CHAPTER

2

Steady-State Conduction—One Dimension

2-2 | THE PLANE WALL

First consider the plane wall where a direct application of Fourier's law [Equation (1-1)] may be made. Integration yields

$$q = -\frac{kA}{\Delta x}(T_2 - T_1) \tag{2-1}$$

when the thermal conductivity is considered constant. The wall thickness is Δx, and T_1 and T_2 are the wall-face temperatures. If the thermal conductivity varies with temperature according to some linear relation $k = k_0(1 + \beta T)$, the resultant equation for the heat flow is

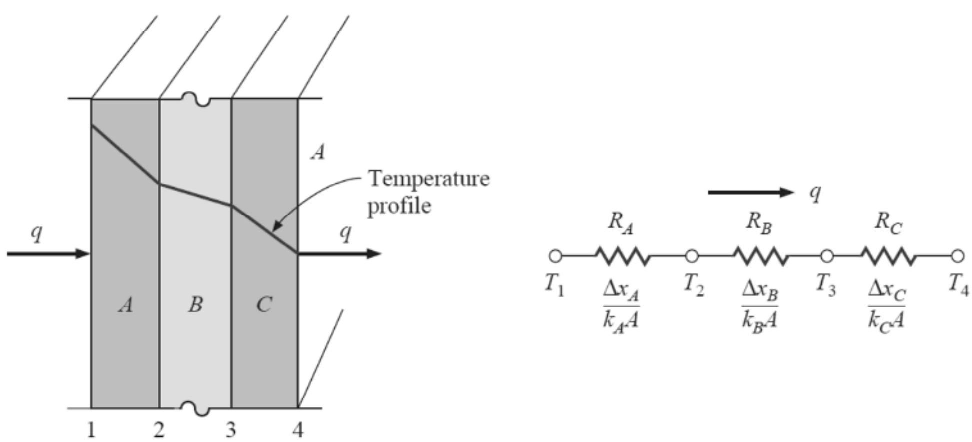
$$q = -\frac{k_0 A}{\Delta x}\left[(T_2 - T_1) + \frac{\beta}{2}(T_2^2 - T_1^2)\right] \tag{2-2}$$

If more than one material is present, as in the multilayer wall shown in Figure 2-1, the analysis would proceed as follows: The temperature gradients in the three materials are shown, and the heat flow may be written

$$q = -k_A A \frac{T_2 - T_1}{\Delta x_A} = -k_B A \frac{T_3 - T_2}{\Delta x_B} = -k_C A \frac{T_4 - T_3}{\Delta x_C}$$

Note that the heat flow must be the same through all sections.

Figure 2-1 | One-dimensional heat transfer through a composite wall and electrical analog.



Solving these three equations simultaneously, the heat flow is written

$$q = \frac{T_1 - T_4}{\Delta x_A/k_A A + \Delta x_B/k_B A + \Delta x_C/k_C A} \qquad [2\text{-}3]$$

$$\text{Heat flow} = \frac{\text{thermal potential difference}}{\text{thermal resistance}} \qquad [2\text{-}4]$$

a relation quite like Ohm's law in electric-circuit theory. In Equation (2-1) the thermal resistance is $\Delta x/kA$, and in Equation (2-3) it is the sum of the three terms in the denominator. We should expect this situation in Equation (2-3) because the three walls side by side act as three thermal resistances in series. The equivalent electric circuit is shown in Figure 2-1*b*.

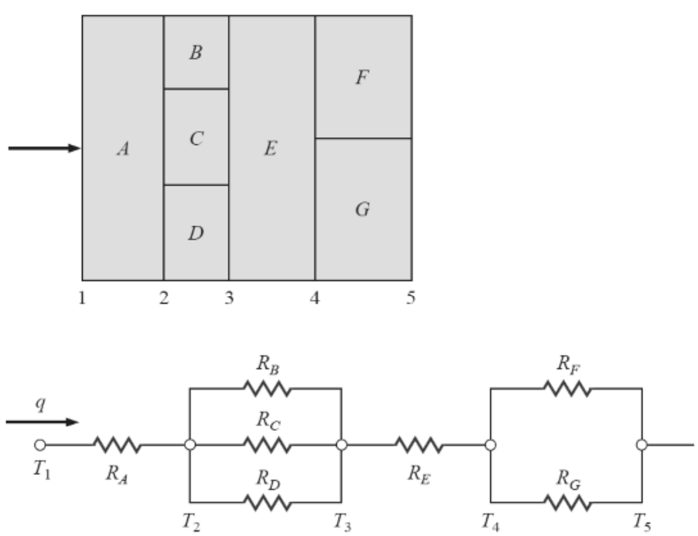
The electrical analogy may be used to solve more complex problems involving both series and parallel thermal resistances. A typical problem and its analogous electric circuit are shown in Figure 2-2. The one-dimensional heat-flow equation for this type of problem may be written

$$q = \frac{\Delta T_{\text{overall}}}{\sum R_{\text{th}}} \qquad [2\text{-}5]$$

where the R_{th} are the thermal resistances of the various materials. The units for the thermal resistance are °C/W or °F · h/Btu.

It is well to mention that in some systems, like that in Figure 2-2, two-dimensional heat flow may result if the thermal conductivities of materials *B*, *C*, and *D* differ by an appreciable amount. In these cases other techniques must be employed to effect a solution.

Figure 2-2 | Series and parallel one-dimensional heat transfer through a composite wall and electrical analog.



2-4 | RADIAL SYSTEMS

Cylinders

Consider a long cylinder of inside radius r_i, outside radius r_o, and length L, such as the one shown in Figure 2-3. We expose this cylinder to a temperature differential $T_i - T_o$ and ask what the heat flow will be. For a cylinder with length very large compared to diameter, it may be assumed that the heat flows only in a radial direction, so that the only space coordinate needed to specify the system is r. Again, Fourier's law is used by inserting the proper area relation. The area for heat flow in the cylindrical system is

$$A_r = 2\pi r L$$

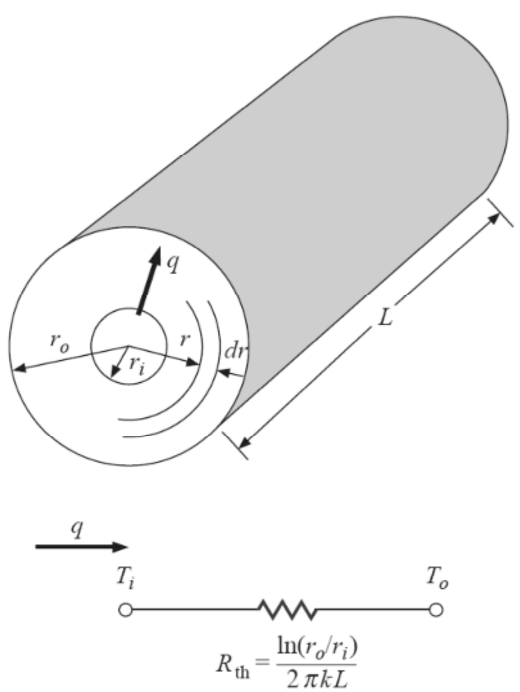
so that Fourier's law is written

$$q_r = -kA_r \frac{dT}{dr} \qquad \textbf{[2-7]}$$

or

$$q_r = -2\pi k r L \frac{dT}{dr}$$

Figure 2-3 | One-dimensional heat flow through a hollow cylinder and electrical analog.



