Chapter Six
Ideal Fluid Flow

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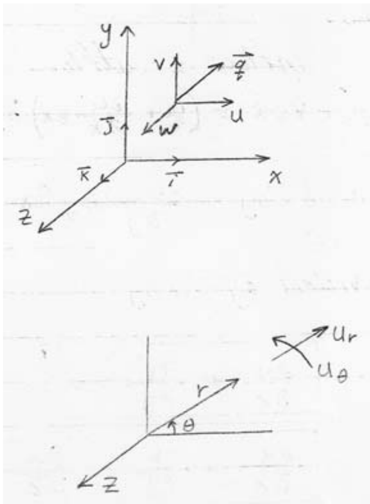
1- Introduction

Velocity vector

$\vec{q} = u\vec{i} + v\vec{j} + w\vec{k}$ In Cartesian coordinates

$\vec{q} = u_r\vec{r} + u_\theta\vec{\theta} + w\vec{k}$ In Polar coordinates



Divergence of $\vec{q} = \nabla \cdot \vec{q}$

$$\nabla \cdot \vec{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Continuity equation

$\nabla \cdot \vec{q} = 0$

Or

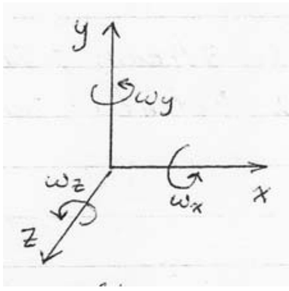
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Curl of $\vec{q} = \nabla \times \vec{q}$

Vorticity equation

$$\nabla \times \vec{q} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

$$\nabla \times \vec{q} = \omega_x \vec{i} + \omega_y \vec{j} + \omega_z \vec{k}$$



If $\nabla \times \vec{q} \neq 0$ the flow is called rotational

If $\nabla \times \vec{q} = 0$ the flow is called irrotational

2- Requirements for ideal- fluid flow

1- non viscous.

2- incompressible.

3- $\nabla \cdot \vec{q} = 0$

4- $\nabla \times \vec{q} = 0$

3- Relationships between stream function (ψ), potential function (φ) and velocity component

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x}$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y}$$

In cylindrical coordinates :

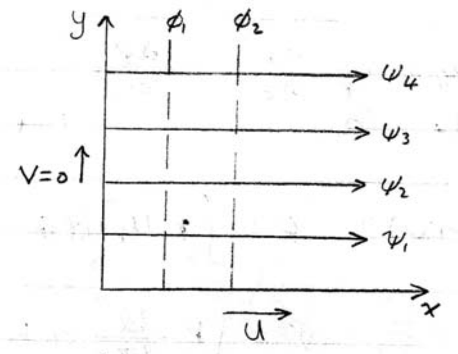
$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r}$$

$$u_\theta = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

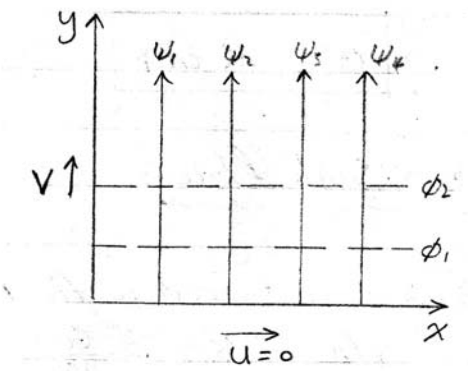
4- Basic flow patterns:

