CHAPTER

3 Steady-State Conduction—Multiple Dimensions

MATHEMATICAL ANALYSIS OF TWO-DIMENSIONAL HEAT CONDUCTION

For steady state with no heat generation, the Laplace equation applies.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad \textbf{[3-1]}$$

The solution to Equation (3-1) will give the temperature in a two-dimensional body as a function of the two independent space coordinates *x* and *y*. Then the heat flow in the *x* and *y* directions may be calculated from the Fourier equations

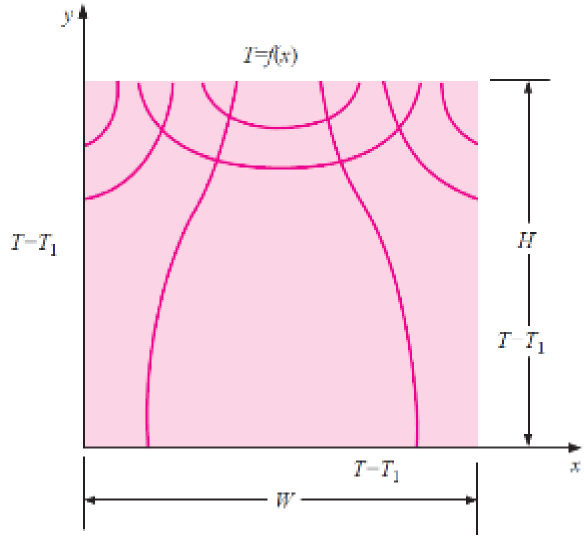
$$q_x = -kA_x \frac{\partial T}{\partial x} \qquad \textbf{[3-2]}$$

$$q_y = -kA_y \frac{\partial T}{\partial y} \qquad \textbf{[3-3]}$$

Consider the rectangular plate shown in Figure 3-2. Three sides of the plate are maintained at the constant temperature T1, and the upper side has some temperature distribution impressed upon it. This distribution could be simply a constant temperature or something more complex, such as a sine-wave distribution. We shall consider both cases.

To solve Equation (3-1), the separation-of-variables method is used. The essential point of this method is that the solution to the differential equation is assumed to take a product form

$$T = XY \quad \text{where} \quad \begin{aligned} X &= X(x) \\ Y &= Y(y) \end{aligned} \qquad \textbf{[3-4]}$$



First consider the boundary conditions with a sine-wave temperature distribution impressed on the upper edge of the plate. Thus

$$\begin{aligned} T &= T_1 && \text{at } y = 0 \\ T &= T_1 && \text{at } x = 0 \\ T &= T_1 && \text{at } x = W \qquad \textbf{[3-5]} \\ T &= T_m \sin\left(\frac{\pi x}{W}\right) + T_1 && \text{at } y = H \end{aligned}$$

Where, T_m is the amplitude of the sine function. Substituting Equation (3-4) in (3-1) gives

$$-\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda^2 \qquad \textbf{[3-6]}$$

Observe that each side of Equation (3-6) is independent of the other because *x* and *y* are independent variables. This requires that each side be equal to some constant. We may thus obtain two ordinary differential equations in terms of this constant,

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \qquad \textbf{[3-7]}$$

$$\frac{d^2Y}{dy^2} - \lambda^2 Y = 0 \qquad \textbf{[3-8]}$$

Where, λ^2 is called the ***separation constant***. Its value must be determined from the boundary conditions. Note that the form of the solution to Equations (3-7) and (3-8) will depend on the sign of λ^2; a different form would also result if λ^2 were zero.

The only way that the correct form can be determined is through an application of the boundary conditions of the problem. So we shall first write down all possible solutions and then see which one fits the problem under consideration.

For $\lambda^2 = 0$:

$$
\begin{aligned}
X &= C_1 + C_2 x \\
Y &= C_3 + C_4 y \\
T &= (C_1 + C_2 x)(C_3 + C_4 y)
\end{aligned}
\qquad \textbf{[3-9]}
$$

This function cannot fit the sine-function boundary condition, so the $\lambda^2 = 0$ solution may be excluded.

For $\lambda^2 < 0$:

$$
\begin{aligned}
X &= C_5 e^{-\lambda x} + C_6 e^{\lambda x} \\
Y &= C_7 \cos \lambda y + C_8 \sin \lambda y \\
T &= (C_5 e^{-\lambda x} + C_6 e^{\lambda x})(C_7 \cos \lambda y + C_8 \sin \lambda y)
\end{aligned}
\qquad \textbf{[3-10]}
$$

Again, the sine-function boundary condition cannot be satisfied, so this solution is excluded also.

For $\lambda^2 > 0$:

$$
\begin{aligned}
X &= C_9 \cos \lambda x + C_{10} \sin \lambda x \\
Y &= C_{11} e^{-\lambda y} + C_{12} e^{\lambda y} \\
T &= (C_9 \cos \lambda x + C_{10} \sin \lambda x)(C_{11} e^{-\lambda y} + C_{12} e^{\lambda y})
\end{aligned}
\qquad \textbf{[3-11]}
$$

Now, it is possible to satisfy the sine-function boundary condition; so we shall attempt to satisfy the other conditions. The algebra is somewhat easier to handle when the substitution ($\boldsymbol{\theta = T - T_1}$) is made.

The differential equation and the solution then retain the same form in the new variable θ, and we need only transform the boundary conditions. Thus

$$
\begin{aligned}
\theta &= 0 && \text{at } y = 0 \\
\theta &= 0 && \text{at } x = 0 \\
\theta &= 0 && \text{at } x = W \\
\theta &= T_m \sin \frac{\pi x}{W} && \text{at } y = H
\end{aligned}
\qquad \textbf{[3-12]}
$$

Applying these conditions, we have

$$0 = (C_9 \cos \lambda x + C_{10} \sin \lambda x)(C_{11} + C_{12}) \qquad [a]$$

$$0 = C_9 (C_{11} e^{-\lambda y} + C_{12} e^{\lambda y}) \qquad [b]$$

$$0=(C_9\cos\lambda W+C_{10}\sin\lambda W)(C_{11}e^{-\lambda y}+C_{12}e^{\lambda y}) \qquad [c]$$

$$T_m\sin\frac{\pi x}{W}=(C_9\cos\lambda x+C_{10}\sin\lambda x)(C_{11}e^{-\lambda H}+C_{12}e^{\lambda H}) \qquad [d]$$

Accordingly;

$C_{11} = - C_{12}$

$C_9 = 0$

And from (c)

$$0=C_{10}C_{12}\sin\lambda W(e^{\lambda y}-e^{-\lambda y})$$

This required that;

$$\sin\lambda W=0 \qquad \textbf{[3-13]}$$

Recall that λ was an undetermined separation constant. Several values will satisfy Equation (3-13), and these may be written

$$\lambda=\frac{n\pi}{W} \qquad \textbf{[3-14]}$$

Where **n** is an integer. The solution to the differential equation may thus be written as a sum of the solutions for each value of **n**. This is an infinite sum, so that the final solution is the infinite series

$$\theta=T-T_1=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi x}{W}\sinh\frac{n\pi y}{W} \qquad \textbf{[3-15]}$$

Where the constants have been combined and the exponential terms converted to the hyperbolic function. The final boundary condition may now be applied:

$$T_m\sin\frac{\pi x}{W}=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi x}{W}\sinh\frac{n\pi H}{W}$$

Which requires that Cn = 0 for n >1. The final solution is therefore

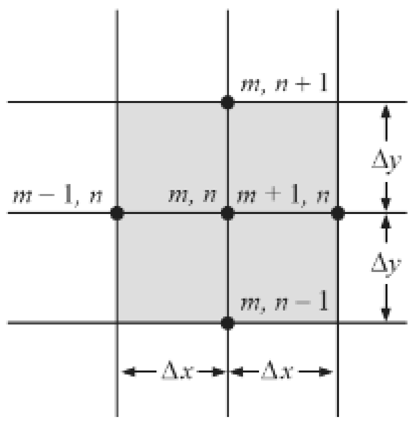
$$T=T_m\frac{\sinh(\pi y/W)}{\sinh(\pi H/W)}\sin\left(\frac{\pi x}{W}\right)+T_1 \qquad \textbf{[3-16]}$$

The temperature field for this problem is shown in Figure 3-2. Note that the heat-flow lines are perpendicular to the isotherms.

NUMERICAL METHOD OF ANALYSIS

Consider a two-dimensional body that is to be divided into equal increments in both the x and y directions, as shown in Figure 3-5. The nodal points are designated as shown, the m locations indicating the x increment and the n locations indicating the y increment. We wish to establish the temperatures at any of these nodal points within the body, using Equation (3-1) as a governing condition. Finite differences are used to approximate differential increments in the temperature and space coordinates; and the smaller we choose these finite increments, the more closely the true temperature distribution will be approximated.

Figure 3-5 | Sketch illustrating nomenclature used in two-dimensional numerical analysis of heat conduction.



The temperature gradients may be written as follows:

$$\left.\frac{\partial T}{\partial x}\right]_{m+1/2,n} \approx \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial x}\right]_{m-1/2,n} \approx \frac{T_{m,n} - T_{m-1,n}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial y}\right]_{m,n+1/2} \approx \frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

$$\left.\frac{\partial T}{\partial y}\right]_{m,n-1/2} \approx \frac{T_{m,n} - T_{m,n-1}}{\Delta y}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right]_{m,n} \approx \frac{\left.\frac{\partial T}{\partial x}\right]_{m+1/2,n} - \left.\frac{\partial T}{\partial x}\right]_{m-1/2,n}}{\Delta x} = \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2}$$

$$\left.\frac{\partial^2 T}{\partial y^2}\right]_{m,n} \approx \frac{\left.\frac{\partial T}{\partial y}\right]_{m,n+1/2} - \left.\frac{\partial T}{\partial y}\right]_{m,n-1/2}}{\Delta y} = \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2}$$

Thus the finite-difference approximation for Equation (3-1) becomes

$$\frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2} + \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2} = 0$$

If $\Delta x = \Delta y$, then

$$T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} = 0 \qquad \textbf{[3-24]}$$

We can also devise a finite-difference scheme to take heat generation into account. We merely add the term $\dot{q}/k$ into the general equation and obtain

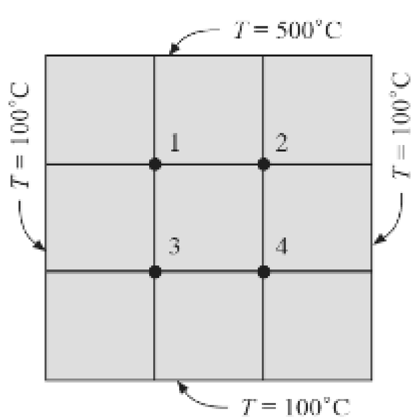
$$\frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2} + \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2} + \frac{\dot{q}}{k} = 0$$

Then for a square grid in which $\Delta x = \Delta y$,

$$T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} + \frac{\dot{q}(\Delta x)^2}{k} - 4T_{m,n-1} = 0 \qquad \textbf{[3-24}\boldsymbol{a}\textbf{]}$$

A very simple example is shown in Figure 3-6, and the four equations for nodes 1, 2, 3, and 4 would be

Figure 3-6 | Four-node problem.



$$100 + 500 + T_2 + T_3 - 4T_1 = 0$$
$$T_1 + 500 + 100 + T_4 - 4T_2 = 0$$
$$100 + T_1 + T_4 + 100 - 4T_3 = 0$$
$$T_3 + T_2 + 100 + 100 - 4T_4 = 0$$

These equations have the solution

$$T_1 = T_2 = 250°\text{C} \qquad T_3 = T_4 = 150°\text{C}$$

When the solid is exposed to some convection boundary condition, the temperatures at the surface must be computed differently from the method given above. Consider the boundary shown in Figure 3-7. The energy balance on node (m, n) is

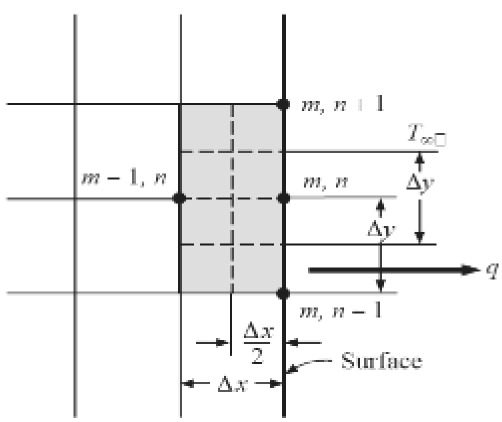
$$-k\,\Delta y \frac{T_{m,n} - T_{m-1,n}}{\Delta x} - k\frac{\Delta x}{2}\frac{T_{m,n} - T_{m,n+1}}{\Delta y} - k\frac{\Delta x}{2}\frac{T_{m,n} - T_{m,n-1}}{\Delta y}$$
$$= h\,\Delta y(T_{m,n} - T_\infty)$$

If $\Delta x = \Delta y$, the boundary temperature is expressed in the equation

$$T_{m,n}\left(\frac{h\,\Delta x}{k} + 2\right) - \frac{h\,\Delta x}{k}T_\infty - \frac{1}{2}(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) = 0 \qquad \textbf{[3-25]}$$

Equation (3-25) applies to a plane surface exposed to a convection boundary condition. It will not apply for other situations, such as an insulated wall or a corner exposed to a convection boundary condition.