with the boundary conditions

$$T = T_i \qquad \text{at } r = r_i$$
$$T = T_o \qquad \text{at } r = r_o$$

The solution to Equation (2-7) is

$$q = \frac{2\pi k L\,(T_i - T_o)}{\ln\,(r_o/r_i)} \qquad \textbf{[2-8]}$$

and the thermal resistance in this case is

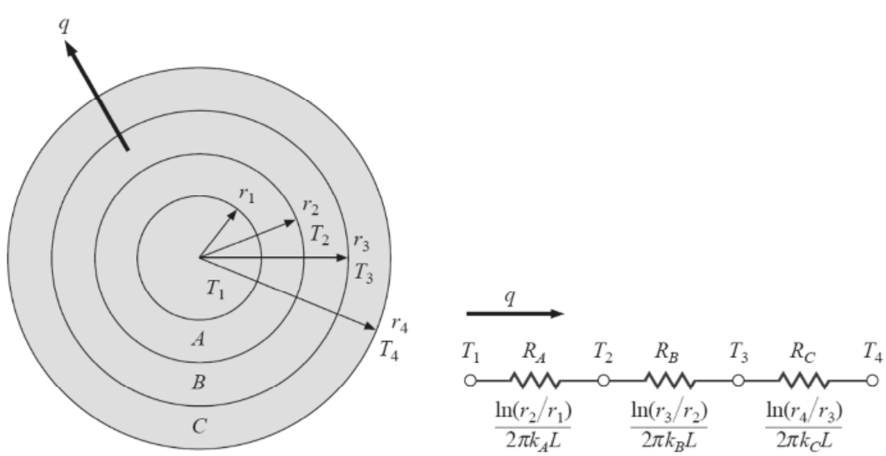
$$R_{\text{th}} = \frac{\ln\,(r_o/r_i)}{2\pi k L}$$

The thermal-resistance concept may be used for multiple-layer cylindrical walls just as it was used for plane walls. For the three-layer system shown in Figure 2-4 the solution is

$$q = \frac{2\pi L\,(T_1 - T_4)}{\ln\,(r_2/r_1)/k_A + \ln\,(r_3/r_2)/k_B + \ln\,(r_4/r_3)/k_C} \qquad \textbf{[2-9]}$$

The thermal circuit is also shown in Figure 2-4.

Figure 2-4 | One-dimensional heat flow through multiple cylindrical sections and electrical analog.



Spheres

Spherical systems may also be treated as one-dimensional when the temperature is a function of radius only. The heat flow is then

$$q = \frac{4\pi k\,(T_i - T_o)}{1/r_i - 1/r_o} \qquad \textbf{[2-10]}$$

The derivation of Equation (2-10) is left as an exercise.

Multilayer Conduction — EXAMPLE 2-1

An exterior wall of a house may be approximated by a 4-in layer of common brick [$k = 0.7\ \mathrm{W/m \cdot {}^\circ C}$] followed by a 1.5-in layer of gypsum plaster [$k = 0.48\ \mathrm{W/m \cdot {}^\circ C}$]. What thickness of loosely packed rock-wool insulation [$k = 0.065\ \mathrm{W/m \cdot {}^\circ C}$] should be added to reduce the heat loss (or gain) through the wall by 80 percent?

Solution

The overall heat loss will be given by

$$q = \frac{\Delta T}{\sum R_{\mathrm{th}}}$$

Because the heat loss with the rock-wool insulation will be only 20 percent (80 percent reduction) of that before insulation

$$\frac{q \text{ with insulation}}{q \text{ without insulation}} = 0.2 = \frac{\sum R_{\mathrm{th}} \text{ without insulation}}{\sum R_{\mathrm{th}} \text{ with insulation}}$$

We have for the brick and plaster, for unit area,

$$R_b = \frac{\Delta x}{k} = \frac{(4)(0.0254)}{0.7} = 0.145\ \mathrm{m^2 \cdot {}^\circ C/W}$$

$$R_p = \frac{\Delta x}{k} = \frac{(1.5)(0.0254)}{0.48} = 0.079\ \mathrm{m^2 \cdot {}^\circ C/W}$$

so that the thermal resistance without insulation is

$$R = 0.145 + 0.079 = 0.224\ \mathrm{m^2 \cdot {}^\circ C/W}$$

Then

$$R \text{ with insulation} = \frac{0.224}{0.2} = 1.122\ \mathrm{m^2 \cdot {}^\circ C/W}$$

and this represents the sum of our previous value and the resistance for the rock wool

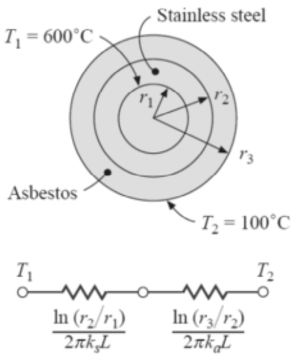
$$1.122 = 0.224 + R_{rw}$$

$$R_{rw} = 0.898 = \frac{\Delta x}{k} = \frac{\Delta x}{0.065}$$

so that

$$\Delta x_{rw} = 0.0584\ \mathrm{m} = 2.30\ \mathrm{in}$$

Figure Example 2-2



EXAMPLE 2-2 Multilayer Cylindrical System

A thick-walled tube of stainless steel [18% Cr, 8% Ni, $k = 19$ W/m · °C] with 2-cm inner diameter (ID) and 4-cm outer diameter (OD) is covered with a 3-cm layer of asbestos insulation [$k = 0.2$ W/m · °C]. If the inside wall temperature of the pipe is maintained at 600°C, calculate the heat loss per meter of length. Also calculate the tube–insulation interface temperature.

■ Solution

Figure Example 2-2 shows the thermal network for this problem. The heat flow is given by

$$\frac{q}{L} = \frac{2\pi (T_1 - T_2)}{\ln (r_2/r_1)/k_s + \ln(r_3/r_2)/k_a} = \frac{2\pi (600 - 100)}{(\ln 2)/19 + (\ln \frac{5}{2})/0.2} = 680 \text{ W/m}$$

This heat flow may be used to calculate the interface temperature between the outside tube wall and the insulation. We have

$$\frac{q}{L} = \frac{T_a - T_2}{\ln (r_3/r_2)/2\pi k_a} = 680 \text{ W/m}$$

where T_a is the interface temperature, which may be obtained as

$$T_a = 595.8°\text{C}$$

The largest thermal resistance clearly results from the insulation, and thus the major portion of the temperature drop is through that material.

Convection Boundary Conditions

We have already seen in Chapter 1 that convection heat transfer can be calculated from

$$q_{\text{conv}} = hA\,(T_w - T_\infty)$$

An electric-resistance analogy can also be drawn for the convection process by rewriting the equation as

$$q_{\text{conv}} = \frac{T_w - T_\infty}{1/hA} \qquad \textbf{[2-11]}$$

where the $1/hA$ term now becomes the convection resistance.