1- Uniform flow

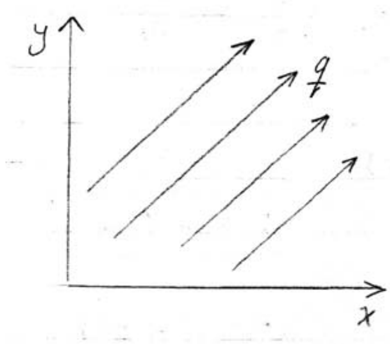
a- Uniform flow in the x- direction



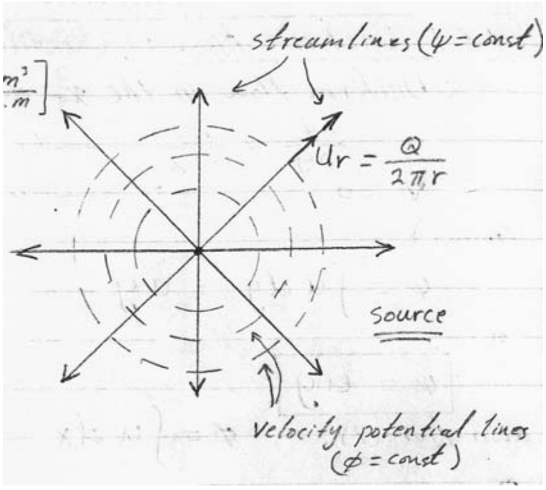
b- Uniform flow in the y- direction



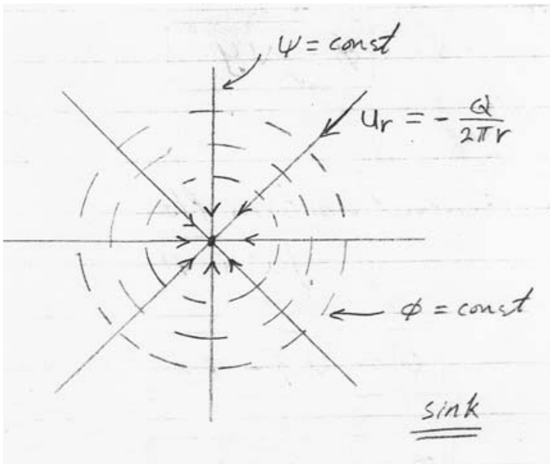
c- General uniform flow



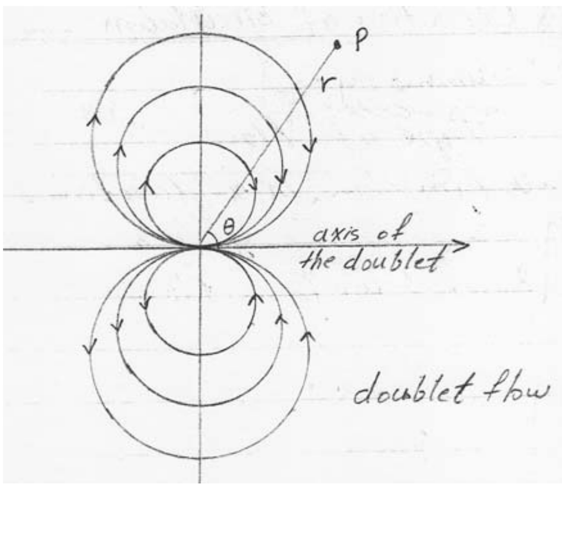
2- Source flow



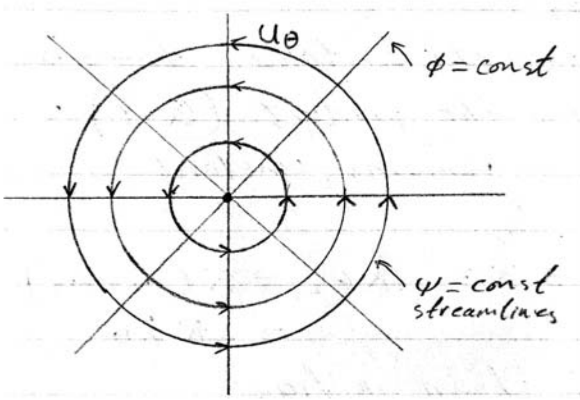
3- Sink flow



4- Doublet flow



5- Free vortex flow



Stream function and Potential function for Basic flow patterns:

Type of flow	ψ	φ
Uniform flow in the x- direction Uniform flow in the y- direction General uniform flow	uy -vx uy-vx	ux vy ux+vy
Source	kθ	k ln r
Sink	-kθ	-k ln r
Doublet	$\frac{-\mu}{2\pi}\frac{\sin\theta}{r}$	$\frac{\mu}{2\pi}\frac{\cos\theta}{r}$
Free vortex	$-\frac{\Gamma}{2\pi}\ln r$	$\frac{\Gamma}{2\pi}\theta$

Note: k = strength of the source = $\frac{Q}{2\pi}$ or (= $\frac{m}{2\pi}$)

Definition of circulation (Γ):

