 90° bend with guide vanes

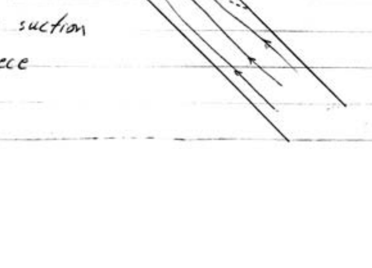


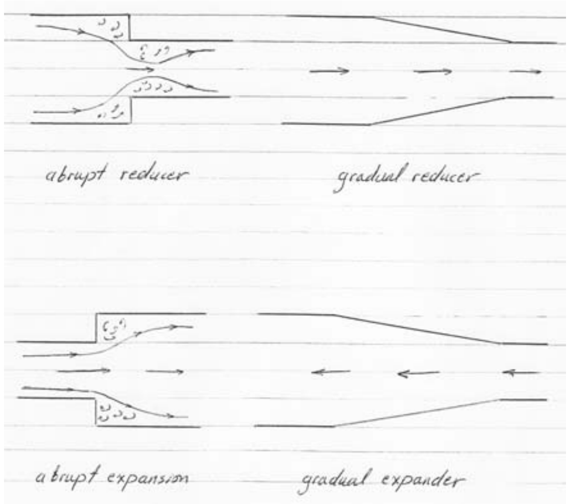
effect of curvature of a 90° bend.

airflow in a supply branch piece



airflow in a suction branch piece





8- Examples:

1- A smooth flat plate 3 m wide and 30 m long is towed through still water (ρ = 998 kg/m^3 , $\nu = 1.007\times10^{-6}$ m^2/s) with speed of 6 m/s. Determine the friction drag on one side of the plate and on the first 3 m of the plate.

Solution:

For the whole plate:-

$$Re_L = \frac{U_\infty L}{\nu} = \frac{6\times 30}{1.007\times10^{-6}} = 1.787\times10^{8} > 5\times10^{5}$$

the B.L is turbulent.

$$C_f = \frac{0.455}{(\log_{10} Re_L)^{2.58}} = 0.00196$$

The drag on one side is:-

$$F_f = \frac{1}{2}\rho U_\infty^2 A C_f$$

$$F_f = \frac{1}{2} * 998 * 6^2 * (3*30) * 0.00196 = \underline{3169\ N} \quad (Ans)$$

For the first 3m of the plate: $Re_x = 1.787\times10^{7} > 10^{7}$

$$F_f = \frac{1}{2} * 998 * 6^2 * (3*3) * \frac{0.455}{[\log_{10}(1.787\times10^{7})]^{2.58}}$$

$$F_f = \underline{443\ N} \quad (Ans)$$

2- Calculate the diameter of a parachute (in the form of a hemispherical shell) to be used for dropping a small object of mass 90 kg so that it touches the earth at a velocity no greater than 6 m/s. The drag coefficient for a hemispherical shell with its concave side upstream is approximately1.32 for Re > 10^3, (ρ = 1.22 kg/m^3).

Solution:

$$F_D = \frac{1}{2} \rho U_\infty^2 A C_D \quad \text{—①}$$

also

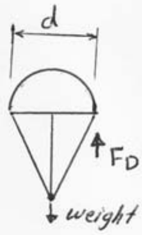
$$F_D = \text{weight} = m.g \quad \text{—②}$$

$$= 90 * 9.81 = 882.9 N$$

eqn ① = eqn ②

$$882.9 = \frac{1}{2} * 1.22 * 6^2 \; \frac{\pi d^2}{4} * 1.32$$

$$\Rightarrow \underline{d = 6.23 \; m} \quad (Ans)$$



3- If $\frac{u}{U_\infty} = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$, find the thickness of the boundary layer, the shear stress at the trailing edge, and the drag force on one side of the plate 1 m long, if it is immersed in water flowing with a velocity of 0.3 m/s (ρ = 1000 kg/m^3 , μ = 0.001 Pa.s)