2-5 | THE OVERALL HEAT-TRANSFER COEFFICIENT

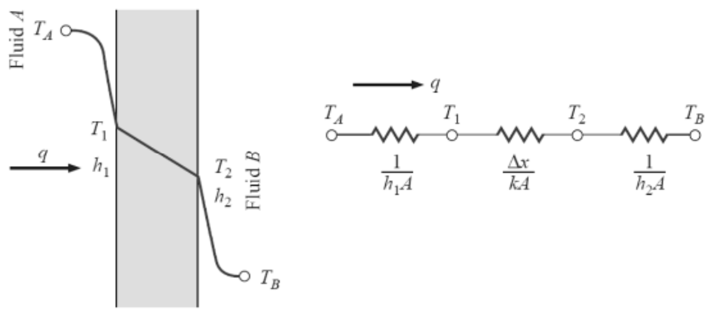
Consider the plane wall shown in Figure 2-5 exposed to a hot fluid A on one side and a cooler fluid B on the other side. The heat transfer is expressed by

$$q = h_1 A\,(T_A - T_1) = \frac{kA}{\Delta x}\,(T_1 - T_2) = h_2 A\,(T_2 - T_B)$$

The heat-transfer process may be represented by the resistance network in Figure 2-5, and the overall heat transfer is calculated as the ratio of the overall temperature difference to the sum of the thermal resistances:

$$q = \frac{T_A - T_B}{1/h_1 A + \Delta x/kA + 1/h_2 A} \qquad \textbf{[2-12]}$$

Figure 2-5 | Overall heat transfer through a plane wall.



Observe that the value $1/hA$ is used to represent the convection resistance. The overall heat transfer by combined conduction and convection is frequently expressed in terms of an overall heat-transfer coefficient U, defined by the relation

$$q = UA\,\Delta T_{\text{overall}} \qquad \textbf{[2-13]}$$

where A is some suitable area for the heat flow. In accordance with Equation (2-12), the overall heat-transfer coefficient would be

$$U = \frac{1}{1/h_1 + \Delta x/k + 1/h_2}$$

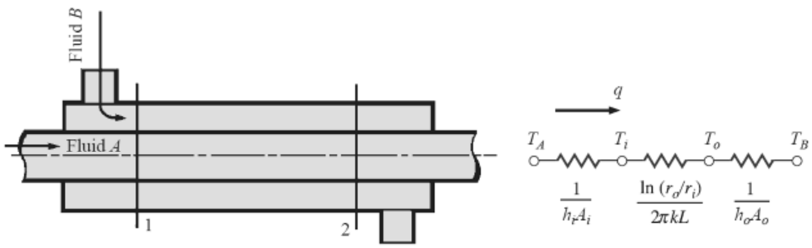
The overall heat-transfer coefficient is also related to the R value of Equation (2-6) through

$$U = \frac{1}{R \text{ value}}$$

For a hollow cylinder exposed to a convection environment on its inner and outer surfaces, the electric-resistance analogy would appear as in Figure 2-6 where, again, T_A and T_B are the two fluid temperatures. Note that the area for convection is not the same for both fluids in this case, these areas depending on the inside tube diameter and wall thickness. The overall heat transfer would be expressed by

$$q = \frac{T_A - T_B}{\dfrac{1}{h_i A_i} + \dfrac{\ln(r_o/r_i)}{2\pi k L} + \dfrac{1}{h_o A_o}} \qquad \textbf{[2-14]}$$

Figure 2-6 | Resistance analogy for hollow cylinder with convection boundaries.



in accordance with the thermal network shown in Figure 2-6. The terms A_i and A_o represent the inside and outside surface areas of the inner tube. The overall heat-transfer coefficient may be based on either the inside or the outside area of the tube. Accordingly,

$$U_i = \frac{1}{\dfrac{1}{h_i} + \dfrac{A_i \ln (r_o/r_i)}{2\pi k L} + \dfrac{A_i}{A_o}\dfrac{1}{h_o}} \qquad \text{[2-15]}$$

$$U_o = \frac{1}{\dfrac{A_o}{A_i}\dfrac{1}{h_i} + \dfrac{A_o \ln (r_o/r_i)}{2\pi k L} + \dfrac{1}{h_o}} \qquad \text{[2-16]}$$

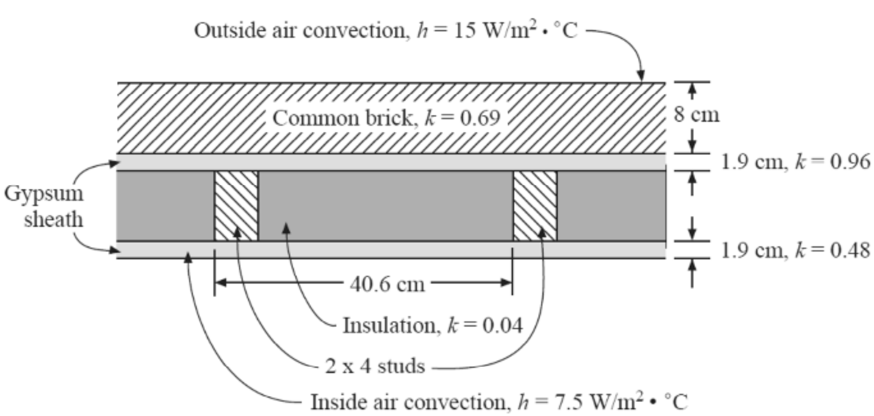
The general notion, for either the plane wall or cylindrical coordinate system, is that

$$UA = 1/\Sigma\, \mathrm{R_{th}} = 1/\mathrm{R_{th,overall}}$$

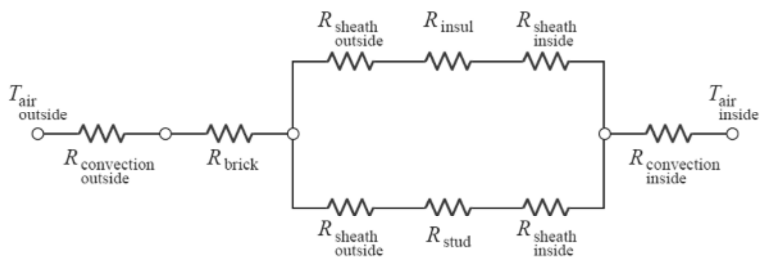
EXAMPLE 2-3 Heat Transfer Through a Composite Wall

"Two-by-four" wood studs have actual dimensions of 4.13×9.21 cm and a thermal conductivity of $0.1\ \mathrm{W/m \cdot {}^\circ C}$. A typical wall for a house is constructed as shown Figure Example 2-3. Calculate the overall heat-transfer coefficient and *R* value of the wall.

Figure Example 2-3 | (*a*) Construction of a dwelling wall; (*b*) thermal resistance model.



(*a*)



(*b*)

■ Solution

The wall section may be considered as having two parallel heat-flow paths: (1) through the studs, and (2) through the insulation. We will compute the thermal resistance for each, and then combine the values to obtain the overall heat-transfer coefficient.