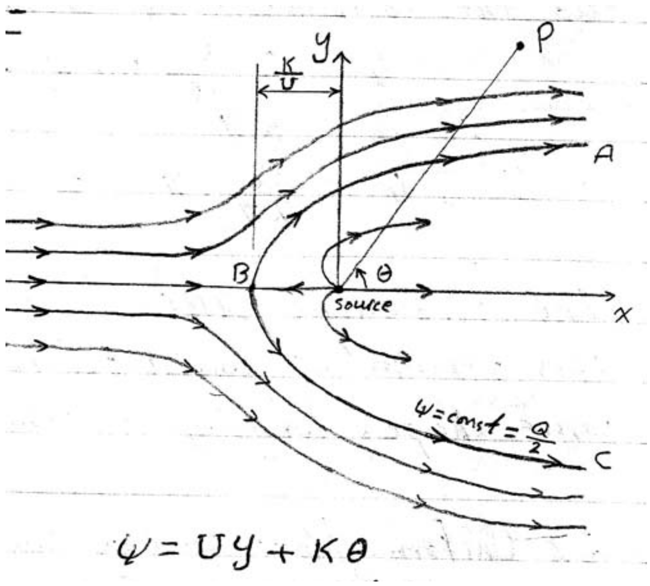
$$\Gamma = \oint_c q_s ds$$

Circulation = vorticity×area

$$\Gamma = \omega_z \times A$$

5- Combination of basic flows:

1- Uniform flow and a source.



The stream function:

$$\psi = Uy + k\theta$$

$$\psi = U.r\sin\theta + k\theta$$

The velocity components:

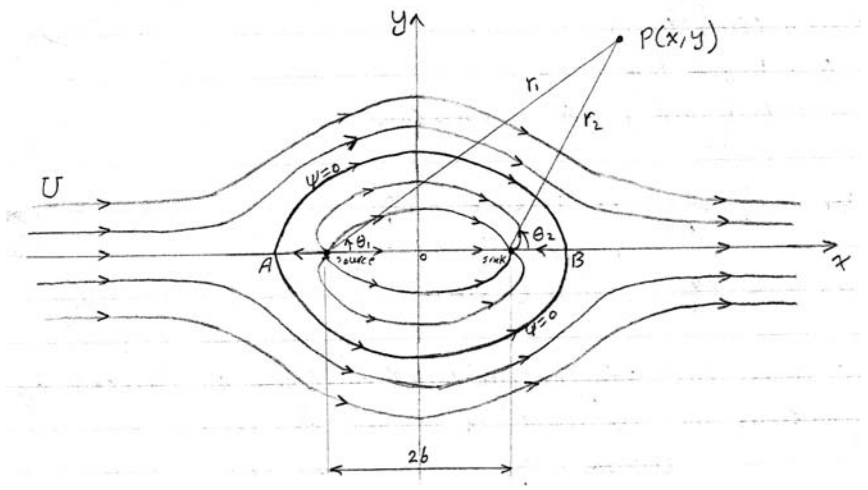
$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U.\cos\theta + \frac{k}{r}$$

and

$$u_\theta = -\frac{\partial\psi}{\partial r} = -U.\sin\theta$$

The dividing streamline ($\psi = \frac{Q}{2}$) could be replaced by a solid surface of the same shape, forming a semi-infinite body (half-body).

2- Uniform flow and a source-sink pair.



The stream function:

$$\psi = Uy + k\theta_1 - k\theta_2$$

$$\psi = Uy + k\tan^{-1}\frac{y}{x+b} - k\tan^{-1}\frac{y}{x-b}$$

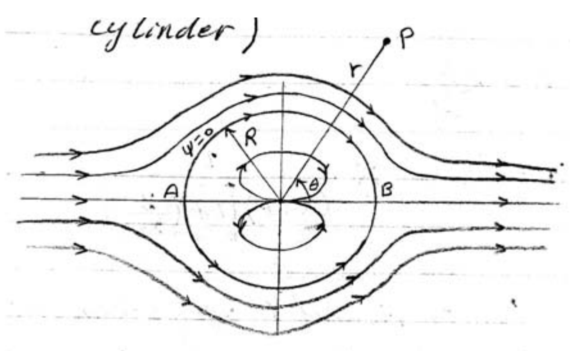
The velocity component:

$$u = \frac{\partial \psi}{\partial y} = U + \frac{k}{(x+b)\left[1+\left(\frac{y}{x+b}\right)^2\right]} - \frac{k}{(x-b)\left[1+\left(\frac{y}{x-b}\right)^2\right]}$$

The dividing streamline ($\psi = 0$) could be replaced by a solid surface of the same shape, forming an oval called a Rankine oval.