Figure 3-7 | Nomenclature for nodal equation with convective boundary condition.



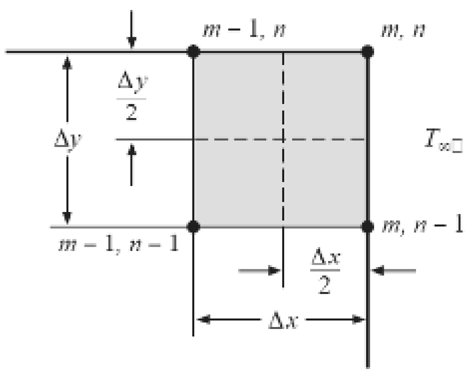
Consider the corner section shown in Figure 3-8. The energy balance for the corner section is

$$-k\frac{\Delta y}{2}\frac{T_{m,n}-T_{m-1,n}}{\Delta x}-k\frac{\Delta x}{2}\frac{T_{m,n}-T_{m,n-1}}{\Delta y}=h\frac{\Delta x}{2}(T_{m,n}-T_\infty)+h\frac{\Delta y}{2}(T_{m,n}-T_\infty)$$

If $\Delta x=\Delta y$,

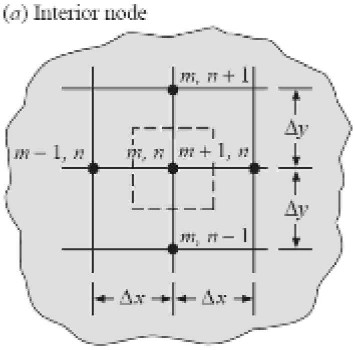
$$2T_{m,n}\left(\frac{h\,\Delta x}{k}+1\right)-2\frac{h\,\Delta x}{k}T_\infty-(T_{m-1,n}+T_{m,n-1})=0 \qquad \textbf{[3-26]}$$

Figure 3-8 | Nomenclature for nodal equation with convection at a corner section.

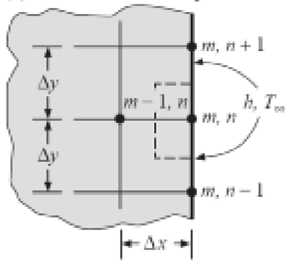


Other boundary conditions may be treated in a similar fashion, and a convenient summary of nodal equations is given in Table 3-2 for different geometrical and boundary situations.

Table 3-2 | Summary of nodal formulas for finite-difference calculations. (Dashed lines indicate element volume.)†

Physical situation	Nodal equation for equal increments in x and y (second equation in situation is in form for Gauss-Seidel iteration)
(*a*) Interior node 	$0=T_{m+1,n}+T_{m,n+1}+T_{m-1,n}+T_{m,n-1}-4T_{m,n}$ $T_{m,n}=(T_{m+1,n}+T_{m,n+1}+T_{m-1,n}+T_{m,n-1})/4$

(*b*) Convection boundary node

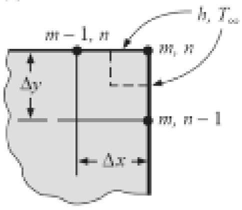


$$0 = \frac{h\Delta x}{k} T_\infty + \frac{1}{2}(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) - \left(\frac{h\Delta x}{k} + 2\right) T_{m,n}$$

$$T_{m,n} = \frac{T_{m-1,n} + (T_{m,n+1} + T_{m,n-1})/2 + \mathrm{Bi}\, T_\infty}{2 + \mathrm{Bi}}$$

$$\mathrm{Bi} = \frac{h\Delta x}{k}$$

(*c*) Exterior corner with convection boundary

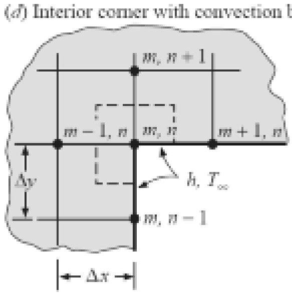


$$0 = 2\frac{h\Delta x}{k} T_\infty + (T_{m-1,n} + T_{m,n-1}) - 2\left(\frac{h\Delta x}{k} + 1\right) T_{m,n}$$

$$T_{m,n} = \frac{(T_{m-1,n} + T_{m,n-1})/2 + \mathrm{Bi}\, T_\infty}{1 + \mathrm{Bi}}$$

$$\mathrm{Bi} = \frac{h\Delta x}{k}$$

(*d*) Interior corner with convection boundary

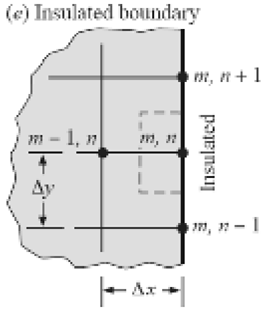


$$0 = 2\frac{h\Delta x}{k} T_\infty + 2T_{m-1,n} + T_{m,n+1} + T_{m+1,n} + T_{m,n-1} - 2\left(3 + \frac{h\Delta x}{k}\right) T_{m,n}$$

$$T_{m,n} = \frac{\mathrm{Bi}\, T_\infty + T_{m,n+1} + T_{m-1,n} + (T_{m+1,n} + T_{m,n-1})/2}{3 + \mathrm{Bi}}$$

$$\mathrm{Bi} = \frac{h\Delta x}{k}$$

(*e*) Insulated boundary



$$0 = T_{m,n+1} + T_{m,n-1} + 2T_{m-1,n} - 4T_{m,n}$$

$$T_{m,n} = (T_{m,n+1} + T_{m,n-1} + 2T_{m-1,n})/4$$

Nine-Node Problem

EXAMPLE 3-5

Consider the square of Figure Example 3-5. The left face is maintained at 100°C and the top face at 500°C, while the other two faces are exposed to an environment at 100°C:

$$h = 10\ \mathrm{W/m^2 \cdot {}^\circ C} \quad \text{and} \quad k = 10\ \mathrm{W/m \cdot {}^\circ C}$$

The block is 1 m square. Compute the temperature of the various nodes as indicated in Figure Example 3-5 and the heat flows at the boundaries.

■ Solution

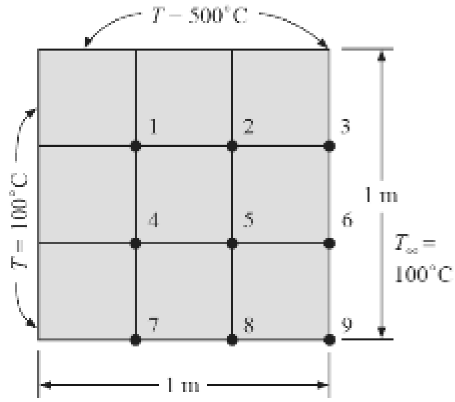
The nodal equation for nodes 1, 2, 4, and 5 is

$$T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} = 0$$

The equation for nodes 3, 6, 7, and 8 is given by Equation (3-25), and the equation for 9 is given by Equation (3-26):

$$\frac{h\Delta x}{k} = \frac{(10)(1)}{(3)(10)} = \frac{1}{3}$$

Figure Example 3-5 | Nomenclature for Example 3-5.



The equations for nodes 3 and 6 are thus written

$$2T_2 + T_6 + 567 - 4.67T_3 = 0$$
$$2T_5 + T_3 + T_9 + 67 - 4.67T_6 = 0$$

The equations for nodes 7 and 8 are given by

$$2T_4 + T_8 + 167 - 4.67T_7 = 0$$
$$2T_5 + T_7 + T_9 + 67 - 4.67T_8 = 0$$

and the equation for node 9 is

$$T_6 + T_8 + 67 - 2.67T_9 = 0$$

We thus have nine equations and nine unknown nodal temperatures. We shall discuss solution techniques shortly, but for now we just list the answers:

Node	Temperature, °C
1	280.67
2	330.30
3	309.38
4	192.38
5	231.15
6	217.19
7	157.70
8	184.71
9	175.62

The heat flows at the boundaries are computed in two ways: as conduction flows for the 100 and 500°C faces and as convection flows for the other two faces. For the 500°C face, the heat flow *into* the face is

$$q = \sum k\Delta x \frac{\Delta T}{\Delta y} = (10)\left[500 - 280.67 + 500 - 330.30 + (500 - 309.38)\left(\tfrac{1}{2}\right)\right]$$
$$= 4843.4 \text{ W/m}$$

The heat flow *out* of the 100°C face is

$$q = \sum k\Delta y \frac{\Delta T}{\Delta x} = (10)\left[280.67 - 100 + 192.38 - 100 + (157.70 - 100)\left(\tfrac{1}{2}\right)\right]$$
$$= 3019 \text{ W/m}$$

The convection heat flow *out* the right face is given by the convection relation

$$q = \sum h\Delta y(T - T_\infty)$$
$$= (10)\left(\tfrac{1}{3}\right)\left[309.38 - 100 + 217.19 - 100 + (175.62 - 100)\left(\tfrac{1}{2}\right)\right]$$
$$= 1214.6 \text{ W/m}$$

Finally, the convection heat flow *out* the bottom face is

$$q = \sum h\Delta x(T - T_\infty)$$
$$= (10)\left(\tfrac{1}{3}\right)\left[(100 - 100)\left(\tfrac{1}{2}\right) + 157.70 - 100 + 184.71 - 100 + (175.62 - 100)\left(\tfrac{1}{2}\right)\right]$$
$$= 600.7 \text{ W/m}$$

The total heat flow out is

$$q_{\text{out}} = 3019 + 1214.6 + 600.7 = 4834.3 \text{ W/m}$$

This compares favorably with the 4843.4 W/m conducted into the top face. A solution of this example using the Excel spreadsheet format is given in Appendix D.

Solution Techniques

From the foregoing discussion we have seen that the numerical method is simply a means of approximating a continuous temperature distribution with the finite nodal elements. The more nodes taken, the closer the approximation; but, of course, more equations mean more cumbersome solutions. Fortunately, computers and even programmable calculators have the capability to obtain these solutions very quickly. In practical problems the selection of a large number of nodes may be unnecessary because of uncertainties in boundary conditions. For example, it is not

uncommon to have uncertainties in h, the convection coefficient, of ±15 to 20 percent. The nodal equations may be written as

$$\begin{aligned}
a_{11}T_1 + a_{12}T_2 + \cdots + a_{1n}T_n &= C_1 \\
a_{21}T_1 + a_{22}T_2 + \cdots \qquad\qquad &= C_2 \\
a_{31}T_1 + \cdots \qquad\qquad\qquad &= C_3 \qquad\qquad \textbf{[3-27]} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{n1}T_1 + a_{n2}T_2 + \cdots + a_{nn}T_n &= C_n
\end{aligned}$$

where $T_1, T_2, \ldots, T_n$ are the unknown nodal temperatures. By using the matrix notation

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \\ a_{31} & & \cdots & \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad [C] = \begin{bmatrix} C_1 \\ C_2 \\ \cdot \\ \cdot \\ \cdot \\ C_n \end{bmatrix} \qquad [T] = \begin{bmatrix} T_1 \\ T_2 \\ \cdot \\ \cdot \\ \cdot \\ T_n \end{bmatrix}$$

Equation (3-27) can be expressed as

$$[A][T] = [C] \qquad \textbf{[3-28]}$$

For example, the matrix notation for the system of Example 3-5 would be

$$\begin{bmatrix}
-4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & -4.67 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & -4.67 & 0 & 0 & 1 \\
0 & 0 & 0 & 2 & 0 & 0 & -4.67 & 1 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 1 & -4.67 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2.67
\end{bmatrix}
\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{bmatrix}
=
\begin{bmatrix} -600 \\ -500 \\ -567 \\ -100 \\ 0 \\ -67 \\ -167 \\ -67 \\ -67 \end{bmatrix}$$

CHAPTER 4 Unsteady-State Conduction

If a solid body is suddenly subjected to a change in environment, some time must elapse before an equilibrium temperature condition will prevail in the body. We refer to the equilibrium condition as the steady state and calculate the temperature distribution and heat transfer by methods described in Chapters 2 and 3. In the transient heating or cooling process that takes place in the interim period before equilibrium is established, the analysis must be modified to take into account the change in internal energy of the body with time, and the boundary conditions must be adjusted to match the physical situation that is apparent in the unsteady-state heat-transfer problem.