1. *Heat transfer through studs* ($A = 0.0413\ \mathrm{m}^2$ for unit depth). This heat flow occurs through six thermal resistances:

a. Convection resistance outside of brick

$$R = \frac{1}{hA} = \frac{1}{(15)(0.0413)} = 1.614\ {}^\circ\mathrm{C/W}$$

b. Conduction resistance in brick

$$R = \Delta x/kA = \frac{0.08}{(0.69)(0.0413)} = 2.807\ {}^\circ\mathrm{C/W}$$

c. Conduction resistance through outer sheet

$$R = \frac{\Delta x}{kA} = \frac{0.019}{(0.96)(0.0413)} = 0.48\ {}^\circ\mathrm{C/W}$$

d. Conduction resistance through wood stud

$$R = \frac{\Delta x}{kA} = \frac{0.0921}{(0.1)(0.0413)} = 22.3\ {}^\circ\mathrm{C/W}$$

e. Conduction resistance through inner sheet

$$R = \frac{\Delta x}{kA} = \frac{0.019}{(0.48)(0.0413)} = 0.96\ {}^\circ\mathrm{C/W}$$

f. Convection resistance on inside

$$R = \frac{1}{hA} = \frac{1}{(7.5)(0.0413)} = 3.23\ {}^\circ\mathrm{C/W}$$

The total thermal resistance through the wood stud section is

$$R_{\text{total}} = 1.614 + 2.807 + 0.48 + 22.3 + 0.96 + 3.23 = 31.39\ {}^\circ\mathrm{C/W} \qquad [a]$$

2. *Insulation section* ($A = 0.406 - 0.0413$ m^2 for unit depth). Through the insulation section, five of the materials are the same, but the resistances involve different area terms, i.e., 40.6 − 4.13 cm instead of 4.13 cm, so that each of the previous resistances must be multiplied by a factor of $4.13/(40.6 - 4.13) = 0.113$. The resistance through the insulation is

$$R = \frac{\Delta x}{kA} = \frac{0.0921}{(0.04)(0.406 - 0.0413)} = 6.31$$

and the total resistance through the insulation section is

$$R_{\text{total}} = (1.614 + 2.807 + 0.48 + 0.96 + 3.23)(0.113) + 6.31 = 7.337\ ^\circ\text{C/W} \qquad [b]$$

The overall resistance for the section is now obtained by combining the parallel resistances in Equations (a) and (b) to give

$$R_{\text{overall}} = \frac{1}{(1/31.39) + (1/7.337)} = 5.947\ ^\circ\text{C/W} \qquad [c]$$

This value is related to the overall heat-transfer coefficient by

$$q = UA\Delta T = \frac{\Delta T}{R_{\text{overall}}} \qquad [d]$$

where A is the area of the total section = 0.406 m^2. Thus,

$$U = \frac{1}{RA} = \frac{1}{(5.947)(0.406)} = 0.414\ \text{W/m}^2 \cdot {}^\circ\text{C}$$

As we have seen, the R value is somewhat different from thermal resistance and is given by

$$R\ \text{value} = \frac{1}{U} = \frac{1}{0.414} = 2.414^\circ\text{C} \cdot \text{m}^2/\text{W}$$

■ Comment

This example illustrates the relationships between the concepts of thermal resistance, the overall heat-transfer coefficient, and the R value. Note that the R value involves a unit area concept, while the thermal resistance does not.

Overall Heat-Transfer Coefficient for a Tube — EXAMPLE 2-5

Water flows at 50°C inside a 2.5-cm-inside-diameter tube such that $h_i = 3500$ W/m$^2 \cdot$°C. The tube has a wall thickness of 0.8 mm with a thermal conductivity of 16 W/m · °C. The outside of the tube loses heat by free convection with $h_o = 7.6$ W/m$^2 \cdot$°C. Calculate the overall heat-transfer coefficient and heat loss per unit length to surrounding air at 20°C.

■ Solution

There are three resistances in series for this problem, as illustrated in Equation (2-14). With $L = 1.0$ m, $d_i = 0.025$ m, and $d_o = 0.025 + (2)(0.0008) = 0.0266$ m, the resistances may be calculated as

$$R_i = \frac{1}{h_i A_i} = \frac{1}{(3500)\pi(0.025)(1.0)} = 0.00364\ ^\circ\text{C/W}$$

$$R_t = \frac{\ln(d_o/d_i)}{2\pi kL} = \frac{\ln(0.0266/0.025)}{2\pi(16)(1.0)} = 0.00062\ ^\circ\mathrm{C/W}$$

$$R_o = \frac{1}{h_o A_o} = \frac{1}{(7.6)\pi(0.0266)(1.0)} = 1.575\ ^\circ\mathrm{C/W}$$

Clearly, the outside convection resistance is the largest, and *overwhelmingly so.* This means that it is the controlling resistance for the total heat transfer because the other resistances (in series) are negligible in comparison. We shall base the overall heat-transfer coefficient on the outside tube area and write

$$q = \frac{\Delta T}{\sum R} = UA_o\Delta T \qquad \textbf{[}\boldsymbol{a}\textbf{]}$$

$$U_o = \frac{1}{A_o \sum R} = \frac{1}{[\pi(0.0266)(1.0)](0.00364 + 0.00062 + 1.575)} = 7.577\ \mathrm{W/m^2 \cdot {}^\circ C}$$

or a value very close to the value of $h_o = 7.6$ for the outside convection coefficient. The heat transfer is obtained from Equation (a), with

$$q = UA_o\,\Delta T = (7.577)\pi(0.0266)(1.0)(50 - 20) = 19\ \mathrm{W}\ (\text{for } 1.0\ \text{m length})$$

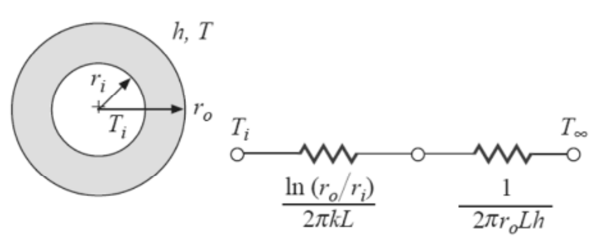
■ **Comment**

This example illustrates the important point that many practical heat-transfer problems involve multiple modes of heat transfer acting in combination; in this case, as a series of thermal resistances. It is not unusual for one mode of heat transfer to dominate the overall problem. In this example, the total heat transfer could have been computed very nearly by just calculating the free convection heat loss from the outside of the tube maintained at a temperature of 50°C. Because the inside convection and tube wall resistances are so small, there are correspondingly small temperature drops, and the outside temperature of the tube will be very nearly that of the liquid inside, or 50°C.

2-6 | CRITICAL THICKNESS OF INSULATION

Let us consider a layer of insulation which might be installed around a circular pipe, as shown in Figure 2-7. The inner temperature of the insulation is fixed at T_i, and the outer

Figure 2-7 | Critical insulation thickness.



surface is exposed to a convection environment at T_∞. From the thermal network the heat transfer is

$$q = \frac{2\pi L\,(T_i - T_\infty)}{\dfrac{\ln(r_o/r_i)}{k} + \dfrac{1}{r_o h}} \tag{2-17}$$

Now let us manipulate this expression to determine the outer radius of insulation r_o, which will maximize the heat transfer. The maximization condition is

$$\frac{dq}{dr_o} = 0 = \frac{-2\pi L\,(T_i - T_\infty)\left(\dfrac{1}{kr_o} - \dfrac{1}{hr_o^2}\right)}{\left[\dfrac{\ln(r_o/r_i)}{k} + \dfrac{1}{r_o h}\right]^2}$$

which gives the result

$$r_o = \frac{k}{h} \tag{2-18}$$

Equation (2-18) expresses the critical-radius-of-insulation concept. If the outer radius is less than the value given by this equation, then the heat transfer will be *increased* by adding more insulation. For outer radii greater than the critical value an increase in insulation thickness will cause a decrease in heat transfer. The central concept is that for sufficiently small values of h the convection heat loss may actually increase with the addition of insulation because of increased surface area.

EXAMPLE 2-6 Critical Insulation Thickness

Calculate the critical radius of insulation for asbestos [$k = 0.17$ W/m · °C] surrounding a pipe and exposed to room air at 20°C with $h = 3.0$ W/m^2 · °C. Calculate the heat loss from a 200°C, 5.0-cm-diameter pipe when covered with the critical radius of insulation and without insulation.