Solution:

$$Re_L = \frac{\rho L U_\infty}{\mu} = \frac{1000 * 1 * 0.3}{0.001} = 3 \times 10^5 < 5 \times 10^5$$

The flow is laminar ; assume the width of the plate = 1m

Velocity profile: $f(\eta) = 2\eta - \eta^2$

$$\alpha = \int_0^1 f(\eta)(1 - f(\eta))\, d\eta = \int_0^1 2\eta - \eta^2 (1 - 2\eta + \eta^2)\, d\eta = \frac{2}{15}$$

$$= 0.133$$

$$\beta = \left[\frac{d f(\eta)}{d\eta}\right]_{\eta=0} = 2$$

Boundary layer thickness at the trailing edge :- $(x = L)$

$$\delta = \sqrt{\frac{2\beta}{\alpha}} \; \frac{L}{\sqrt{Re_L}} = 0.01 m$$

$$= \underline{10 \; mm} \quad (Ans)$$

Shear stress at the trailing edge :-

$$\tau_o = \frac{\mu U_\infty \beta}{\delta} = \frac{0.001 * 0.3 * 2}{0.01} = \underline{0.06 \; N/m^2} \quad (Ans)$$

$$F_f = \rho b U_\infty^2 \delta \alpha$$

$$F_f = 1000 * 1 * (0.3)^2 * 0.01 * 0.133$$

$$= \underline{0.12 \; N} \quad (Ans)$$

University of Technology	Sheet No. 2
Mechanical Engineering Dep.	Boundary Layer Theory
Fluid Mechanics II (3 rd year)	2011/2012

1- Calculate the displacement thickness and momentum thickness for the following velocity profiles in the boundary layer:

a- $\frac{u}{U_\infty} = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$; b- $\frac{u}{U_\infty} = \left(\frac{y}{\delta}\right)^{\frac{1}{9}}$ [$\frac{1}{3}\delta ; \frac{2}{15}\delta ; 0.1\delta ; \frac{9}{110}\delta$]

2- Air ($\nu = 1.8\times10^{-5}$ m^2/s) flows along a flat plate with a velocity of 150 km/hr. How long does the plate have to be to obtain a laminar boundary layer thickness of 8 mm.

[6.146 m]

3- Assuming that the velocity distribution in the laminar boundary layer: $\frac{u}{U_\infty} = \sin\left(\frac{\pi y}{2\delta}\right)$.

Determine the total friction coefficient in terms of the Reynolds number. [$1.31/\sqrt{\mathrm{Re}_L}$]

4- A thin plate 2 m wide is placed in a uniform air stream of velocity 100 m/s, ($\rho = 1.2$ kg/m^3). If the skin friction drag force is 60 N, calculate the displacement thickness of the boundary layer at trailing edge of the plate. Assume that the velocity profile at all points in the boundary layer is: $f(\eta) = \eta^{1/6}$. [3.3 mm]

5- A river barge which is 50 m long and 12 m wide has flat bottom; therefore, its resistance is similar to one side of a flat plate. If the barge is towed at speed of 3 m/s through still water, what towing force is required to overcome viscous resistance and what is the boundary layer thickness at mid length? Assume the boundary layer is turbulent for the entire length. ($\rho = 1000$ kg/m^3 ; $\nu = 1.21\times10^{-6}$ m^2/s) [5.57 kN ; 0.26 m]

6- A uniform free stream of air at 0.8 m/s flows over a flat plate (4 m long × 1 m wide). Assuming the boundary layer to be laminar on the plate and the velocity profile is:

$\frac{u}{U_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$. Find the ratio of the drag force on the front half portion to the drag force on the rear half portion of the plate. ($\rho = 1.2$ kg/m^3 ; $\nu = 1.51\times10^{-5}$ m^2/s) [2.42]