To analyze a transient heat-transfer problem, we could proceed by solving the general heat-conduction equation by the separation-of-variables method, similar to the analytical treatment used for the two-dimensional steady-state problem.

The differential equation of one dimensional unsteady conduction is;

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \qquad \textbf{[4-1]}$$

LUMPED-HEAT-CAPACITY SYSTEM

We continue our discussion of transient heat conduction by analyzing systems that may be considered uniform in temperature. This type of analysis is called the ***lumped-heat-capacity*** method. Such systems are obviously idealized because a temperature gradient must exist in a material if heat is to be conducted into or out of the material.

If a hot steel ball were immersed in a cool pan of water, the lumped-heat-capacity method of analysis might be used if we could justify an assumption of uniform ball temperature during the cooling process. Clearly, the temperature distribution in the ball would depend on the thermal conductivity of the ball material and the heat-transfer conditions from the surface of the ball to the surrounding fluid (i.e., the surface-convection heat transfer coefficient).We should obtain a reasonably uniform temperature distribution in the ball if the resistance to heat transfer by

conduction were small compared with the convection resistance at the surface, so that the major temperature gradient would occur through the fluid layer at the surface. The lumped-heat-capacity analysis, then, is one that assumes that the internal resistance of the body is negligible in comparison with the external resistance. The convection heat loss from the body is evidenced as a decrease in the internal energy of the body, as shown in Figure 4-2. Thus,

$$q = hA(T - T_\infty) = -c\rho V \frac{dT}{d\tau} \qquad \textbf{[4-4]}$$

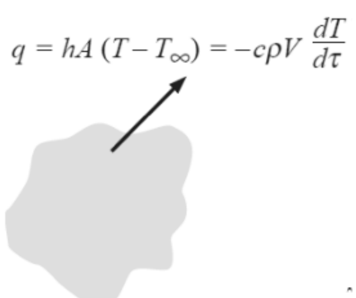
Where **A** is the surface area for convection and **V** is the volume. The initial condition is written

$T = T_0$ at $\tau = 0$

So that the solution to Equation (4-4) is

$$\frac{T - T_\infty}{T_0 - T_\infty} = e^{-[hA/\rho c V]\tau} \qquad \textbf{[4-5]}$$

Figure 4-2 | Nomenclature for single-lump heat-capacity analysis.



We have already noted that the lumped-capacity type of analysis assumes a uniform temperature distribution throughout the solid body and that the assumption is equivalent to saying that the surface-convection resistance is large compared with the internal-conduction resistance.

Such an analysis may be expected to yield reasonable estimates within about 5 percent when the following condition is met:

$$\frac{h(V/A)}{k} < 0.1 \qquad \textbf{[4-6]}$$

Where, **k** is the thermal conductivity of the solid. If one considers the ratio **V/A=s** as a characteristic dimension of the solid, the dimensionless group in Equation (4-6) is called the ***Biot number****:*

$$\frac{hs}{k} = \text{Biot number} = \text{Bi}$$

Steel Ball Cooling in Air **EXAMPLE 4-1**

A steel ball [$c = 0.46$ kJ/kg · °C, $k = 35$ W/m · °C] 5.0 cm in diameter and initially at a uniform temperature of 450°C is suddenly placed in a controlled environment in which the temperature is maintained at 100°C. The convection heat-transfer coefficient is 10 W/m^2 · °C. Calculate the time required for the ball to attain a temperature of 150°C.

■ **Solution**

We anticipate that the lumped-capacity method will apply because of the low value of h and high value of k. We can check by using Equation (4-6):

$$\frac{h(V/A)}{k} = \frac{(10)[(4/3)\pi(0.025)^3]}{4\pi(0.025)^2(35)} = 0.0023 < 0.1$$

so we may use Equation (4-5). We have

$$\begin{aligned} T &= 150^\circ\text{C} \quad & \rho &= 7800\ \text{kg/m}^3 \quad & &[486\ \text{lb}_m/\text{ft}^3] \\ T_\infty &= 100^\circ\text{C} \quad & h &= 10\ \text{W/m}^2 \cdot {}^\circ\text{C} \quad & &[1.76\text{Btu/h} \cdot \text{ft}^2 \cdot {}^\circ\text{F}] \\ T_0 &= 450^\circ\text{C} \quad & c &= 460\ \text{J/kg} \cdot {}^\circ\text{C} \quad & &[0.11\ \text{Btu/lb}_m \cdot {}^\circ\text{F}] \end{aligned}$$

$$\frac{hA}{\rho c V} = \frac{(10)4\pi(0.025)^2}{(7800)(460)(4\pi/3)(0.025)^3} = 3.344 \times 10^{-4}\ \text{s}^{-1}$$

$$\frac{T - T_\infty}{T_0 - T_\infty} = e^{-[hA/\rho c V]\tau}$$

$$\frac{150 - 100}{450 - 100} = e^{-3.344\times 10^{-4}\tau}$$

$$\tau = 5819\ \text{s} = 1.62\ \text{h}$$

TRANSIENT NUMERICAL METHOD

Consider a two-dimensional body divided into increments as shown in Figure 4-19. The subscript **m** denotes the **x** position, and the subscript **n** denotes the **y** position. Within the solid body the differential equation that governs the heat flow is

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = \rho c \frac{\partial T}{\partial \tau} \qquad \textbf{[4-24]}$$

Assuming constant properties, we recall from Chapter 3 that the second partial derivatives may be approximated by

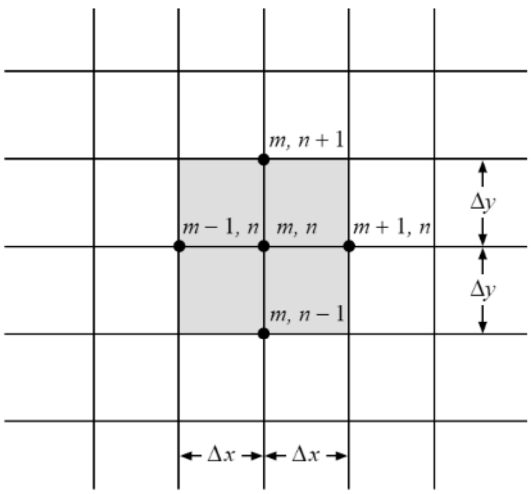
$$\frac{\partial^2 T}{\partial x^2} \approx \frac{1}{(\Delta x)^2}(T_{m+1,n} + T_{m-1,n} - 2T_{m,n}) \qquad \textbf{[4-25]}$$

$$\frac{\partial^2 T}{\partial y^2} \approx \frac{1}{(\Delta y)^2}(T_{m,n+1} + T_{m,n-1} - 2T_{m,n}) \qquad \textbf{[4-26]}$$

The time derivative in Equation (4-24) is approximated by

$$\frac{\partial T}{\partial \tau} \approx \frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta \tau} \qquad \textbf{[4-27]}$$

Figure 4-19 | Nomenclature for numerical solution of two-dimensional unsteady-state conduction problem.



In this relation the superscripts designate the time increment. Combining the relations above gives the difference equation equivalent to Equation (4-24)

$$\frac{T_{m+1,n}^{p} + T_{m-1,n}^{p} - 2T_{m,n}^{p}}{(\Delta x)^2} + \frac{T_{m,n+1}^{p} + T_{m,n-1}^{p} - 2T_{m,n}^{p}}{(\Delta y)^2} = \frac{1}{\alpha}\frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta \tau} \qquad \textbf{[4-28]}$$

Thus, if the temperatures of the various nodes are known at any particular time, the temperatures after a time increment **Δτ** may be calculated by writing an equation like Equation (4-28) for each node and obtaining the values of $T_{m,n}^{p+1}$. The procedure may be repeated to obtain the distribution after any desired number of time increments. If the increments of space coordinates are chosen such that

$\Delta x = \Delta y$ the resulting equation for $T_{m,n}^{p+1}$ becomes

$$T_{m,n}^{p+1} = \frac{\alpha\,\Delta\tau}{(\Delta x)^2}\left(T_{m+1,n}^{p} + T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p}\right) + \left[1 - \frac{4\alpha\,\Delta\tau}{(\Delta x)^2}\right] T_{m,n}^{p} \qquad \textbf{[4-29]}$$

If the time and distance increments are conveniently chosen so that

$$\frac{(\Delta x)^2}{\alpha\,\Delta\tau} = 4 \qquad \textbf{[4-30]}$$

It is seen that the temperature of node **(m, n)** after a time increment is simply the arithmetic average of the four surrounding nodal temperatures at the beginning of the time increment. When a one-dimensional system is involved, the equation becomes

$$T_{m}^{p+1} = \frac{\alpha\,\Delta\tau}{(\Delta x)^2}\left(T_{m+1}^{p} + T_{m-1}^{p}\right) + \left[1 - \frac{2\alpha\,\Delta\tau}{(\Delta x)^2}\right] T_{m}^{p} \qquad \textbf{[4-31]}$$

And if the time and distance increments are chosen so that

$$\frac{(\Delta x)^2}{\alpha\,\Delta\tau} = 2 \qquad \textbf{[4-32]}$$

The temperature of node **m** after the time increment is given as the arithmetic average of the two adjacent nodal temperatures at the beginning of the time increment

<u>Example</u>

Time constant	Fluid temperature ^oC	T_1	T_2	T_3	T_4	Fluid temperature ^oC
0	0	100	100	100	100	0
1	0	50	100	100	50	0
2	0	50	75	75	50	0
3	0	37.5	62.5	62.5	37.5	0
4	0	31.75	50	50	31.75	0

CHAPTER

5 Principles of Convection

The subject of convection heat transfer requires an energy balance along with an analysis of the fluid dynamics of the problems concerned. Our discussion in this chapter will first consider some of the simple relations of fluid dynamics and boundary layer analysis that are important for a basic understanding of convection heat transfer. Next, we shall impose an energy balance on the flow system and determine the influence of the flow on the temperature gradients in the fluid. Finally, having obtained knowledge of the temperature distribution, the heat-transfer rate from a heated surface to a fluid that is forced over it may be determined.

VISCOUS FLOW

Consider the flow over a flat plate as shown in Figures 5-1 and 5-2. Beginning at the leading edge of the plate, a region develops where the influence of viscous forces is felt. These viscous forces are described in terms of a shear stress τ between the fluid layers. If this stress is assumed to be proportional to the normal velocity gradient, we have the defining equation for the viscosity,

$$\tau = \mu \frac{du}{dy} \qquad \textbf{[5-1]}$$

The constant of proportionality $\boldsymbol{\mu}$ is called the *dynamic viscosity.* A typical set of units is newton-seconds per square meter; however, many sets of units are used for the viscosity, and care must be taken to select the proper group that will be consistent with the formulation at hand. The region of flow that develops from the leading edge of the plate in which the effects of viscosity are observed is called the *boundary layer*. Some arbitrary point is used to designate the $\boldsymbol{y}$ position where the boundary layer ends; this point is usually chosen as the $\boldsymbol{y}$ coordinate where the velocity becomes 99 percent of the free-stream value. Initially, the boundary-layer development is laminar, but at some critical distance from the leading edge, depending on the flow field and fluid properties, small disturbances in the flow begin to become amplified, and a transition process takes place until the flow

becomes turbulent. The turbulent-flow region may be pictured as a random churning action with chunks of fluid moving to and fro in all directions.
The transition from laminar to turbulent flow occurs when

$$\frac{u_\infty x}{\nu} = \frac{\rho u_\infty x}{\mu} > 5 \times 10^5$$

Where
u_∞ = free-stream velocity, m/s
x = distance from leading edge, m
$\nu = \mu/\rho$ = kinematic viscosity, m^2/s

Figure 5-1 | Sketch showing different boundary-layer flow regimes on a flat plate.

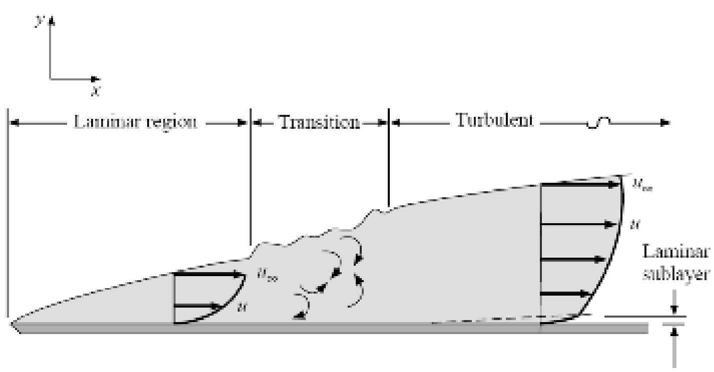
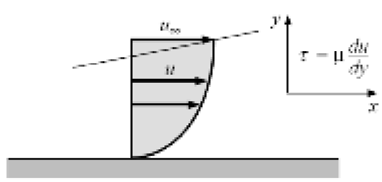


Figure 5-2 | Laminar velocity profile on a flat plate.