■ Solution

From Equation (2-18) we calculate r_o as

$$r_o = \frac{k}{h} = \frac{0.17}{3.0} = 0.0567 \text{ m} = 5.67 \text{ cm}$$

The inside radius of the insulation is $5.0/2 = 2.5$ cm, so the heat transfer is calculated from Equation (2-17) as

$$\frac{q}{L} = \frac{2\pi\,(200 - 20)}{\dfrac{\ln(5.67/2.5)}{0.17} + \dfrac{1}{(0.0567)(3.0)}} = 105.7 \text{ W/m}$$

Without insulation the convection from the outer surface of the pipe is

$$\frac{q}{L} = h(2\pi r)(T_i - T_o) = (3.0)(2\pi)(0.025)(200 - 20) = 84.8 \text{ W/m}$$

So, the addition of 3.17 cm $(5.67 - 2.5)$ of insulation actually *increases* the heat transfer by 25 percent.

As an alternative, fiberglass having a thermal conductivity of 0.04 W/m · °C might be employed as the insulation material. Then, the critical radius would be

$$r_o = \frac{k}{h} = \frac{0.04}{3.0} = 0.0133 \text{ m} = 1.33 \text{ cm}$$

Now, the value of the critical radius is less than the outside radius of the pipe (2.5 cm), so addition of *any* fiberglass insulation would cause a *decrease* in the heat transfer. In a practical pipe insulation problem, the total heat loss will also be influenced by radiation as well as convection from the outer surface of the insulation.

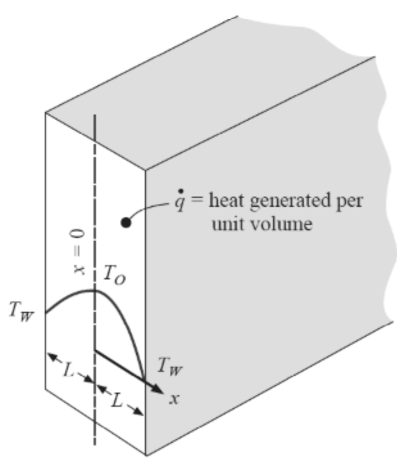
2-7 | HEAT-SOURCE SYSTEMS

A number of interesting applications of the principles of heat transfer are concerned with systems in which heat may be generated internally. Nuclear reactors are one example; electrical conductors and chemically reacting systems are others. At this point we shall confine our discussion to one-dimensional systems, or, more specifically, systems where the temperature is a function of only one space coordinate.

Plane Wall with Heat Sources

Consider the plane wall with uniformly distributed heat sources shown in Figure 2-8. The thickness of the wall in the x direction is $2L$, and it is assumed that the dimensions in the other directions are sufficiently large that the heat flow may be considered as one-dimensional. The heat generated per unit volume is $\dot{q}$, and we assume that the thermal conductivity does not vary with temperature. This situation might be produced in a practical situation by passing a current through an electrically conducting material. From Chapter 1,

Figure 2-8 | Sketch illustrating one-dimensional conduction problem with heat generation.



the differential equation that governs the heat flow is

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0 \qquad \textbf{[2-19]}$$

For the boundary conditions we specify the temperatures on either side of the wall, i.e.,

$$T = T_w \qquad \text{at } x = \pm L \qquad \textbf{[2-20]}$$

The general solution to Equation (2-19) is

$$T = -\frac{\dot{q}}{2k}x^2 + C_1 x + C_2 \qquad \textbf{[2-21]}$$

Because the temperature must be the same on each side of the wall, C_1 must be zero. The temperature at the midplane ($x = 0$) is denoted by T_0 and from Equation (2-21)

$$T_0 = C_2$$

The temperature distribution is therefore

$$T - T_0 = -\frac{\dot{q}}{2k}x^2 \qquad \textbf{[2-22}\boldsymbol{a}\textbf{]}$$

or

$$\frac{T - T_0}{T_w - T_0} = \left(\frac{x}{L}\right)^2 \qquad \textbf{[2-22}\boldsymbol{b}\textbf{]}$$

a parabolic distribution. An expression for the midplane temperature T_0 may be obtained through an energy balance. At steady-state conditions the total heat generated must equal the heat lost at the faces. Thus

$$2\left(-kA\left.\frac{dT}{dx}\right]_{x=L}\right) = \dot{q}A\,2L$$

where A is the cross-sectional area of the plate. The temperature gradient at the wall is obtained by differentiating Equation (2-22b):

$$\left.\frac{dT}{dx}\right]_{x=L} = (T_w - T_0)\left.\left(\frac{2x}{L^2}\right)\right]_{x=L} = (T_w - T_0)\,\frac{2}{L}$$

Then

$$-k(T_w - T_0)\,\frac{2}{L} = \dot{q}L$$

and

$$T_0 = \frac{\dot{q}L^2}{2k} + T_w \qquad \textbf{[2-23]}$$

This same result could be obtained by substituting $T = T_w$ at $x = L$ into Equation (2-22a).

The equation for the temperature distribution could also be written in the alternative form

$$\frac{T - T_w}{T_0 - T_w} = 1 - \frac{x^2}{L^2} \qquad \textbf{[2-22}\boldsymbol{c}\textbf{]}$$

2-8 | CYLINDER WITH HEAT SOURCES

Consider a cylinder of radius R with uniformly distributed heat sources and constant thermal conductivity. If the cylinder is sufficiently long that the temperature may be considered a function of radius only, the appropriate differential equation may be obtained by neglecting the axial, azimuth, and time-dependent terms in Equation (1-3b),

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + \frac{\dot{q}}{k} = 0 \qquad \textbf{[2-24]}$$

The boundary conditions are

$$T = T_w \qquad \text{at } r = R$$

and heat generated equals heat lost at the surface:

$$\dot{q}\pi R^2 L = -k2\pi RL \left.\frac{dT}{dr}\right]_{r=R}$$

Since the temperature function must be continuous at the center of the cylinder, we could specify that

$$\frac{dT}{dr} = 0 \qquad \text{at } r = 0$$

However, it will not be necessary to use this condition since it will be satisfied automatically when the two boundary conditions are satisfied.

We rewrite Equation (2-24)

$$r\frac{d^2T}{dr^2} + \frac{dT}{dr} = \frac{-\dot{q}r}{k}$$

and note that

$$r\frac{d^2T}{dr^2} + \frac{dT}{dr} = \frac{d}{dr}\left(r\frac{dT}{dr}\right)$$

Then integration yields

$$r\frac{dT}{dr} = \frac{-\dot{q}r^2}{2k} + C_1$$

and

$$T = \frac{-\dot{q}r^2}{4k} + C_1 \ln r + C_2$$

From the second boundary condition above,

$$\left.\frac{dT}{dr}\right]_{r=R} = \frac{-\dot{q}R}{2k} = \frac{-\dot{q}R}{2k} + \frac{C_1}{R}$$

Thus

$$C_1 = 0$$

We could also note that C_1 must be zero because at $r = 0$ the logarithm function becomes infinite.

From the first boundary condition,

$$T = T_w = \frac{-\dot{q}R^2}{4k} + C_2 \qquad \text{at } r = R$$

so that

$$C_2 = T_w + \frac{\dot{q}R^2}{4k}$$

The final solution for the temperature distribution is then

$$T - T_w = \frac{\dot{q}}{4k}(R^2 - r^2) \tag{2-25a}$$

or, in dimensionless form,

$$\frac{T - T_w}{T_0 - T_w} = 1 - \left(\frac{r}{R}\right)^2 \tag{2-25b}$$

where T_0 is the temperature at $r = 0$ and is given by

$$T_0 = \frac{\dot{q}R^2}{4k} + T_w \tag{2-26}$$

It is left as an exercise to show that the temperature gradient at $r = 0$ is zero.