3- Uniform flow and a doublet:

(Non lifting flow over a cylinder)



The stream function:

$$\psi = Uy - \frac{\mu}{2\pi}\frac{\sin\theta}{r}$$

$$\psi = U.r\sin\theta\left(1 - \frac{R^2}{r^2}\right)$$

where

$$R^2 = \frac{\mu}{2\pi U}$$

The velocity components:

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U.\cos\theta\left(1 - \frac{R^2}{r^2}\right)$$

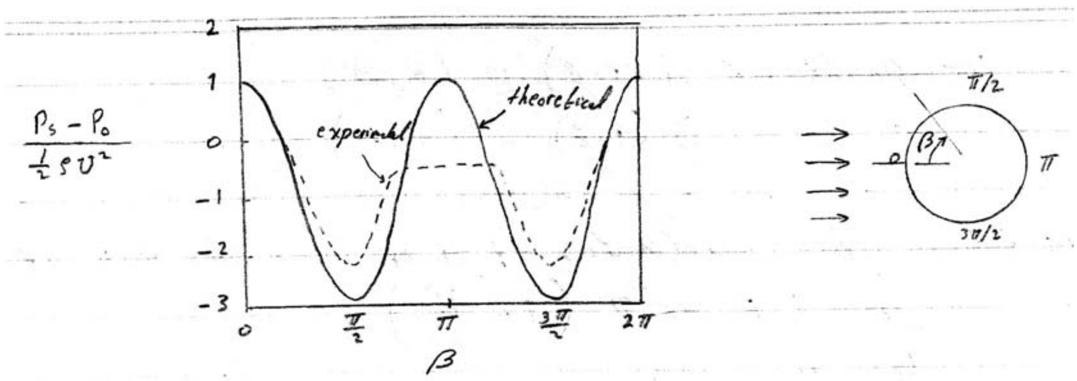
and

$$u_\theta = -\frac{\partial\psi}{\partial r} = -U.\sin\theta\left(1 + \frac{R^2}{r^2}\right)$$

The dividing streamline ($\psi = 0$) could be replaced by a solid surface of the same shape, forming a circular cylinder with radius $R = \sqrt{\frac{\mu}{2\pi U}}$.

The pressure distribution on the cylinder surface is obtained from:

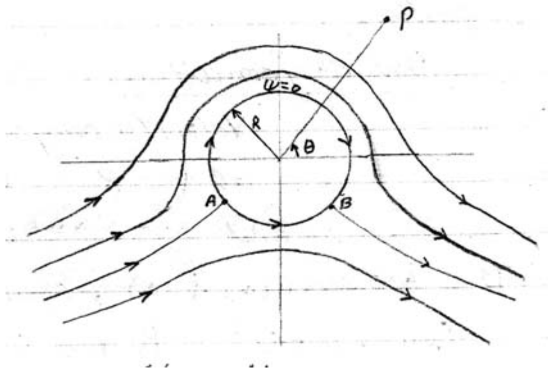
$$P_s = P_o + \frac{1}{2}\rho U^2\left(1 - 4\sin^2\theta\right)$$



The pressure distribution is symmetrical around the cylinder and the resultant force developed on the cylinder = zero.

<u>4- Doublet and free vortex in a uniform flow</u>:

(Lifting flow over a cylinder)



The stream function:

$$\psi = U.r\sin\theta\left(1-\frac{R^2}{r^2}\right)+\frac{\Gamma}{2\pi}\ln r$$

The velocity components:

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U.\cos\theta\left(1-\frac{R^2}{r^2}\right)$$

and

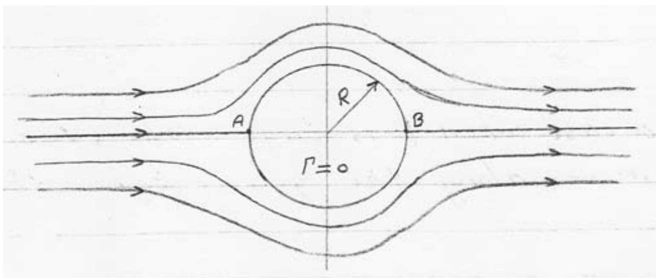
$$u_\theta = -\frac{\partial\psi}{\partial r} = -U.\sin\theta\left(1+\frac{R^2}{r^2}\right)-\frac{\Gamma}{2\pi r}$$

The location of the stagnation points is given by:

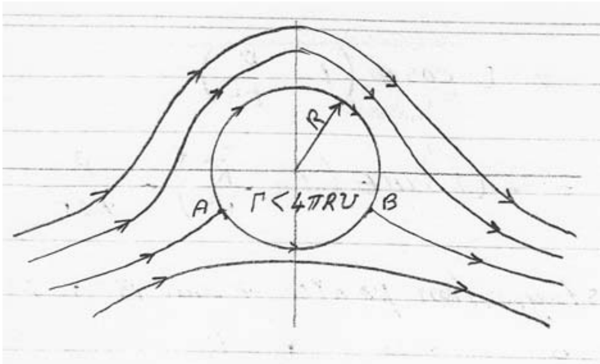
$$r = R \;;\; \sin\theta = \left(\frac{-\Gamma}{4\pi RU}\right)$$

There are four possible cases:

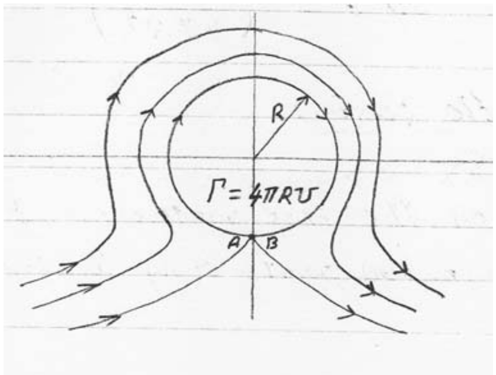
a- $(\Gamma = 0)$



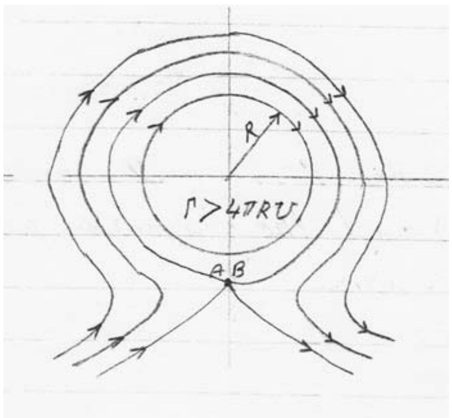
b- (Γ < 4πRU)



c- (Γ = 4πRU)



d- (Γ > 4πRU)

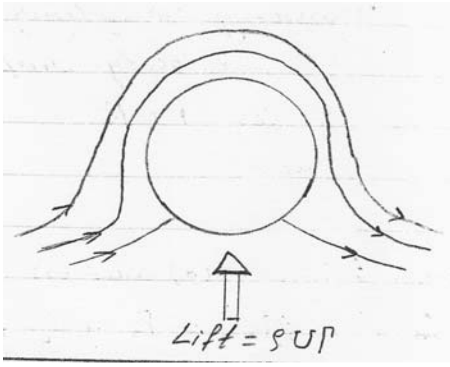


The pressure distribution on the cylinder surface is obtained from:

$$P_s = P_o + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho\left(-2U.\sin\theta - \frac{\Gamma}{2\pi R}\right)^2$$

The lift force on the cylinder is

Lift = $\rho U \Gamma L$ where L = length of the cylinder



6- Examples:

1- Does the stream function ($\psi = xy$) represent a physically possible flow? If so, determine the velocity at a point (2,3).

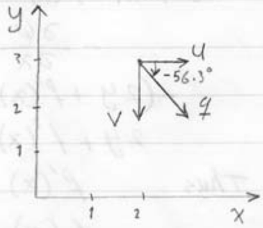
Solution:

To prove that the flow is possible, the continuity equation must be satisfied.