This particular grouping of terms is called the Reynolds number, and is dimensionless if a consistent set of units is used for all the properties:

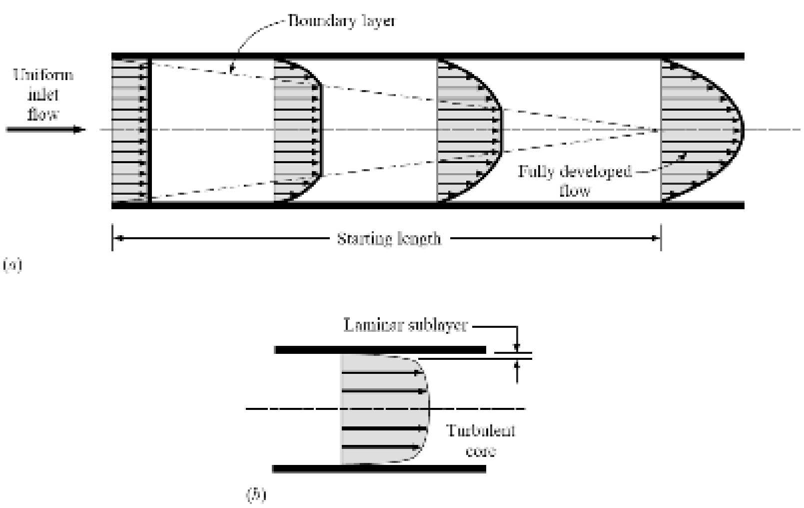
$$\mathrm{Re}_x = \frac{u_\infty x}{\nu} \quad \textbf{[5-2]}$$

Consider the flow in a tube as shown in Figure 5-3. A boundary layer develops at the entrance, as shown. Eventually the boundary layer fills the entire tube, and the flow is said to be fully developed. If the flow is laminar, a parabolic velocity profile is experienced, as shown in Figure 5-3*a*. When the flow is turbulent, a somewhat blunter profile is observed, as in Figure 5-3*b*. In a tube, the Reynolds number is again used as a criterion for laminar and turbulent flow. For the flow is

usually observed to be turbulent d is the tube diameter. Again, a range of Reynolds numbers for transition may be observed, depending on the pipe roughness and smoothness of the flow. The generally accepted range for transition is
$2000 < Re_d < 4000$

$$Re_d = \frac{u_m d}{\nu} > 2300 \qquad \textbf{[5-3]}$$

Figure 5-3 Velocity profile for (*a*) laminar flow in a tube and (*b*) turbulent tube flow.



LAMINAR BOUNDARY LAYER ON A FLAT PLATE

Consider the elemental control volume shown in Figure 5-4. We derive the equation of motion for the boundary layer by making a force-and-momentum balance on this element.
To simplify the analysis we assume:
1. The fluid is incompressible and the flow is steady.
2. There are no pressure variations in the direction perpendicular to the plate.
3. The viscosity is constant.
4. Viscous-shear forces in the ***y*** direction are negligible.
We apply Newton's second law of motion,

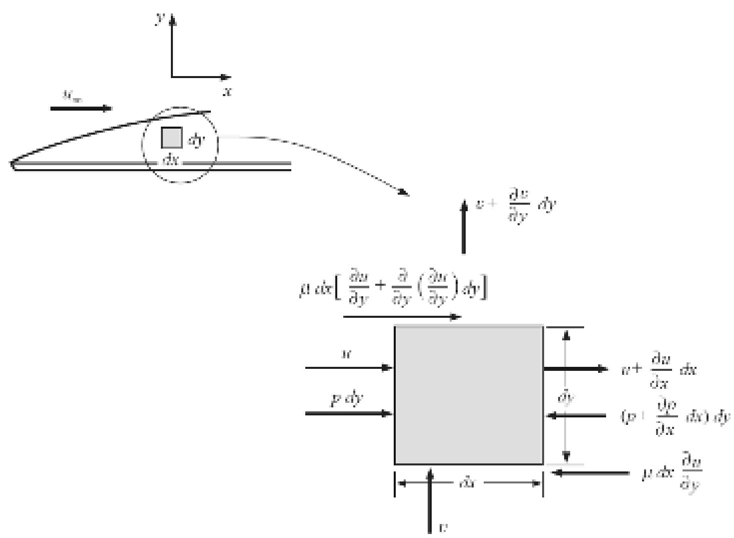
$$\sum F_x = \frac{d(mV)_x}{d\tau}$$

The above form of Newton's second law of motion applies to a system of constant mass. In fluid dynamics it is not usually convenient to work with elements of mass; rather, we deal with elemental control volumes such as that shown in Figure 5-4,

where mass may flow in or out of the different sides of the volume, which is fixed in space. For this system the force balance is then written

$$\sum F_x = \text{increase in momentum flux in } x \text{ direction}$$

Figure 5-4 | Elemental control volume for force balance on laminar boundary layer.



The result of the net force in **x** direction is

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\mu\frac{\partial^2 u}{\partial y^2}-\frac{\partial p}{\partial x} \qquad \textbf{[5-13]}$$

This is the momentum equation of the laminar boundary layer with constant properties.

The integral boundary-layer equation is

$$\rho\frac{d}{dx}\int_0^{\delta}(u_\infty-u)u\,dy=\tau_w=\mu\left.\frac{\partial u}{\partial y}\right]_{y=0} \qquad \textbf{[5 -14]}$$

If the velocity profile were known, the appropriate function could be inserted in Equation (5-14) to obtain an expression for the boundary-layer thickness. For our approximate analysis we first write down some conditions that the velocity function must satisfy:

$$u = 0 \qquad \text{at } y = 0 \qquad \textbf{[a]}$$

$$u = u_\infty \qquad \text{at } y = \delta \qquad \textbf{[b]}$$

$$\frac{\partial u}{\partial y} = 0 \qquad \text{at } y = \delta \qquad \textbf{[c]}$$

$$\frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{at } y = 0 \qquad \textbf{[d]}$$

The simplest function that we can choose to satisfy these conditions is a polynomial with four arbitrary constants. Thus

$$u = C_1 + C_2 y + C_3 y^2 + C_4 y^3$$

Applying the four conditions (*a*) to (*d)*,

$$\frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

Apply this velocity profile in equation 5-14, yields

$$\delta = 4.64\sqrt{\frac{\nu x}{u_\infty}}$$

Or in dimensionless form

$$\frac{\delta}{x} = \frac{4.64}{\mathrm{Re}_x^{1/2}}$$

Where

$$\mathrm{Re}_x = \frac{u_\infty x}{\nu}$$

The exact solution of the boundary-layer equations as given in Appendix B yields

$$\frac{\delta}{x} = \frac{5.0}{\mathrm{Re}_x^{1/2}}$$

ENERGY EQUATION OF THE BOUNDARY LAYER

The foregoing analysis considered the fluid dynamics of a laminar-boundary-layer flow system. We shall now develop the energy equation for this system and then proceed to an integral method of solution.
Consider the elemental control volume shown in Figure 5-6. To simplify the analysis we assume

1. Incompressible steady flow
2. Constant viscosity, thermal conductivity, and specific heat
3. Negligible heat conduction in the direction of flow (x direction), i.e.

$$\frac{\partial T}{\partial x} \ll \frac{\partial T}{\partial y}$$

Then, for the element shown, the energy balance may be written as;

Energy convected in left face + energy convected in bottom face + heat conducted in bottom face + net viscous work done on element =energy convected out right face+energy convected out top face + heat conducted out top face

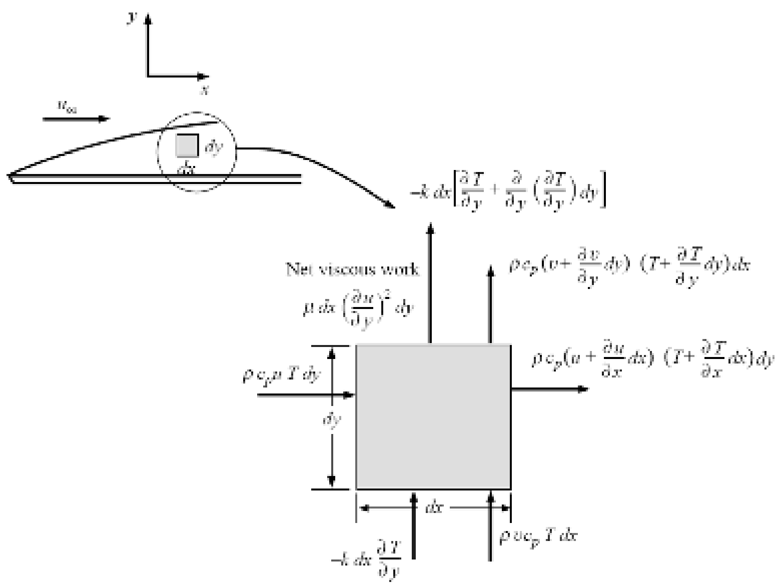
The convective and conduction energy quantities are indicated in Figure 5-6, and the energy term for the viscous work may be derived as follows. The viscous work may be computed as a product of the net viscous-shear force and the distance this force moves in unit time. The viscous-shear force is the product of the shear-stress and the area **dx**,

$$\mu \frac{\partial u}{\partial y} dx$$

And the distance through which it moves per unit time in respect to the elemental control volume **dx dy** is

$$\frac{\partial u}{\partial y} dy$$

Figure 5-6 | Elemental volume for energy analysis of laminar boundary layer



So that the net viscous energy delivered to the element is

$$\mu \left(\frac{\partial u}{\partial y} \right)^2 dx\, dy$$

Writing the energy balance corresponding to the quantities shown in Figure 5-6, assuming unit depth in the ***z*** direction, and neglecting second-order differentials yields

$$\rho c_p \left[u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + T \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right] dx\, dy = k \frac{\partial^2 T}{\partial y^2} dx\, dy + \mu \left(\frac{\partial u}{\partial y} \right)^2 dx\, dy$$

Using the continuity relation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

And dividing by $\boldsymbol{\rho c_p}$ gives

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p} \left(\frac{\partial u}{\partial y} \right)^2 \qquad \textbf{[5-22]}$$

This is the energy equation of the laminar boundary layer. The left side represents the net transport of energy into the control volume, and the right side represents the sum of the net heat conducted out of the control volume and the net viscous work done on the element. The viscous-work term is of importance only at high velocities since its magnitude will be small compared with the other terms when low-velocity flow is studied. This may be shown with an order-of-magnitude analysis of the two terms on the right side of Equation (5-22). For this order-of-magnitude analysis we might consider the velocity as having the order of the free-stream velocity $u\infty$ and the y dimension of the order of **δ**. Thus

$$u \sim u_\infty \qquad \text{and} \qquad y \sim \delta$$

$$\alpha \frac{\partial^2 T}{\partial y^2} \sim \alpha \frac{T}{\delta^2}$$

$$\frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^2 \sim \frac{\mu}{\rho c_p}\frac{u_\infty^2}{\delta^2}$$

If the ratio of these quantities is small, that is,

$$\frac{\mu}{\rho c_p \alpha}\frac{u_\infty^2}{T} \ll 1 \qquad \textbf{[5-23]}$$

Then the viscous dissipation is small in comparison with the conduction term. Let us rearrange Equation (5-23) by introducing

$$\Pr = \frac{\nu}{\alpha} = \frac{c_p \mu}{k}$$

Where, **Pr** is called the Prandtl number, which we shall discuss later. Equation (5-23) becomes

$$\Pr \frac{u_\infty^2}{c_p T} \ll 1 \qquad \textbf{[5-24]}$$

As an example, consider the flow of air at

$$u_\infty = 70 \text{ m/s} \qquad T = 20°\text{C} = 293 \text{ K} \qquad p = 1 \text{ atm}$$

For these conditions $c_p = 1005 = \text{J/kg} \cdot °\text{C}$ and $\Pr = 0.7$ so that

$$\Pr \frac{u_\infty^2}{c_p T} = \frac{(0.7)(70)^2}{(1005)(293)} = 0.012 \ll 1.0$$

This value indicating that the viscous dissipation is small for even this rather large flow velocity of **70 m/s**. Thus, for low-velocity incompressible flow, we have

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2} \qquad \textbf{[5-25]}$$

There is a striking similarity between Equation (5-25) and the momentum equation for constant pressure

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \qquad \textbf{[5-26]}$$

The solution to the two equations will have exactly the same form when $\boldsymbol{\alpha = \nu}$. Thus we should expect that the relative magnitudes of the thermal diffusivity and kinematic viscosity would have an important influence on convection heat transfer since these magnitudes relate the velocity distribution to the temperature distribution. This is exactly the case, and we shall see the role that these parameters play in the subsequent discussion.