For a hollow cylinder with uniformly distributed heat sources the appropriate boundary conditions would be

$$T = T_i \quad \text{at } r = r_i \text{ (inside surface)}$$
$$T = T_o \quad \text{at } r = r_o \text{ (outside surface)}$$

The general solution is still

$$T = -\frac{\dot{q}r^2}{4k} + C_1 \ln r + C_2$$

Application of the new boundary conditions yields

$$T - T_o = \frac{\dot{q}}{4k}(r_o^2 - r^2) + C_1 \ln \frac{r}{r_o} \tag{2-27}$$

where the constant C_1 is given by

$$C_1 = \frac{T_i - T_o + \dot{q}(r_i^2 - r_o^2)/4k}{\ln(r_i/r_o)} \tag{2-28}$$

EXAMPLE 2-7 Heat Source with Convection

A current of 200 A is passed through a stainless-steel wire [$k = 19$ W/m · °C] 3 mm in diameter. The resistivity of the steel may be taken as 70 $\mu\Omega \cdot$cm, and the length of the wire is 1 m. The wire is submerged in a liquid at 110 °C and experiences a convection heat-transfer coefficient of 4 k W/m^2 · °C. Calculate the center temperature of the wire.

■ **Solution**

All the power generated in the wire must be dissipated by convection to the liquid:

$$P = I^2 R = q = hA(T_w - T_\infty) \tag{a}$$

The resistance of the wire is calculated from

$$R = \rho \frac{L}{A} = \frac{(70 \times 10^{-6})(100)}{\pi(0.15)^2} = 0.099\ \Omega$$

where ρ is the resistivity of the wire. The surface area of the wire is πdL, so from Equation (a),

$$(200)^2(0.099) = 4000\pi(3 \times 10^{-3})(1)(T_w - 110) = 3960 \text{ W}$$

and

$$T_w = 215°\text{C} \quad [419°\text{F}]$$

The heat generated per unit volume $\dot{q}$ is calculated from

$$P = \dot{q}V = \dot{q}\pi r^2 L$$

so that

$$\dot{q} = \frac{3960}{\pi\,(1.5 \times 10^{-3})^2(1)} = 560.2 \text{ MW/m}^3 \quad [5.41 \times 10^7 \text{ Btu/h} \cdot \text{ft}^3]$$

Finally, the center temperature of the wire is calculated from Equation (2-26):

$$T_0 = \frac{\dot{q}r_o^2}{4k} + T_w = \frac{(5.602 \times 10^8)(1.5 \times 10^{-3})^2}{(4)(19)} + 215 = 231.6°\text{C} \quad [449°\text{F}]$$

2-9 CONDUCTION - CONVECTION SYSTEMS

The heat that is conducted through a body must frequently be removed (or delivered) by some convection process. For example, the heat lost by conduction through a furnace wall must be dissipated to the surroundings through convection. In heat-exchanger applications a finned-tube arrangement might be used to remove heat from a hot liquid. The heat transfer from the liquid to the finned tube is by convection. The heat is conducted through the material and finally dissipated to the surroundings by convection. Obviously, an analysis of combined conduction-convection systems is very important from a practical standpoint.
Consider the one-dimensional fin exposed to a surrounding fluid at a temperature $\mathbf{T_\infty}$ as shown in **Figure 2-9**. The temperature of the base of the fin is $\mathbf{T_0}$. We approach the problem by making an energy balance on an element of the fin of thickness **dx** as shown in the figure. Thus:

Energy in left face=energy out right face + energy lost by convection

Let the cross-sectional area of the fin be **A** and the perimeter be **P**. Then the energy quantities are

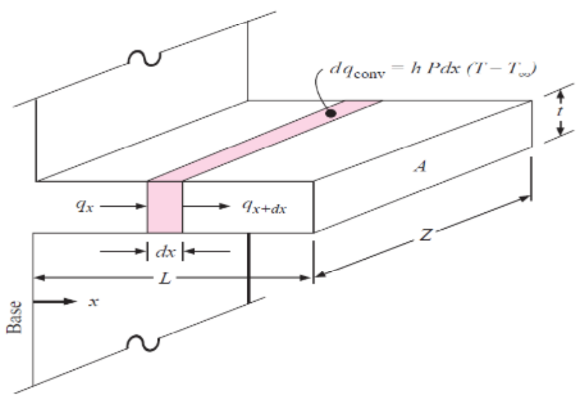
$$\text{Energy in left face} = q_x = -kA\frac{dT}{dx}$$

$$\text{Energy out right face} = q_{x+dx} = -kA \left.\frac{dT}{dx}\right]_{x+dx}$$

$$= -kA\left(\frac{dT}{dx} + \frac{d^2T}{dx^2}\,dx\right)$$

$$\text{Energy lost by convection} = hP\,dx\,(T - T_\infty)$$

Figure 2-9 | Sketch illustrating one-dimensional conduction and convection through a rectangular fin.



Here it is noted that the differential surface area for convection is the product of the perimeter of the fin **P** and the differential length **dx**. When we combine the quantities, the energy balance yields

$$\frac{d^2T}{dx^2} - \frac{hP}{kA}(T - T_\infty) = 0 \qquad \textbf{[2-30}\boldsymbol{a}\textbf{]}$$

Let **θ =T −T**$_\infty$. Then Equation (2-30*a*) becomes

$$\frac{d^2\theta}{dx^2} - \frac{hP}{kA}\theta = 0 \qquad \textbf{[2-30}\boldsymbol{b}\textbf{]}$$

If we let $\mathbf{m^2 = hP/kA}$, the general solution for Equation (2-30*b*) may be written

$$\theta = C_1e^{-mx} + C_2e^{mx} \qquad \textbf{[2-31]}$$

In order to find the constants C1 and C2, we have to apply two boundary conditions. One boundary condition is:

$\boldsymbol{\theta = \theta_0 = T_0 - T_\infty}$ **at x = 0**

The other boundary condition depends on the physical situation. Several cases may be considered:

CASE 1 The fin is very long, and the temperature at the end of the fin is essentially that of the surrounding fluid.

For case 1 the boundary conditions are

$\boldsymbol{\theta = \theta_0}$ **at x = 0**

$\boldsymbol{\theta = 0}$ **at x =** ∞

And the solution becomes;

$$\frac{\theta}{\theta_0} = \frac{T - T_\infty}{T_0 - T_\infty} = e^{-mx} \qquad \textbf{[2-32]}$$

CASE 2 The fin is of finite length and loses heat by convection from its end.

In this case the second boundary condition is;

$\boldsymbol{q_{conv} = q_{cond} = -KA(dT/dx)}$ at **x = L**

The solution for case 2 is more involved algebraically, and the result is;

$$\frac{T - T_\infty}{T_o - T_\infty} = \frac{\cosh m(L - x) + (h/mk)\sinh m(L - x)}{\cosh mL + (h/mk)\sinh mL} \qquad \textbf{[2-34]}$$

CASE 3 The end of the fin is insulated so that

dT/dx = 0 at x = L.