7- Air flows over a horizontal smooth flat plate at speed 14.5 m/s. The plate length is 1.5 m and its width is 0.8 m. The boundary layer is turbulent from the leading edge. The velocity profile is: $\frac{u}{U_\infty} = \eta^{\frac{1}{6}}$ where $\eta = \frac{y}{\delta}$. Evaluate the boundary layer thickness and the wall shear stress at the trailing edge of the plate. ($\rho = 1.21$ kg/m^3 ; $\nu = 1.5\times10^{-5}$ m^2/s)

[30.75 mm ; 0.447 N/m^2]

8- Air (ρ = 1.21 kg/m^3) flows over a thin flat plate 1 m long and 0.3 m wide. The flow is uniform at the leading edge of the plate. Assume the velocity profile in the boundary layer is linear, and the free stream velocity is 2.7 m/s. Using control volume (abcd) shown in figure, compute the mass flow rate across surface (ab). Determine the magnitude and direction of the x- component of the force required to hold plate stationary.

[3.9×10^{-3} kg/s ; -3.5×10^{-3} N]

9- Estimate the power required to move a flat plate 9 m long and 3 m wide in oil (ρ = 920 kg/m^3 ; μ = 0.067 Pa.s) at 8 m/s. For the following cases:
a- the boundary layer is laminar over the surface of the plate.
b- the boundary layer is turbulent over the surface of the plate from the leading edge.
c- transition from laminar to turbulent at $Re_c = 5\times10^5$.
(Assume the velocity profile for the turbulent boundary layer is $f(\eta) = \eta^{1/9}$).

[8.5 kW ; 28.55 kW ; 18.05 kW]

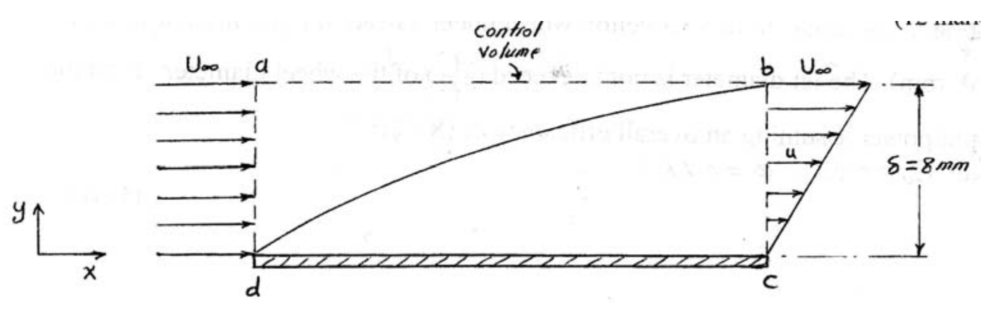
10- For the velocity profile: $\frac{u}{U_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$, determine whether the flow has separated or not separated or will attach with the surface after separation.

11- A honeycomb type of flow straightener is formed from perpendicular flat metal strips to give 25 mm square passages, 150 mm long. Water of kinematic viscosity 1.21 mm^2/s approaches the straightener at 1.8 m/s. Calculate the displacement thickness of the boundary layer and the velocity of the main stream at the outlet end of the straightener. Applying Bernoulli's equation to the main stream, deduce the pressure drop through the straightener. [0.546 mm ; 1.968 m/s ; 316.5 Pa]

12- Air of kinematic viscosity 15 mm^2/s and density 1.21 kg/m^3 flows past a smooth 150 mm diameter sphere at 60 m/s. Determine the drag force. What would be the drag force on a 150 mm diameter circular disc held perpendicular to this air stream.

[3 N ; 42 N]

13- The chimney of a boiler house is 50 m tall and has an outside diameter of 3 m. Compute the overturning moment about the base if a 30 m/s wind blows past it at the standard atmospheric conditions. (ρ = 1.21 kg/m^3 ; $\nu = 15\times10^{-6}$ m^2/s) [1430 kN.m]



Problem No. 8