THE THERMAL BOUNDARY LAYER

Consider the system shown in Figure 5-7. The temperature of the wall is $\mathbf{T_w}$, the temperature of the fluid outside the thermal boundary layer is $\mathbf{T_\infty}$, and the thickness of the thermal boundary layer is designated as $\boldsymbol{\delta_t}$. At the wall, the velocity is zero, and the heat transfer into the fluid takes place by conduction. Thus the local heat flux per unit area, q'' , is

$$\frac{q}{A}=q''=-k\left.\frac{\partial T}{\partial y}\right]_{\text{wall}} \qquad \textbf{[5-27]}$$

From Newton's law of cooling [Equation (1-8)],

$$q''=h(T_w-T_\infty) \qquad \textbf{[5-28]}$$

where h is the convection heat-transfer coefficient. Combining these equations, we have

$$h=\frac{-k(\partial T/\partial y)_{\text{wall}}}{T_w-T_\infty} \qquad \textbf{[5-29]}$$

So that we need only find the temperature gradient at the wall in order to evaluate the heat-transfer coefficient. This means that we must obtain an expression for the

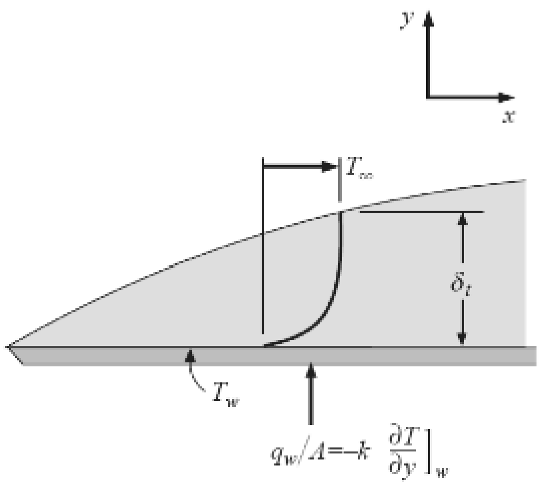
temperature distribution. To do this, an approach similar to that used in the momentum analysis of the boundary layer is followed.
The conditions that the temperature distribution must satisfy are

$$T = T_w \quad \text{at } y = 0 \qquad [a]$$

$$\frac{\partial T}{\partial y} = 0 \quad \text{at } y = \delta_t \qquad [b]$$

$$T = T_\infty \quad \text{at } y = \delta_t \qquad [c]$$

Figure 5-7 | Temperature profile in the thermal boundary layer.



and by writing Equation (5-25) at $y = 0$ with no viscous heating we find

$$\frac{\partial^2 T}{\partial y^2} = 0 \quad \text{at } y = 0 \qquad [d]$$

since the velocities must be zero at the wall.

Conditions (a) to (d) may be fitted to a cubic polynomial as in the case of the velocity profile, so that

$$\frac{\theta}{\theta_\infty} = \frac{T - T_w}{T_\infty - T_w} = \frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3 \qquad [5\text{-}30]$$

Where $\boldsymbol{\theta = T - T_w}$. There now remains the problem of finding an expression for $\boldsymbol{\delta_t}$, the thermal-boundary-layer thickness. This may be obtained by an integral analysis of the energy equation for the boundary layer.
Consider the control volume bounded by the planes **1, 2, A-A**, and the wall as shown in Figure 5-8. It is assumed that the thermal boundary layer is thinner than the hydrodynamic boundary layer, as shown. The wall temperature is $\boldsymbol{T_w}$, the free-

stream temperature is $\mathbf{T_\infty}$, and the heat given up to the fluid over the length **dx** is $\mathbf{dq_w}$. We wish to make the energy balance

Energy convected in + viscous work within element + heat transfer at wall = energy convected out [5-31]

The energy convected in through plane 1 is

$$\rho c_p \int_0^H uT\, dy$$

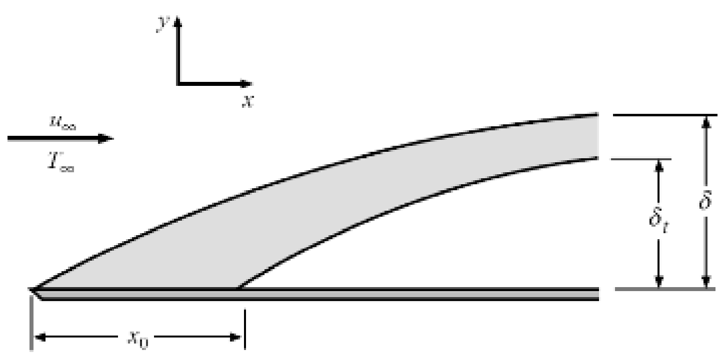
And the energy convected out through plane 2 is

$$\rho c_p \left(\int_0^H uT\, dy \right) + \frac{d}{dx} \left(\rho c_p \int_0^H uT\, dy \right) dx$$

The mass flow through plane *A*-*A* is

$$\frac{d}{dx} \left(\int_0^H \rho u\, dy \right) dx$$

Figure 5-9 | Hydrodynamic and thermal boundary layers on a flat plate. Heating starts at $x = x_0$.



and this carries with it an energy equal to

$$c_p T_\infty \frac{d}{dx} \left(\int_0^H \rho u\, dy \right) dx$$

The net viscous work done within the element is

$$\mu \left[\int_0^H \left(\frac{du}{dy} \right)^2 dy \right] dx$$

and the heat transfer at the wall is

$$dq_w = -k\,dx\,\frac{\partial T}{\partial y}\bigg]_w$$

Combining these energy quantities according to Equation (5-31) and collecting terms gives

$$\frac{d}{dx}\left[\int_0^H (T_\infty - T)u\,dy\right] + \frac{\mu}{\rho c_p}\left[\int_0^H \left(\frac{du}{dy}\right)^2 dy\right] = \alpha\,\frac{\partial T}{dy}\bigg]_w \qquad \textbf{[5-32]}$$

To calculate the heat transfer at the wall, we need to derive an expression for the thermal boundary layer thickness that may be used in conjunction with Equations (5-29) and (5-30) to determine the heat-transfer coefficient. For now, we neglect the viscous-dissipation term; this term is very small unless the velocity of the flow field becomes very large. The plate under consideration need not be heated over its entire length. The situation that we shall analyze is shown in Figure 5-9, where the hydrodynamic boundary layer develops from the leading edge of the plate, while heating does not begin until $x=x_0$. Inserting the temperature distribution Equation (5-30) and the velocity distribution Equation (5-19) into Equation (5-32) and neglecting the viscous-dissipation term, gives

$$\begin{aligned}\frac{d}{dx}\left[\int_0^H (T_\infty - T)u\,dy\right] &= \frac{d}{dx}\left[\int_0^H (\theta_\infty - \theta)u\,dy\right]\\ &= \theta_\infty u_\infty \frac{d}{dx}\left\{\int_0^H \left[1 - \frac{3}{2}\frac{y}{\delta_t} + \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3\right]\left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right] dy\right\}\\ &= \alpha\,\frac{\partial T}{\partial y}\bigg]_{y=0} = \frac{3\alpha\theta_\infty}{2\delta_t}\end{aligned}$$

Where $\boldsymbol{\delta_t}$ is the thermal boundary layer thickness

The integration result shows that

$$\zeta = \frac{\delta_t}{\delta} = \frac{1}{1.026}\mathrm{Pr}^{-1/3}\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3} \qquad \textbf{[5-36]}$$

Where

$$\mathrm{Pr} = \frac{\nu}{\alpha} \qquad \textbf{[5-37]}$$

has been introduced. The ratio $\boldsymbol{\nu/\alpha}$ is called the Prandtl number after Ludwig Prandtl, the German scientist who introduced the concepts of boundary-layer theory.

When the plate is heated over the entire length, $x_0 = 0$, and

$$\frac{\delta_t}{\delta} = \zeta = \frac{1}{1.026}\mathrm{Pr}^{-1/3} \qquad \textbf{[5-38]}$$

In the foregoing analysis the assumption was made that **ζ<1**. This assumption is satisfactory for fluids having Prandtl numbers greater than about **0.7**. Fortunately, most gases and liquids fall within this category. Liquid metals are a notable exception, however, since they have Prandtl numbers of the order of **0.01**.
In the foregoing analysis the assumption was made that **ζ<1**. This assumption is satisfactory for fluids having Prandtl numbers greater than about **0.7**. Fortunately, most gases and liquids fall within this category. Liquid metals are a notable exception, however, since they have Prandtl numbers of the order of **0.01**.

$$\mathrm{Pr} = \frac{\nu}{\alpha} = \frac{\mu/\rho}{k/\rho c_p} = \frac{c_p \mu}{k} \qquad \textbf{[5-39]}$$

Returning now to the analysis, we have

$$h = \frac{-k(\partial T/\partial y)_w}{T_w - T_\infty} = \frac{3}{2}\frac{k}{\delta_t} = \frac{3}{2}\frac{k}{\zeta\delta} \qquad \textbf{[5-40]}$$

Substituting for the hydrodynamic-boundary-layer thickness from Equation (5-21) and using Equation (5-36) gives

$$h_x = 0.332k\,\mathrm{Pr}^{1/3}\left(\frac{u_\infty}{\nu x}\right)^{1/2}\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{-1/3} \qquad \textbf{[5-41]}$$

The equation may be nondimensionalized by multiplying both sides by **x/k**, producing the dimensionless group on the left side,

$$\mathrm{Nu}_x = \frac{h_x x}{k} \qquad \textbf{[5-42]}$$

called the Nusselt number after Wilhelm Nusselt, who made significant contributions to the theory of convection heat transfer. Finally,

$$\mathrm{Nu}_x = 0.332\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{-1/3} \qquad \textbf{[5-43]}$$

Or, for the plate heated over its entire length, $\mathbf{x_0 = 0}$ and

$$\mathrm{Nu}_x = 0.332\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2} \qquad \textbf{[5-44]}$$

Equations (5-41), (5-43), and (5-44) express the local values of the heat-transfer coefficient in terms of the distance from the leading edge of the plate and the fluid properties. For the case where $x_0 = 0$ the average heat-transfer coefficient and Nusselt number may be obtained by integrating over the length of the plate:

$$\bar{h} = \frac{\int_0^L h_x\,dx}{\int_0^L dx} = 2h_{x=L} \qquad \textbf{[5-45a]}$$

For a plate where heating starts at $x = x_0$, it can be shown that the average heat transfer coefficient can be expressed as

$$\frac{\bar{h}_{x_0-L}}{h_{x=L}} = 2L\frac{1-(x_0/L)^{3/4}}{L-x_0} \qquad \textbf{[5-45b]}$$

In this case, the total heat transfer for the plate would be

$$\frac{\bar{h}_{x_0-L}}{h_{x=L}} = 2L\frac{1-(x_0/L)^{3/4}}{L-x_0} \qquad \textbf{[5-45b]}$$

Assuming the heated section is at the constant temperature T_w. For the plate heated over the entire length,

$$\overline{\mathrm{Nu}}_L = \frac{\bar{h}L}{k} = 2\,\mathrm{Nu}_{x=L} \qquad \textbf{[5-46a]}$$

Or

$$\overline{\mathrm{Nu}}_L = \frac{\bar{h}L}{k} = 0.664\,\mathrm{Re}_L^{1/2}\mathrm{Pr}^{1/3} \qquad \textbf{[5-46b]}$$

Where

$$\mathrm{Re}_L = \frac{\rho u_\infty L}{\mu}$$

The foregoing analysis was based on the assumption that the fluid properties were constant throughout the flow. When there is an appreciable variation between wall and free-stream conditions, it is recommended that the properties be evaluated at

the so-called *film temperature* $\mathbf{T_f}$, defined as the arithmetic mean between the wall and free-stream temperature,

$$T_f = \frac{T_w + T_\infty}{2} \qquad \textbf{[5-47]}$$

For the constant-heat-flux case it can be shown that the local Nusselt number is given by

$$\mathrm{Nu}_x = \frac{hx}{k} = 0.453\ \mathrm{Re}_x^{1/2}\ \mathrm{Pr}^{1/3} \qquad \textbf{[5-48]}$$

which may be expressed in terms of the wall heat flux and temperature difference as

$$\mathrm{Nu}_x = \frac{q_w x}{k(T_w - T_\infty)} \qquad \textbf{[5-49]}$$

The average temperature difference along the plate, for the constant-heat-flux condition, may be obtained by performing the integration

$$\overline{T_w - T_\infty} = \frac{1}{L}\int_0^L (T_w - T_\infty)\,dx = \frac{1}{L}\int_0^L \frac{q_w x}{k\ \mathrm{Nu}_x}\,dx$$
$$= \frac{q_w L/k}{0.6795\ \mathrm{Re}_L^{1/2}\ \mathrm{Pr}^{1/3}} \qquad \textbf{[5-50]}$$

Or

$$q_w = \tfrac{3}{2} h_{x=L}(\overline{T_w - T_\infty})$$

Isothermal Flat Plate Heated Over Entire Length — **EXAMPLE 5-4**

For the flow system in Example 5-3 assume that the plate is heated over its entire length to a temperature of 60°C. Calculate the heat transferred in (*a*) the first 20 cm of the plate and (*b*) the first 40 cm of the plate.