For case 3 the boundary conditions are

$$\theta = \theta_0 \quad \text{at } x = 0$$
$$\frac{d\theta}{dx} = 0 \quad \text{at } x = L$$

Thus

$$\theta_0 = C_1 + C_2$$
$$0 = m(-C_1 e^{-mL} + C_2 e^{mL})$$

Solving for the constants C_1 and C_2, we obtain

$$\frac{\theta}{\theta_0} = \frac{e^{-mx}}{1+e^{-2mL}} + \frac{e^{mx}}{1+e^{2mL}} \qquad \textbf{[2-33a]}$$

$$= \frac{\cosh[m(L-x)]}{\cosh mL} \qquad \textbf{[2-33b]}$$

The hyperbolic functions are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

All of the heat lost by the fin must be conducted into the base at x = 0. Using the equations for the temperature distribution, we can compute the heat loss from

$$q = -kA \left.\frac{dT}{dx}\right]_{x=0}$$

An alternative method of integrating the convection heat loss could be used:

$$q = \int_0^L hP(T - T_\infty)\,dx = \int_0^L hP\,\theta\,dx$$

In most cases, however, the first equation is easier to apply. For case 1,

$$q = -kA\,(-m\theta_0 e^{-m(0)}) = \sqrt{hPkA}\,\theta_0 \qquad \textbf{[2-35]}$$

For case 3:

$$q = -kA\theta_0 m \left(\frac{1}{1+e^{-2mL}} - \frac{1}{1+e^{+2mL}} \right) \qquad \textbf{[2-36]}$$
$$= \sqrt{hPkA}\,\theta_0 \tanh mL$$

The heat flow for case 2 is:

$$q = \sqrt{hPkA}\,(T_0 - T_\infty)\frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL} \qquad \textbf{[2-37]}$$

2-10 Fin efficiency

$$\text{Fin efficiency} = \frac{\text{actual heat transferred}}{\begin{array}{c}\text{heat that would be transferred}\\ \text{if entire fin area were}\\ \text{at base temperature}\end{array}} = \eta_f$$

For case 1, the fin efficiency;

$$\eta_f = \frac{\sqrt{hpKA}\,\theta_0}{hpL\theta_0} = \frac{1}{mL} \qquad [2\text{-}38]$$

For case 3, the fin efficiency becomes

$$\eta_f = \frac{\sqrt{hPkA}\,\theta_0 \tanh mL}{hPL\theta_0} = \frac{\tanh mL}{mL} \qquad [2\text{-}39]$$

Influence of Thermal Conductivity on Fin Temperature Profiles

EXAMPLE 2-8

Compare the temperature distributions in a straight cylindrical rod having a diameter of 2 cm and a length of 10 cm and exposed to a convection environment with $h = 25$ W/m$^2\cdot$°C, for three fin materials: copper [$k = 385$ W/m $\cdot$ °C], stainless steel [$k = 17$ W/m $\cdot$ °C], and glass [$k = 0.8$ W/m $\cdot$ °C]. Also compare the relative heat flows and fin efficiencies.

■ Solution

We have

$$\frac{hP}{kA} = \frac{(25)\pi(0.02)}{k\pi(0.01)^2} = \frac{5000}{k}$$

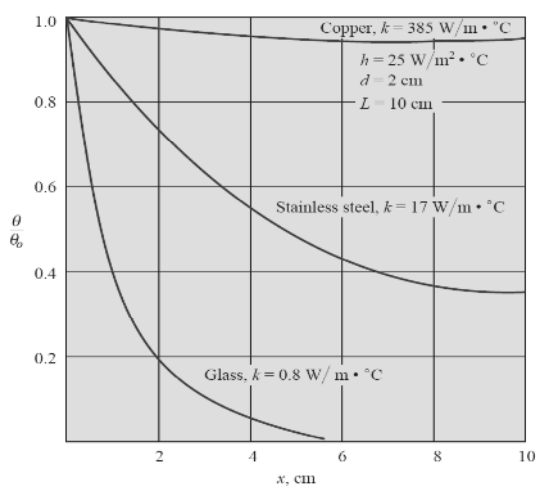
The terms of interest are therefore

Material	$\frac{hP}{kA}$	m	mL
Copper	12.99	3.604	0.3604
Stainless steel	294.1	17.15	1.715
Glass	6250	79.06	7.906

These values may be inserted into Equation (2-33a) to calculate the temperatures at different x locations along the rod, and the results are shown in Figure Example 2-8. We notice that the glass behaves as a "very long" fin, and its behavior could be calculated from Equation (2-32). The fin efficiencies are calculated from Equation (2-38) by using the corrected length approximation of Equation (2-42). We have

$$L_c = L + \frac{d}{4} = 10 + \frac{2}{4} = 10.5 \text{ cm}$$

Figure Example 2-8



The parameters of interest for the heat-flow and efficiency comparisons are now tabulated as

Material	$hPkA$	mL_c
Copper	0.190	0.3784
Stainless steel	0.0084	1.8008
Glass	3.9×10^{-4}	8.302

To compare the heat flows we could either calculate the values from Equation (2-36) for a unit value of θ_0 or observe that the fin efficiency gives a relative heat-flow comparison because the maximum heat transfer is the same for all three cases; i.e., we are dealing with the same fin size, shape, and value of h. We thus calculate the values of η_f from Equation (2-38) and the above values of mL_c.

Material	η_f	q relative to copper, %
Copper	0.955	100
Stainless steel	0.526	53.1
Glass	0.124	12.6

The temperature profiles in the accompanying figure can be somewhat misleading. The glass has the steepest temperature gradient at the base, but its much lower value of k produces a lower heat-transfer rate.

Straight Aluminum Fin

EXAMPLE 2-9

An aluminum fin [$k = 200$ W/m · °C] 3.0 mm thick and 7.5 cm long protrudes from a wall, as in Figure 2-9. The base is maintained at 300°C, and the ambient temperature is 50°C with $h = 10$ W/m^2 · °C. Calculate the heat loss from the fin per unit depth of material.

■ Solution

We may use the approximate method of solution by extending the fin a fictitious length $t/2$ and then computing the heat transfer from a fin with insulated tip as given by Equation (2-36). We have

$$L_c = L + t/2 = 7.5 + 0.15 = 7.65 \text{ cm } [3.01 \text{ in}]$$

$$m = \sqrt{\frac{hP}{kA}} = \left[\frac{h(2z + 2t)}{ktz}\right]^{1/2} \approx \sqrt{\frac{2h}{kt}}$$

when the fin depth $z \gg t$. So,

$$m = \left[\frac{(2)(10)}{(200)(3 \times 10^{-3})}\right]^{1/2} = 5.774$$

From Equation (2-36), for an insulated-tip fin

$$q = (\tanh mL_c)\sqrt{hPkA}\,\theta_0$$

For a 1 m depth

$$A = (1)(3 \times 10^{-3}) = 3 \times 10^{-3} \text{ m}^2 \ [4.65 \text{ in}^2]$$

and

$$q = (5.774)(200)(3 \times 10^{-3})(300 - 50) \tanh[(5.774)(0.0765)]$$
$$= 359 \text{ W/m } [373.5 \text{ Btu/h} \cdot \text{ft}]$$

Circumferential Aluminum Fin

EXAMPLE 2-10

Aluminum fins 1.5 cm wide and 1.0 mm thick are placed on a 2.5-cm-diameter tube to dissipate the heat. The tube surface temperature is 170°, and the ambient-fluid temperature is 25°C. Calculate the heat loss per fin for $h = 130$ W/m^2 · °C. Assume $k = 200$ W/m · °C for aluminum.