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

$$1 - 1 = 0$$

$u = \frac{\partial \psi}{\partial y} = x$; $v = -\frac{\partial \psi}{\partial x} = -y$



at point (2,3) $u = x = 2$

$v = -y = -3$

$q = \sqrt{u^2 + v^2} = \sqrt{4+9} = 3.6$ units (Ans)

$\theta = \tan^{-1} \frac{v}{u} = \frac{-3}{2} \Longrightarrow \theta = -56.3^\circ$ (Ans)

2- A velocity potential in two-dimensional flow is given by ($\varphi = y+x^2-y^2$); find the stream function for this flow.

Solution:

a)- $\phi = y + x^2 - y^2$

$$\frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = 2x$$

$$\partial \psi = 2x \, \partial y$$

$$\therefore \quad \psi = 2xy + f(x) \qquad \text{———} (*) \qquad f(x) = \text{constant of integration.}$$

To find $f(x)$

$$-\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y}$$

$$-(2y + f'(x)) = 1 - 2y$$

$$2y + f'(x) = -1 + 2y$$

Thus $\quad f'(x) = -1$

$$f(x) = -x + C$$

sub in eqn $(*)$ $\implies$ $\psi = 2xy - x + C$ (Ans)

3- A stream function in two-dimensional flow is ($\psi = 9+6x-4y+7xy$); find the velocity potential for this flow.

Solution:

$\psi = 9 + 6x - 4y + 7xy$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} = -4 + 7x$$

$$\therefore \quad \phi = -4x + \frac{7}{2} x^2 + f(y) \qquad \text{where } f(y) = \text{constant of integration.}$$

to find $f(y)$

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$f'(y) = -(6 + 7y) = -6 - 7y$$

$$\therefore \quad f(y) = -6y - \frac{7}{2} y^2 + C$$

$$\therefore \quad \phi = -4x - 6y + \frac{7}{2}(x^2 - y^2) + C \qquad \text{(Ans)}$$

University of Technology	Sheet No. 6
Mechanical Engineering Dep.	Ideal Fluid Flow
Fluid Mechanics II (3 rd year)	2011/2012

1- Show that the two-dimensional flow described by the equation $\psi = x + 2x^2 - 2y^2$ is irrotational. Find the velocity potential for this flow. [$\phi = -y - 4xy + c$]

2- A certain flow field is described by the velocity potential $\phi = A\ln r + Br\cos\theta$ where A and B are positive constants. Determine the corresponding stream function and locate any stagnation points in this flow field. [$\psi = A\theta + Br\sin\theta + c$; $\left(\frac{A}{B},0\right)$ $\left(\frac{A}{B},\pi\right)$]

3- The velocity components in a two-dimensional flow field for an incompressible fluid are expressed as: $u = \frac{y^3}{3} + 2x - x^2y$; $v = xy^2 - 2y - \frac{x^3}{3}$.

a) show that these functions represents a possible case of irrotational flow.

b) obtain expressions for the stream function and velocity potential.

4- The formula $\phi = 0.04x^3 + axy^2 + by^3$ represent the velocity potential of a two-dimensional ideal flow. Evaluate the constants a and b, and calculate the pressure difference between the points (0,0) and (3,4)m, if the fluid has density of 1300 kg/m^3.

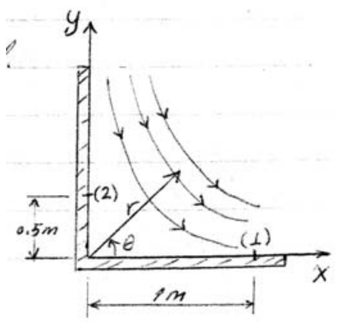
[a = -0.12, b = 0 ; 5.85 kN/m^2]

5- The two-dimensional flow of a non-viscous, incompressible fluid in the vicinity of the 90^o corner of figure is described by the stream function $\psi = 2r^2 \sin 2\theta$.