■ Solution

The total heat transfer over a certain length of the plate is desired; so we wish to calculate average heat-transfer coefficients. For this purpose we use Equations (5-44) and (5-45), evaluating the properties at the film temperature:

$$T_f = \frac{27 + 60}{2} = 43.5°\mathrm{C} = 316.5\ \mathrm{K} \quad [110.3°\mathrm{F}]$$

From Appendix A the properties are

$$\nu = 17.36 \times 10^{-6}\ \mathrm{m^2/s} \quad [1.87 \times 10^{-4}\ \mathrm{ft^2/s}]$$
$$k = 0.02749\ \mathrm{W/m \cdot °C} \quad [0.0159\ \mathrm{Btu/h \cdot ft \cdot °F}]$$
$$\mathrm{Pr} = 0.7$$
$$c_p = 1.006\ \mathrm{kJ/kg \cdot °C} \quad [0.24\ \mathrm{Btu/lb_m \cdot °F}]$$

At $x = 20$ cm

$$\mathrm{Re}_x = \frac{u_\infty x}{\nu} = \frac{(2)(0.2)}{17.36 \times 10^{-6}} = 23{,}041$$

$$\mathrm{Nu}_x = \frac{h_x x}{k} = 0.332 \mathrm{Re}_x^{1/2} \mathrm{Pr}^{1/3}$$

$$= (0.332)(23{,}041)^{1/2}(0.7)^{1/3} = 44.74$$

$$h_x = \mathrm{Nu}_x \left(\frac{k}{x}\right) = \frac{(44.74)(0.02749)}{0.2}$$

$$= 6.15 \ \mathrm{W/m^2 \cdot {}^\circ C} \quad [1.083 \ \mathrm{Btu/h \cdot ft^2 \cdot {}^\circ F}]$$

The average value of the heat-transfer coefficient is twice this value, or

$$\bar{h} = (2)(6.15) = 12.3 \ \mathrm{W/m^2 \cdot {}^\circ C} \quad [2.17 \ \mathrm{Btu/h \cdot ft^2 \cdot {}^\circ F}]$$

The heat flow is

$$q = \bar{h} A (T_w - T_\infty)$$

If we assume unit depth in the z direction,

$$q = (12.3)(0.2)(60 - 27) = 81.18 \ \mathrm{W} \quad [277 \ \mathrm{Btu/h}]$$

At $x = 40$ cm

$$\mathrm{Re}_x = \frac{u_\infty x}{\nu} = \frac{(2)(0.4)}{17.36 \times 10^{-6}} = 46{,}082$$

$$\mathrm{Nu}_x = (0.332)(46{,}082)^{1/2}(0.7)^{1/3} = 63.28$$

$$h_x = \frac{(63.28)(0.02749)}{0.4} = 4.349 \ \mathrm{W/m^2 \cdot {}^\circ C}$$

$$\bar{h} = (2)(4.349) = 8.698 \ \mathrm{W/m^2 \cdot {}^\circ C} \quad [1.53 \ \mathrm{Btu/h \cdot ft^2 \cdot {}^\circ F}]$$

$$q = (8.698)(0.4)(60 - 27) = 114.8 \ \mathrm{W} \quad [392 \ \mathrm{Btu/h}]$$

EXAMPLE 5-5 Flat Plate with Constant Heat Flux

A 1.0-kW heater is constructed of a glass plate with an electrically conducting film that produces a constant heat flux. The plate is 60 cm by 60 cm and placed in an airstream at 27°C, 1 atm with $u_\infty = 5$ m/s. Calculate the average temperature difference along the plate and the temperature difference at the trailing edge.

■ Solution

Properties should be evaluated at the film temperature, but we do not know the plate temperature. So for an initial calculation, we take the properties at the free-stream conditions of

$$T_\infty = 27^\circ\mathrm{C} = 300 \ \mathrm{K}$$

$$\nu = 15.69 \times 10^{-6} \ \mathrm{m^2/s} \qquad \mathrm{Pr} = 0.708 \qquad k = 0.02624 \ \mathrm{W/m \cdot {}^\circ C}$$

$$\mathrm{Re}_L = \frac{(0.6)(5)}{15.69 \times 10^{-6}} = 1.91 \times 10^5$$

From Equation (5-50) the average temperature difference is

$$\overline{T_w - T_\infty} = \frac{[1000/(0.6)^2](0.6)/0.02624}{0.6795(1.91 \times 10^5)^{1/2}(0.708)^{1/3}} = 240^\circ\mathrm{C}$$

Now, we go back and evaluate properties at

$$T_f = \frac{240 + 27 + 27}{2} = 147°\text{C} = 420\text{ K}$$

and obtain

$$\nu = 28.22 \times 10^{-6}\ \text{m}^2/\text{s} \qquad \text{Pr} = 0.687 \qquad k = 0.035\ \text{W/m} \cdot °\text{C}$$

$$\text{Re}_L = \frac{(0.6)(5)}{28.22 \times 10^{-6}} = 1.06 \times 10^5$$

$$\overline{T_w - T_\infty} = \frac{[1000/(0.6)^2](0.6)/0.035}{0.6795(1.06 \times 10^5)^{1/2}(0.687)^{1/3}} = 243°\text{C}$$

At the end of the plate ($x = L = 0.6$ m) the temperature difference is obtained from Equations (5-48) and (5-50) with the constant 0.453 to give

$$(T_w - T_\infty)_{x=L} = \frac{(243.6)(0.6795)}{0.453} = 365.4°\text{C}$$

An alternate solution would be to base the Nusselt number on Equation (5-51).

Plate with Unheated Starting Length

EXAMPLE 5-6

Air at 1 atm and 300 K flows across a 20-cm-square plate at a free-stream velocity of 20 m/s. The last half of the plate is heated to a constant temperature of 350 K. Calculate the heat lost by the plate.

■ **Solution**

First we evaluate the air properties at the film temperature

$$T_f = (T_w + T_\infty)/2 = 325\text{ K}$$

and obtain

$$\nu = 18.23 \times 10^{-6}\text{m}^2/\text{s} \qquad k = 0.02814\ \text{W/m} \cdot °\text{C} \qquad \text{Pr} = 0.7$$

At the trailing edge of the plate the Reynolds number is

$$\text{Re}_L = u_\infty L/\nu = (20)(0.2)/18.23 \times 10^{-6} = 2.194 \times 10^5$$

or, laminar flow over the length of the plate.

Heating does not start until the last half of the plate, or at a position $x_0 = 0.1$ m. The local heat-transfer coefficient for this condition is given by Equation (5-41):

$$h_x = 0.332k\,\text{Pr}^{1/3}(u_\infty/\nu x)^{1/2}[1 - (x_0/x)^{0.75}]^{-1/3} \qquad [a]$$

Inserting the property values along with $x_0 = 0.1$ gives

$$h_x = 8.6883x^{-1/2}(1 - 0.17783x^{-0.75})^{-1/3} \qquad [b]$$

The plate is 0.2 m wide so the heat transfer is obtained by integrating over the heated length $x_0 < x < L$

$$q = (0.2)(T_w - T_\infty)\int_{x_0=0.1}^{L=0.2} h_x\,dx \qquad [c]$$

Inserting Equation (b) in Equation (c) and performing the numerical integration gives

$$q = (0.2)(8.6883)(0.4845)(350 - 300) = 421\text{ W} \qquad [d]$$

The average value of the heat-transfer coefficient *over the heated length* is given by

$$\bar{h} = q/(T_w - T_\infty)(L - x_0)W = 421/(350 - 300)(0.2 - 0.1)(0.2) = 421\ \text{W/m}^2 \cdot °\text{C}$$

where W is the width of the plate.

An easier calculation can be made by applying Equation (5-45b) to determine the average heat transfer coefficient over the heated portion of the plate. The result is

$$\bar{h} = 425.66\ \text{W/m}^2 \cdot °\text{C} \quad \text{and} \quad q = 425.66\text{ W}$$

which indicates, of course, only a small error in the numerical integration.

THE RELATION BETWEEN FLUID FRICTION AND HEAT TRANSFER

We have already seen that the temperature and flow fields are related. Now we seek an expression whereby the frictional resistance may be directly related to heat transfer.
The shear stress at the wall may be expressed in terms of a friction coefficient $\mathbf{C_f}$:

$$\tau_w = C_f \frac{\rho u_\infty^2}{2} \qquad \textbf{[5-52]}$$

Equation (5-52) is the defining equation for the friction coefficient. The shear stress may also be calculated from the relation

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right]_w$$

Using the velocity distribution given by Equation (5-19), we have

$$\tau_w = \frac{3}{2}\frac{\mu u_\infty}{\delta}$$

and making use of the relation for the boundary-layer thickness gives

$$\tau_w = \frac{3}{2}\frac{\mu u_\infty}{4.64}\left(\frac{u_\infty}{\nu x}\right)^{1/2} \qquad \textbf{[5-53]}$$

Combining Equations (5-52) and (5-53) leads to

$$\frac{C_{fx}}{2} = \frac{3}{2}\frac{\mu u_\infty}{4.64}\left(\frac{u_\infty}{\nu x}\right)^{1/2}\frac{1}{\rho u_\infty^2} = 0.323\ \mathrm{Re}_x^{-1/2} \qquad \textbf{[5-54]}$$

The exact solution of the boundary-layer equations yields

$$\frac{C_{fx}}{2} = 0.332\ \mathrm{Re}_x^{-1/2} \qquad \textbf{[5-54}\boldsymbol{a}\textbf{]}$$

Equation (5-44) may be rewritten in the following form:

$$\frac{\mathrm{Nu}_x}{\mathrm{Re}_x\,\mathrm{Pr}} = \frac{h_x}{\rho c_p u_\infty} = 0.332\ \mathrm{Pr}^{-2/3}\ \mathrm{Re}_x^{-1/2}$$

The group on the left is called the Stanton number,

$$\mathrm{St}_x = \frac{h_x}{\rho c_p u_\infty}$$

So that

$$\mathrm{St}_x\ \mathrm{Pr}^{2/3} = 0.332\ \mathrm{Re}_x^{1/2} \qquad \textbf{[5-55]}$$

Upon comparing Equations (5-54) and (5-55), we note that the right sides are alike except for a difference of about 3 percent in the constant, which is the result of the approximate nature of the integral boundary-layer analysis. We recognize this approximation and write

$$\mathrm{St}_x \mathrm{Pr}^{2/3} = \frac{C_{fx}}{2} \qquad \textbf{[5-56]}$$

Equation (5-56), called the *Reynolds-Colburn analogy,* expresses the relation between fluid friction and heat transfer for laminar flow on a flat plate. The heat-transfer coefficient thus could be determined by making measurements of the frictional drag on a plate under conditions in which no heat transfer is involved.

EXAMPLE 5-8 Drag Force on a Flat Plate

For the flow system in Example 5-4 compute the drag force exerted on the first 40 cm of the plate using the analogy between fluid friction and heat transfer.