■ Solution

For this example we can compute the heat transfer by using the fin-efficiency curves in Figure 2-12. The parameters needed are

$$L_c = L + t/2 = 1.5 + 0.05 = 1.55 \text{ cm}$$
$$r_1 = 2.5/2 = 1.25 \text{ cm}$$
$$r_{2c} = r_1 + L_c = 1.25 + 1.55 = 2.80 \text{ cm}$$

$$r_{2c}/r_1 = 2.80/1.25 = 2.24$$

$$A_m = t(r_{2c} - r_1) = (0.001)(2.8 - 1.25)(10^{-2}) = 1.55 \times 10^{-5}\ \text{m}^2$$

$$L_c^{3/2}\left(\frac{h}{kA_m}\right)^{1/2} = (0.0155)^{3/2}\left[\frac{130}{(200)(1.55 \times 10^{-5})}\right]^{1/2} = 0.396$$

From Figure 2-12, $\eta_f = 82$ percent. The heat that would be transferred if the entire fin were at the base temperature is (both sides of fin exchanging heat)

$$\begin{aligned} q_{\max} &= 2\pi(r_{2c}^2 - r_1^2)h(T_0 - T_\infty) \\ &= 2\pi(2.8^2 - 1.25^2)(10^{-4})(130)(170 - 25) \\ &= 74.35\ \text{W}\ [253.7\ \text{Btu/h}] \end{aligned}$$

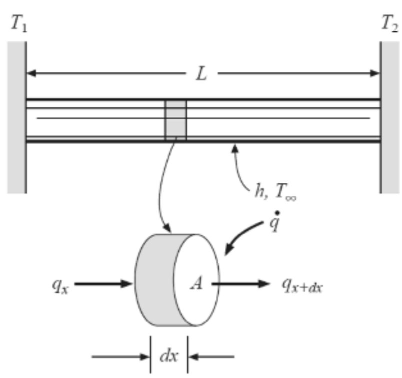
The actual heat transfer is then the product of the heat flow and the fin efficiency:

$$q_{\text{act}} = (0.82)(74.35) = 60.97\ \text{W}\ [208\ \text{Btu/h}]$$

EXAMPLE 2-11 Rod with Heat Sources

A rod containing uniform heat sources per unit volume $\dot{q}$ is connected to two temperatures as shown in Figure Example 2-11. The rod is also exposed to an environment with convection coefficient h and temperature T_∞. Obtain an expression for the temperature distribution in the rod.

Figure Example 2-11



■ Solution

We first must make an energy balance on the element of the rod shown, similar to that used to derive Equation (2-30). We have

Energy in left face + heat generated in element
= energy out right face + energy lost by convection

or

$$-kA\frac{dT}{dx} + \dot{q}A\,dx = -kA\left(\frac{dT}{dx} + \frac{d^2T}{dx^2}dx\right) + hP\,dx\,(T - T_\infty)$$

Simplifying, we have

$$\frac{d^2T}{dx^2} - \frac{hP}{kA}(T - T_\infty) + \frac{\dot{q}}{k} = 0 \qquad \textbf{[a]}$$

or, with $\theta = T - T_\infty$ and $m^2 = hP/kA$

$$\frac{d^2\theta}{dx} - m^2\theta + \frac{\dot{q}}{k} = 0 \qquad \textbf{[b]}$$

We can make a further variable substitution as

$$\theta' = \theta - \dot{q}/km^2$$

so that our differential equation becomes

$$\frac{d^2\theta'}{dx^2} - m^2\theta' = 0 \qquad \textbf{[c]}$$

which has the general solution

$$\theta' = C_1 e^{-mx} + C_2 e^{mx} \qquad \textbf{[d]}$$

The two end temperatures are used to establish the boundary conditions:

$$\theta' = \theta_1' = T_1 - T_\infty - \dot{q}/km^2 = C_1 + C_2$$
$$\theta' = \theta_2' = T_2 - T_\infty - \dot{q}/km^2 = C_1 e^{-mL} + C_2 e^{mL}$$

Solving for the constants C_1 and C_2 gives

$$\theta' = \frac{(\theta_1' e^{2mL} - \theta_2' e^{mL})e^{-mx} + (\theta_2' e^{mL} - \theta_1')e^{mx}}{e^{2mL} - 1} \qquad \textbf{[e]}$$

For an infinitely long heat-generating fin with the left end maintained at T_1, the temperature distribution becomes

$$\theta'/\theta_1' = e^{-mx} \qquad \textbf{[f]}$$

a relation similar to Equation (2-32) for a non-heat-generating fin.

■ **Comment**

PROBLEMS

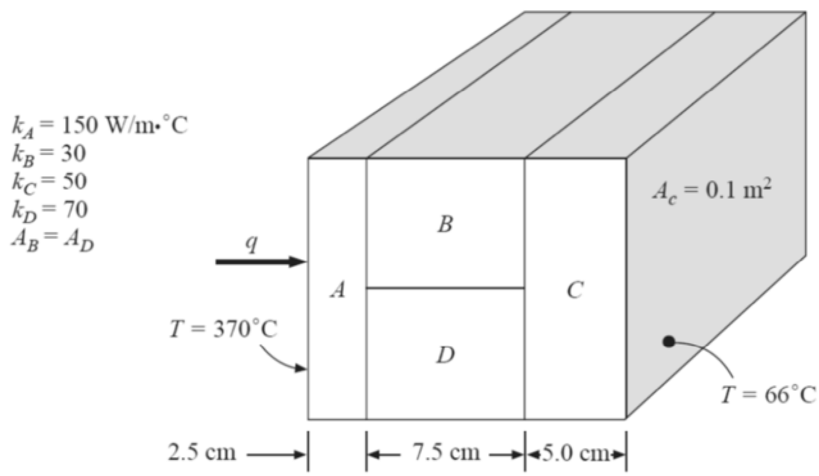
2-1 A wall 2 cm thick is to be constructed from material that has an average thermal conductivity of 1.3 W/m · °C. The wall is to be insulated with material having an average thermal conductivity of 0.35 W/m · °C, so that the heat loss per square meter will not exceed 1830 W. Assuming that the inner and outer surface temperatures of the insulated wall are 1300 and 30°C, calculate the thickness of insulation required.

2-2 A certain material 2.5 cm thick, with a cross-sectional area of 0.1 m^2, has one side maintained at 35°C and the other at 95°C. The temperature at the center plane of the material is 62°C, and the heat flow through the material is 1 kW. Obtain an expression for the thermal conductivity of the material as a function of temperature.

2-3 A composite wall is formed of a 2.5-cm copper plate, a 3.2-mm layer of asbestos, and a 5-cm layer of fiberglass. The wall is subjected to an overall temperature difference of 560°C. Calculate the heat flow per unit area through the composite structure.

2-4 Find the heat transfer per unit area through the composite wall in Figure P2-4. Assume one-dimensional heat flow.

Figure P2-4



2-5 One side of a copper block 5 cm thick is maintained at 250°C. The other side is covered with a layer of fiberglass 2.5 cm thick. The outside of the fiberglass is maintained at 35°C, and the total heat flow through the copper-fiberglass combination is 52 kW. What is the area of the slab?

2-6 An outside wall for a building consists of a 10-cm layer of common brick and a 2.5-cm layer of fiberglass [$k = 0.05$ W/m · °C]. Calculate the heat flow through the wall for a 25°C temperature differential.