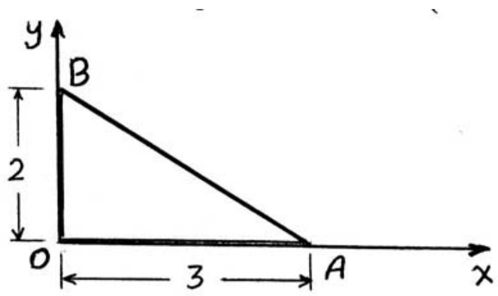
a) determine the corresponding velocity potential.

b) if the pressure at point(1,0) on the wall is 30kPa, what is the pressure at point (0,0.5) , assume ρ = 1000 kg/m^3 , and x-y plane is horizontal. [$\phi = 2r^2 \cos 2\theta + c$; 36 kPa]

6- The stream function for an incompressible flow filed is given by the equation $\psi = 3tx^2y - ty^3$. Find the potential function and determine the flow rates across the faces of the triangular prism OAB shown in figure having a thickness of 5 units in the z-direction at time t = 1. [$\phi = tx^3 - 3txy^2 + c$; 40; 0; 40]



Problem No. 5



Problem No.6

7- Prove that for a two-dimensional flow, the vorticity at a point is twice the rotation (angular velocity).

8- The pressure far from an irrotational vortex in the atmosphere is zero gage. If the velocity at r = 20 m is 20 m/s, find the velocity and pressure at r = 2 m. ($\rho = 1.2$ kg/m^3)
[200 m/s ; -23.76 kPa]

9- A non viscous incompressible fluid flow between wedge shaped-wall into small opening as shown in figure. The velocity potential which described the flow is $\phi = -2 \ln r$. Determine the volume rate of flow (per unit length) in the opening. [$-\pi/3$ m^3/s per m]

10- A source with strength $0.2/2\pi$ m^3/s.m and a vortex with strength $1/2\pi$ m^2/s are located at the origin. Determine the equations for velocity potential and stream function. What are the velocity components at x = 1 m , y = 0.5 m? [0.0285 m/s ; 0.143 m/s]

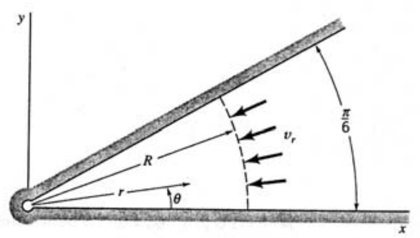
11- In an infinite two-dimensional flow filed, a sink of strength $3/2\pi$ m^3/s.m is located at the origin, and another of strength $4/2\pi$ m^3/s.m at (2 , 0). What is the magnitude and direction of the velocity at point (0 , 2). [0.429 m/s ; -68.22^o]

12- Flow over a plane half-body is studied by utilizing a free-stream at 5 m/s superimposed on a source at the origin. The body has a maximum width 2 m. Calculate:
a) the coordinates of the stagnation point.
b) the width of the body at the origin.
c) the velocity at a point (0.5 , $\pi/2$). [(0.32 , π) ; 1 m ; 5.93 m/s]

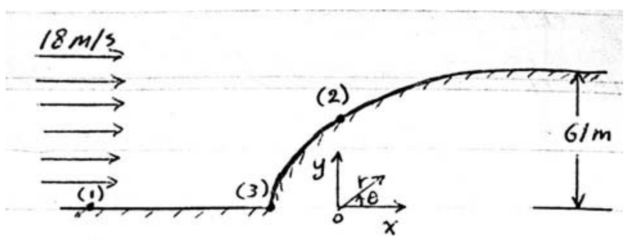
13- The shape of a hill arising from a plain can be approximated with the top section of a half-body as is shown in figure. The height of the hill approaches 61 m. When a 18 m/s wind blows toward the hill, what is the magnitude of the air velocity at point (2) above the origin. What is the elevation of point (2) and what is the difference in pressure between point (1) and point (2). ($\rho_{air} = 1.23$ kg/m^3) [21.34 m/s ; 30.5 m ; 448.83 Pa]

14- A circular cylinder 0.5 m diameter rotates at 600 rpm in a uniform stream of 15 m/s. Locate the stagnation points. Calculate the minimum rotational speed for detached stagnation point in the same uniform flow. [-31.6^o and -148.4^o ; 1146 rpm]

15- A circular cylinder 20 m long is placed in a uniform stream of 100 m/s (ρ= 0.7 kg/m^3). The lift force generated by the cylinder is 2100 kN. The stagnation points are at (-60^o and -120^o). Derive a relationship between the locations of the stagnation points and the circulation around the cylinder. Calculate the diameter of the cylinder. [2.75 m]



Problem No. 9



Problem No. 13