■ **Solution**

We use Equation (5-56) to compute the friction coefficient and then calculate the drag force. An average friction coefficient is desired, so

$$\overline{\mathrm{St}}\ \mathrm{Pr}^{2/3} = \frac{\overline{C}_f}{2} \qquad [a]$$

The density at 316.5 K is

$$\rho = \frac{p}{RT} = \frac{1.0132 \times 10^5}{(287)(316.5)} = 1.115\ \mathrm{kg/m^3}$$

For the 40-cm length

$$\overline{\mathrm{St}} = \frac{\bar{h}}{\rho c_p u_\infty} = \frac{8.698}{(1.115)(1006)(2)} = 3.88 \times 10^{-3}$$

Then from Equation (a)

$$\frac{\overline{C}_f}{2} = (3.88 \times 10^{-3})(0.7)^{2/3} = 3.06 \times 10^{-3}$$

The average shear stress at the wall is computed from Equation (5-52):

$$\overline{\tau}_w = \overline{C}_f \rho \frac{u_\infty^2}{2}$$
$$= (3.06 \times 10^{-3})(1.115)(2)^2$$
$$= 0.0136 \text{ N/m}^2$$

The drag force is the product of this shear stress and the area,

$$D = (0.0136)(0.4) = 5.44 \text{ mN} \quad [1.23 \times 10^{-3} \text{ lb}_f]$$

The Bulk Temperature

In tube flow the convection heat-transfer coefficient is usually defined by

$$\text{Local heat flux} = q'' = h(T_w - T_b) \qquad \textbf{[5-101]}$$

Where $\mathbf{T_w}$ is the wall temperature and $\mathbf{T_b}$ is the so-called *bulk temperature* or energy-average fluid temperature across the tube, which may be calculated from

$$T_b = \overline{T} = \frac{\int_0^{r_o} \rho 2\pi r\, dr\, u c_p T}{\int_0^{r_o} \rho 2\pi r\, dr\, u c_p} \qquad \textbf{[5-102]}$$

The reason for using the bulk temperature in the definition of heat-transfer coefficients for tube flow may be explained as follows. In a tube flow there is no easily discernible free stream condition as is present in the flow over a flat plate. Even the centerline temperature $\mathbf{T_c}$ is not easily expressed in terms of the inlet flow variables and the heat transfer. At any $\mathbf{x}$ position, the temperature that is indicative of the total energy of the flow is an integrated mass-energy average temperature over the entire flow area. The numerator of Equation (5-102) represents the total energy flow through the tube, and the denominator represents the product of mass flow and specific heat integrated over the flow area. The bulk temperature is thus representative of the total energy of the flow at the particular location.

Summery

Table 5-2 | Summary of equations for flow over flat plates. Properties evaluated at $T_f = (T_w + T_\infty)/2$ unless otherwise r

Flow regime	Restrictions	Equation	Equ
		Heat transfer	
Laminar, local	$T_w = \text{const}$, $\mathrm{Re}_x < 5 \times 10^5$, $0.6 < \mathrm{Pr} < 50$	$\mathrm{Nu}_x = 0.332\,\mathrm{Pr}^{1/3}\mathrm{Re}_x^{1/2}$	
Laminar, local	$T_w = \text{const}$, $\mathrm{Re}_x < 5 \times 10^5$, $\mathrm{Re}_x\,\mathrm{Pr} > 100$	$\mathrm{Nu}_x = \dfrac{0.3387\,\mathrm{Re}_x^{1/2}\,\mathrm{Pr}^{1/3}}{\left[1+\left(\dfrac{0.0468}{\mathrm{Pr}}\right)^{2/3}\right]^{1/4}}$	
Laminar, local	$q_w = \text{const}$, $\mathrm{Re}_x < 5 \times 10^5$, $0.6 < \mathrm{Pr} < 50$	$\mathrm{Nu}_x = 0.453\,\mathrm{Re}_x^{1/2}\,\mathrm{Pr}^{1/3}$	
Laminar, local	$q_w = \text{const}$, $\mathrm{Re}_x < 5 \times 10^5$	$\mathrm{Nu}_x = \dfrac{0.4637\,\mathrm{Re}_x^{1/2}\,\mathrm{Pr}^{1/3}}{\left[1+\left(\dfrac{0.0207}{\mathrm{Pr}}\right)^{2/3}\right]^{1/4}}$	
Laminar, average	$\mathrm{Re}_L < 5 \times 10^5$, $T_w = \text{const}$	$\overline{\mathrm{Nu}}_L = 2\,\mathrm{Nu}_{x=L} = 0.664\,\mathrm{Re}_L^{1/2}\,\mathrm{Pr}^{1/3}$	
Laminar, local	$T_w = \text{const}$, $\mathrm{Re}_x < 5 \times 10^5$, $\mathrm{Pr} \ll 1$ (liquid metals)	$\mathrm{Nu}_x = 0.564(\mathrm{Re}_x\,\mathrm{Pr})^{1/2}$	
Laminar, local	$T_w = \text{const}$, starting at $x = x_0$, $\mathrm{Re}_x < 5 \times 10^5$, $0.6 < \mathrm{Pr} < 50$	$\mathrm{Nu}_x = 0.332\,\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}\left[1-\left(\frac{x_0}{x}\right)^{3/4}\right]^{-1/3}$	
Turbulent, local	$T_w = \text{const}$, $5 \times 10^5 < \mathrm{Re}_x < 10^7$	$\mathrm{St}_x\,\mathrm{Pr}^{2/3} = 0.0296\,\mathrm{Re}_x^{-0.2}$	
Turbulent, local	$T_w = \text{const}$, $10^7 < \mathrm{Re}_x < 10^9$	$\mathrm{St}_x\,\mathrm{Pr}^{2/3} = 0.185(\log \mathrm{Re}_x)^{-2.584}$	
Turbulent, local	$q_w = \text{const}$, $5 \times 10^5 < \mathrm{Re}_x < 10^7$	$\mathrm{Nu}_x = 1.04\,\mathrm{Nu}_{x\,T_w=\text{const}}$	
Laminar-turbulent, average	$T_w = \text{const}$, $\mathrm{Re}_x < 10^7$, $\mathrm{Re}_{\text{crit}} = 5 \times 10^5$	$\overline{\mathrm{St}}\,\mathrm{Pr}^{2/3} = 0.037\,\mathrm{Re}_L^{-0.2} - 871\,\mathrm{Re}_L^{-1}$ $\overline{\mathrm{Nu}}_L = \mathrm{Pr}^{1/3}(0.037\,\mathrm{Re}_L^{0.8} - 871)$	
Laminar-turbulent, average	$T_w = \text{const}$, $\mathrm{Re}_x < 10^7$, liquids, μ at T_∞, μ_w at T_w	$\overline{\mathrm{Nu}}_L = 0.036\,\mathrm{Pr}^{0.43}(\mathrm{Re}_L^{0.8} - 9200)\left(\frac{\mu_\infty}{\mu_w}\right)^{1/4}$	
High-speed flow	$T_w = \text{const}$, $q = hA(T_w - T_{aw})$ $r = (T_{aw} - T_\infty)/(T_o - T_\infty)$ = recovery factor = $\mathrm{Pr}^{1/2}$ (laminar) = $\mathrm{Pr}^{1/3}$ (turbulent)	Same as for low-speed flow with properties evaluated at $T^* = T_\infty + 0.5(T_w - T_\infty) + 0.22(T_{aw} - T_\infty)$	

CHAPTER 6 Empirical and Practical Relations for Forced-Convection Heat Transfer

EMPIRICAL RELATIONS FOR PIPE AND TUBE FLOW

For design and engineering purposes, empirical correlations are usually of greatest practical utility. In this section we present some of the more important and useful empirical relations and point out their limitations.

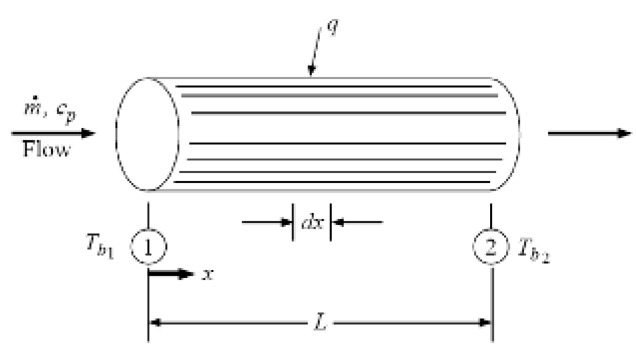
The Bulk Temperature

First let us give some further consideration to the bulk-temperature concept that is important in all heat-transfer problems involving flow inside closed channels. In Chapter 5 we noted that the bulk temperature represents energy average or "mixing cup" conditions. Thus, for the tube flow depicted in Figure 6-1 the total energy added can be expressed in terms of a bulk-temperature difference by

$$q = \dot{m} c_p (T_{b_2} - T_{b_1}) \qquad \textbf{[6-1]}$$

Provided **cp** is reasonably constant over the length. In some differential length **dx** the heat added **dq** can be expressed either in terms of a bulk-temperature difference or in terms of the heat-transfer coefficient.

Figure 6-1 | Total heat transfer in terms of bulk-temperature difference.



$$dq = \dot{m} c_p dT_b = h(2\pi r)dx(T_w - T_b) \qquad \textbf{[6-2]}$$

Where, T_w and T_b are the wall and bulk temperatures at the particular **x** location. The total heat transfer can also be expressed as

$$q = hA(T_w - T_b)_{av} \qquad \textbf{[6-3]}$$

Where, **A** is the total surface area for heat transfer. Because both T_w and T_b can vary along the length of the tube, a suitable averaging process must be adopted for use with Equation (6-3).
A traditional expression for calculation of heat transfer in fully developed turbulent flow in smooth tubes is that recommended by Dittus and Boelter

$$\mathrm{Nu}_d = 0.023\,\mathrm{Re}_d^{0.8}\mathrm{Pr}^n \qquad \textbf{[6-4}\boldsymbol{a}\textbf{]}$$

The properties in this equation are evaluated at the average fluid bulk temperature, and the exponent **n** has the following values:

$$n = \begin{cases} 0.4 & \text{for heating of the fluid} \\ 0.3 & \text{for cooling of the fluid} \end{cases}$$

We wish to generalize the results of these experiments by arriving at one empirical equation that represents all the data. As described above, we may anticipate that the heat-transfer data will be dependent on the Reynolds and Prandtl numbers. A power function for each of these parameters is a simple type of relation to use, so we assume

$$\mathrm{Nu}_d = C\mathrm{Re}_d{}^m\mathrm{Pr}^n$$

Where, **C, m,** and **n** are constants to be determined from the experimental data.

To take into account the property variations, Sieder and Tate [2] recommend the following relation:

$$\mathrm{Nu}_d = 0.027\,\mathrm{Re}_d^{0.8}\mathrm{Pr}^{1/3}\left(\frac{\mu}{\mu_w}\right)^{0.14} \qquad \textbf{[6-5]}$$

All properties are evaluated at bulk-temperature conditions, except μ_w, which is evaluated at the wall temperature.
In the entrance region the flow is not developed, and Nusselt [3] recommended the following equation:

$$\mathrm{Nu}_d = 0.036\,\mathrm{Re}_d^{0.8}\mathrm{Pr}^{1/3}\left(\frac{d}{L}\right)^{0.055} \qquad \text{for } 10 < \frac{L}{d} < 400 \qquad \textbf{[6-6]}$$

Where, **L** is the length of the tube and **d** is the tube diameter. The properties in Equation (6-6) are evaluated at the mean bulk temperature.
If the channel through which the fluid flows is not of circular cross section, it is recommended that the heat-transfer correlations be based on the hydraulic diameter **D_H**, defined by

$$D_H = \frac{4A}{P} \qquad \textbf{[6-14]}$$

Where, **A** is the cross-sectional area of the flow and **P** is the wetted perimeter. This particular grouping of terms is used because it yields the value of the physical diameter when applied to a circular cross section. The hydraulic diameter should be used in calculating the Nusselt and Reynolds numbers.

Turbulent Heat Transfer in a Tube — EXAMPLE 6-1

Air at 2 atm and 200°C is heated as it flows through a tube with a diameter of 1 in (2.54 cm) at a velocity of 10 m/s. Calculate the heat transfer per unit length of tube if a constant-heat-flux condition is maintained at the wall and the wall temperature is 20°C above the air temperature, all along the length of the tube. How much would the bulk temperature increase over a 3-m length of the tube?

■ Solution

We first calculate the Reynolds number to determine if the flow is laminar or turbulent, and then select the appropriate empirical correlation to calculate the heat transfer. The properties of air at a bulk temperature of 200°C are

$$\rho = \frac{p}{RT} = \frac{(2)(1.0132\times10^5)}{(287)(473)} = 1.493\ \mathrm{kg/m^3} \qquad [0.0932\ \mathrm{lb_m/ft^3}]$$

$$\mathrm{Pr} = 0.681$$

$$\mu = 2.57\times10^{-5}\ \mathrm{kg/m\cdot s} \qquad [0.0622\ \mathrm{lb_m/h\cdot ft}]$$

$$k = 0.0386\ \mathrm{W/m\cdot {}^\circ C} \qquad [0.0223\ \mathrm{Btu/h\cdot ft\cdot {}^\circ F}]$$

$$c_p = 1.025\ \mathrm{kJ/kg\cdot {}^\circ C}$$

$$\mathrm{Re}_d = \frac{\rho u_m d}{\mu} = \frac{(1.493)(10)(0.0254)}{2.57\times10^{-5}} = 14{,}756$$

so that the flow is turbulent. We therefore use Equation (6-4*a*) to calculate the heat-transfer coefficient.

$$\mathrm{Nu}_d = \frac{hd}{k} = 0.023\,\mathrm{Re}_d^{0.8}\,\mathrm{Pr}^{0.4} = (0.023)(14{,}756)^{0.8}(0.681)^{0.4} = 42.67$$

$$h = \frac{k}{d}\mathrm{Nu}_d = \frac{(0.0386)(42.67)}{0.0254} = 64.85\ \mathrm{W/m^2 \cdot {}^\circ C} \qquad [11.42\ \mathrm{Btu/h \cdot ft^2 \cdot {}^\circ F}]$$

The heat flow per unit length is then

$$\frac{q}{L} = h\pi d(T_w - T_b) = (64.85)\pi(0.0254)(20) = 103.5\ \mathrm{W/m} \qquad [107.7\ \mathrm{Btu/ft}]$$

We can now make an energy balance to calculate the increase in bulk temperature in a 3.0-m length of tube:

$$q = \dot{m}c_p\Delta T_b = L\left(\frac{q}{L}\right)$$

We also have

$$\dot{m} = \rho u_m \frac{\pi d^2}{4} = (1.493)(10)\pi\frac{(0.0254)^2}{4}$$
$$= 7.565 \times 10^{-3}\ \mathrm{kg/s} \qquad [0.0167\ \mathrm{lb}_m/\mathrm{s}]$$

so that we insert the numerical values in the energy balance to obtain

$$(7.565 \times 10^{-3})(1025)\Delta T_b = (3.0)(103.5)$$

and

$$\Delta T_b = 40.04^\circ\mathrm{C} \qquad [104.07^\circ\mathrm{F}]$$

EXAMPLE 6-2 Heating of Water in Laminar Tube Flow

Water at 60°C enters a tube of 1-in (2.54-cm) diameter at a mean flow velocity of 2 cm/s. Calculate the exit water temperature if the tube is 3.0 m long and the wall temperature is constant at 80°C.

■ Solution

We first evaluate the Reynolds number at the inlet bulk temperature to determine the flow regime. The properties of water at 60°C are

$$\rho = 985\ \mathrm{kg/m^3} \qquad c_p = 4.18\ \mathrm{kJ/kg \cdot {}^\circ C}$$
$$\mu = 4.71 \times 10^{-4}\ \mathrm{kg/m \cdot s} \qquad [1.139\ \mathrm{lb}_m/\mathrm{h \cdot ft}]$$
$$k = 0.651\ \mathrm{W/m \cdot {}^\circ C} \qquad \mathrm{Pr} = 3.02$$
$$\mathrm{Re}_d = \frac{\rho u_m d}{\mu} = \frac{(985)(0.02)(0.0254)}{4.71 \times 10^{-4}} = 1062$$

so the flow is laminar. Calculating the additional parameter, we have

$$\mathrm{Re}_d\mathrm{Pr}\frac{d}{L} = \frac{(1062)(3.02)(0.0254)}{3} = 27.15 > 10$$

so Equation (6-10) is applicable. We do not yet know the mean bulk temperature to evaluate properties so we first make the calculation on the basis of 60°C, determine an exit bulk temperature, and then make a second iteration to obtain a more precise value. When inlet and outlet conditions are designated with the subscripts 1 and 2, respectively, the energy balance becomes

$$q = h\pi dL\left(T_w - \frac{T_{b_1} + T_{b_2}}{2}\right) = \dot{m}c_p\left(T_{b_2} - T_{b_1}\right) \qquad \textbf{[}\boldsymbol{a}\textbf{]}$$

At the wall temperature of 80°C we have

$$\mu_w = 3.55 \times 10^{-4}\ \mathrm{kg/m\cdot s}$$

From Equation (6-10)

$$\mathrm{Nu}_d = (1.86)\left[\frac{(1062)(3.02)(0.0254)}{3}\right]^{1/3}\left(\frac{4.71}{3.55}\right)^{0.14} = 5.816$$

$$h = \frac{k\mathrm{Nu}_d}{d} = \frac{(0.651)(5.816)}{0.0254} = 149.1\ \mathrm{W/m^2\cdot{}^\circ C} \qquad [26.26\ \mathrm{Btu/h\cdot ft^2\cdot{}^\circ F}]$$

The mass flow rate is

$$\dot{m} = \rho\frac{\pi d^2}{4}u_m = \frac{(985)\pi(0.0254)^2(0.02)}{4} = 9.982 \times 10^{-3}\ \mathrm{kg/s}$$

Inserting the value for h into Equation (a) along with $\dot{m}$ and $T_{b_1} = 60°\mathrm{C}$ and $T_w = 80°\mathrm{C}$ gives

$$(149.1)\pi(0.0254)(3.0)\left(80 - \frac{T_{b_2} + 60}{2}\right) = (9.982 \times 10^{-3})(4180)\left(T_{b_2} - 60\right) \qquad \textbf{[}\boldsymbol{b}\textbf{]}$$

This equation can be solved to give

$$T_{b_2} = 71.98°\mathrm{C}$$

Thus, we should go back and evaluate properties at

$$T_{b,\mathrm{mean}} = \frac{71.98 + 60}{2} = 66°\mathrm{C}$$

We obtain

$$\rho = 982\ \mathrm{kg/m^3} \qquad c_p = 4185\ \mathrm{J/kg\cdot{}^\circ C} \qquad \mu = 4.36 \times 10^{-4}\ \mathrm{kg/m\cdot s}$$

$$k = 0.656\ \mathrm{W/m\cdot{}^\circ C} \qquad \mathrm{Pr} = 2.78$$

$$\mathrm{Re}_d = \frac{(1062)(4.71)}{4.36} = 1147$$

$$\mathrm{Re\,Pr}\frac{d}{L} = \frac{(1147)(2.78)(0.0254)}{3} = 27.00$$

$$\mathrm{Nu}_d = (1.86)(27.00)^{1/3}\left(\frac{4.36}{3.55}\right)^{0.14} = 5.743$$

$$h = \frac{(0.656)(5.743)}{0.0254} = 148.3\ \mathrm{W/m^2\cdot{}^\circ C}$$

We insert this value of h back into Equation (a) to obtain

$$T_{b_2} = 71.88°\text{C} \qquad [161.4°\text{F}]$$

The iteration makes very little difference in this problem. If a large bulk-temperature difference had been encountered, the change in properties could have had a larger effect.

CHAPTER 7 Natural Convection Systems

INTRODUCTION

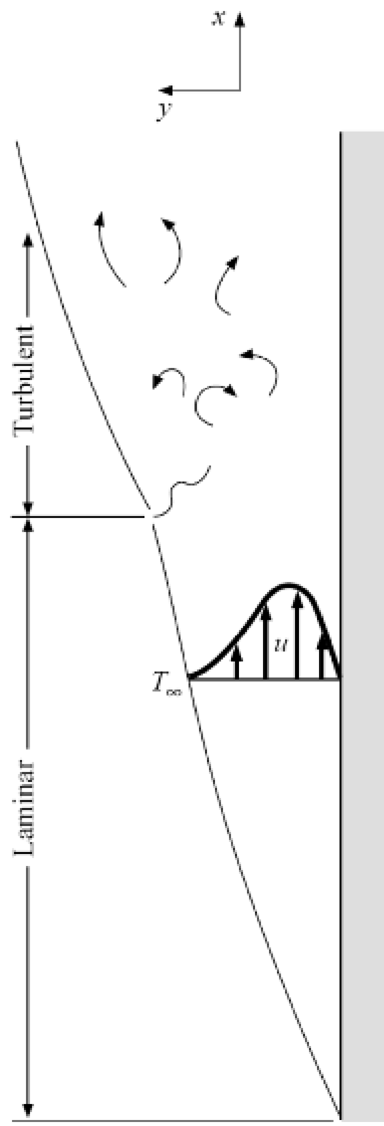
Natural, or free, convection is observed as a result of the motion of the fluid due to density changes arising from the heating process .A hot radiator used for heating a room is one example of a practical device that transfers heat by free convection. The movement of the fluid in free convection, whether it is a gas or a liquid, results from the buoyancy forces imposed on the fluid when its density in the proximity of the heat-transfer surface is decreased as a result of the heating process. The buoyancy forces would not be present if the fluid were not acted upon by some external force field such as gravity, although gravity is not the only type of force field that can produce the free-convection currents; a fluid enclosed in a rotating machine is acted upon by a centrifugal force field, and thus could experience free-convection currents if one or more of the surfaces in contact with the fluid were heated. The buoyancy forces that give rise to the free-convection currents are called *body forces.*

FREE-CONVECTION HEAT TRANSFER ON AVERTICAL FLAT PLATE

Consider the vertical flat plate shown in Figure 7-1. When the plate is heated, a free convection boundary layer is formed, as shown. The velocity profile in this boundary layer is quite unlike the velocity profile in a forced-convection boundary layer. At the wall the velocity is zero because of the no-slip condition; it increases to some maximum value and then decreases to zero at the edge of the boundary layer since the "free-stream" conditions are at rest in the free-convection system. The initial boundary-layer development is laminar; but at some distance from the leading edge, depending on the fluid properties and the temperature difference between wall and environment, turbulent eddies are formed, and transition to a turbulent boundary layer begins. Farther up the plate the boundary layer may become fully turbulent.

To analyze the heat-transfer problem, we must first obtain the differential equation of motion for the boundary layer. For this purpose we choose the x coordinate along the plate and the y coordinate perpendicular to the plate as in the analyses of Chapter 5. The only new force that must be considered in the derivation is the weight of the element of fluid.

Figure 7-1 | Boundary layer on a vertical flat plate.



As before, we equate the sum of the external forces in the **x** direction to the change in momentum flux through the control volume **dx dy**. There results

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}-\rho g+\mu\frac{\partial^2 u}{\partial y^2} \qquad \textbf{[7-1]}$$

Where, the term **−ρg** represents the weight force exerted on the element. The pressure gradient in the **x** direction results from the change in elevation up the plate. Thus

$$\frac{\partial p}{\partial x}=-\rho_\infty g \qquad \textbf{[7-2]}$$

In other words, the change in pressure over a height **dx** is equal to the weight per unit area of the fluid element. Substituting Equation (7-2) into Equation (7-1) gives

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=g(\rho_\infty-\rho)+\mu\frac{\partial^2 u}{\partial y^2} \qquad \textbf{[7-3]}$$

The density difference $\rho_\infty - \rho$ may be expressed in terms of the volume coefficient of expansion **β**, defined by

$$\beta=\frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p=\frac{1}{V_\infty}\frac{V-V_\infty}{T-T_\infty}=\frac{\rho_\infty-\rho}{\rho(T-T_\infty)}$$

So that

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=g\rho\beta(T-T_\infty)+\mu\frac{\partial^2 u}{\partial y^2} \qquad \textbf{[7-4]}$$

This is the equation of motion for the free-convection boundary layer. Notice that the solution for the velocity profile demands knowledge of the temperature distribution. The energy equation for the free-convection system is the same as that for a forced-convection system at low velocity:

$$\rho c_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=k\frac{\partial^2 T}{\partial y^2} \qquad \textbf{[7-5]}$$

The volume coefficient of expansion **β** may be determined from tables of properties for the specific fluid. For ideal gases it may be calculated from

$$\beta = \frac{1}{T}$$

Where, **T** is the absolute temperature of the gas.

EMPIRICAL RELATIONS FOR FREE CONVECTION

Over the years it has been found that average free-convection heat-transfer coefficients can be represented in the following functional form for a variety of circumstances:

$$\overline{\mathrm{Nu}}_f = C(\mathrm{Gr}_f \, \mathrm{Pr}_f)^m \qquad \textbf{[7-25]}$$

The Prandtl number **Pr =ν/α** has been introduced in the above expressions along with a new dimensionless group called the Grashof number **Gr**.
Where, the subscript **f** indicates that the properties in the dimensionless groups are evaluated at the film temperature

$$T_f = \frac{T_\infty + T_w}{2}$$

The product of the Grashof and Prandtl numbers is called the Rayleigh number:

$$\mathrm{Ra} = \mathrm{Gr}\,\mathrm{Pr} \qquad \textbf{[7-26]}$$

The Grashof number may be interpreted physically as a dimensionless group representing the ratio of the buoyancy forces to the viscous forces in the free-convection flow system. It has a role similar to that played by the Reynolds number in forced-convection systems and is the primary variable used as a criterion for transition from laminar to turbulent boundary layer flow. For air in free convection on a vertical flat plate, the critical Grashof number is approximately 4×10^8. Values ranging between 10^8 and 10^9 may be observed for different fluids and environment "turbulence levels."

Characteristic Dimensions

The characteristic dimension to be used in the Nusselt and Grashof numbers depends on the geometry of the problem.

- For a vertical plate it is the height of the plate **L**.
- For a horizontal cylinder it is the diameter **d**.
- For a vertical cylinder it is the height of the cylinder **L**.

For horizontal:

- It is the side of square plate (**a**).
- It is the mean of the two dimensions for a rectangular plate **((a+b) / 2**)
- It is the (**0.9d)** for a circular disk.
- It is the (**A/P**) for unsymmetrical plan forms (where **A** is the area and **P** is the perimeter of the surface).