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المصادر

1. التحليلات العددية والهندسية (المؤلف د. حسن الدلفي ، د. محمود عطائه)
2. Numerical methods for engineering (steven chapra)
3. Numerical analysis (gerald)

(المصادر اعلاه متوفرة في المكتبة)

Laplace Transformation :-

1. Definition

The Laplace transform is one of mathematical tools by solving of ordinary linear differential equations. In comparison with the classical method of solving L.D.E, the Laplace transform method has the following two attractive features :

- (a) The homogeneous equation (C.F) and the particular Integral (P.I) are solved in one operator.
- (b) The Laplace transform converts the diff. eq. into an algebraic equation in s domain. It is then possible to manipulate the algebraic eq. by simple algebraic rules to obtain the solution in the s -domain. The final solution is obtained by taking the inverse Laplace transform.

② Mathematical definition for Laplace transform

Let $f(t)$ be a given function which defined for all $t \geq 0$. We multiply $f(t)$ by e^{-st} and integrate t from zero to infinity. Then, if the resulting integral exists, it is a function of s as $F(s)$:-

$$F(s) = \int_0^{\infty} e^{-st} \cdot f(t) \cdot dt$$

The function $F(s)$ is called the Laplace transform of the original function $f(t)$ and will be denoted by $\mathcal{L}(f(t))$. Thus:

$$F(s) = \mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} \cdot f(t) \cdot dt \quad \text{--- (1)}$$

* the operation just described, which yields $F(s)$ from a given $f(t)$, is called the Laplace transform

* the variable s is referred to as the Laplace operator, it may be real. Later it will be found useful to consider s complex.

Example 1:-

Let $f(t) = 1$ when $t \geq 0$. Find $F(s)$ or $\mathcal{L}(f(t))$

sol.

$$\begin{aligned} \mathcal{L}(f(t)) &= \mathcal{L}(1) = \int_0^{\infty} e^{-st} \cdot 1 \cdot dt = \frac{-1}{s} e^{-st} \Big|_0^{\infty} \quad \text{For } s > 0 \\ &= \left(\frac{-1}{s} e^{-s \cdot \infty} \right) - \left(\frac{-1}{s} e^{-s \cdot 0} \right) = \frac{1}{s} \end{aligned}$$

$$\therefore \boxed{\mathcal{L}(1) = \frac{1}{s}}$$

Example 2: Let $f(t) = e^{at}$ when $t \geq 0$ and $a = \text{const}$. Find $\mathcal{L}(e^{at}) =$

$$\begin{aligned} \text{sol.} : \mathcal{L}(e^{at}) &= \int_0^{\infty} e^{-st} \cdot e^{at} \cdot dt = \int_0^{\infty} e^{(a-s)t} \cdot dt \\ &= \frac{1}{a-s} e^{(a-s)t} \Big|_0^{\infty} = \frac{1}{a-s} \end{aligned}$$

$$\therefore \boxed{\mathcal{L}(e^{at}) = \frac{1}{a-s}} \quad \text{for } a-s > 0$$

Example ③:- Let $f(t) = \sin at$ when $t \geq 0$ and $a = \text{constant}$
Find $\mathcal{L}(\sin at)$

Sol. $\mathcal{L}(\sin at) = \int_0^{\infty} e^{-st} \cdot \sin at \, dt = \left[\lim_{T \rightarrow \infty} \int_0^T e^{-st} \cdot \sin at \, dt \right]$
by parts

let $u = \sin at \quad du = a \cos at$
 $dv = e^{-st} \, dt \quad v = -\frac{1}{s} e^{-st}$

$$\int u \, dv = uv - \int v \, du$$

$$\int e^{-st} \cdot \sin at = -\frac{1}{s} e^{-st} \cdot \sin at + \frac{a}{s} \int e^{-st} \cos at$$

let $u = \cos at$
 $du = -\sin at, dv = e^{-st}, v = -\frac{1}{s} e^{-st}$

$$\Rightarrow \int e^{-st} \cdot \sin at = -\frac{1}{s} e^{-st} \cdot \sin at + \frac{a}{s} \left[-\frac{1}{s} e^{-st} \cos at - \frac{a}{s} \int e^{-st} \cdot \sin at \, dt \right]$$

$$\int e^{-st} \cdot \sin at = -\frac{1}{s} e^{-st} \sin at - \frac{a}{s^2} e^{-st} \cos at - \frac{a^2}{s^2} \int e^{-st} \sin at \, dt$$

Let $I = \int e^{-st} \cdot \sin at \, dt$

$$\left(1 + \frac{a^2}{s^2}\right) I = -\frac{1}{s} e^{-st} \sin at - \frac{a}{s^2} e^{-st} \cos at$$

$$\left(\frac{s^2 + a^2}{s^2}\right) I = -\frac{1}{s} e^{-st} \sin at - \frac{a}{s^2} e^{-st} \cos at$$

$$\Rightarrow I = \frac{s^2}{s^2 + a^2} \left(-\frac{1}{s} e^{-st} \cdot \sin at \right) - \frac{s^2}{s^2 + a^2} \left(\frac{a}{s^2} e^{-st} \cdot \cos at \right)$$

$$\therefore \int e^{-st} \cdot \sin at \cdot dt = \frac{1}{s^2 + a^2} \left[-e^{-st} (s \sin at + a \cos at) \right]_0^{\infty}$$

$$\therefore \mathcal{L}(\sin at) = \lim_{T \rightarrow \infty} \left[\frac{1}{s^2 + a^2} (-e^{-sT} (s \sin aT + a \cos aT)) \right]_0^T$$

$$= \lim_{T \rightarrow \infty} \left[\frac{-e^{-sT} (s \sin aT + a \cos aT)}{s^2 + a^2} - \frac{-e^{-s(0)} (s \sin a(0) + a \cos a(0))}{s^2 + a^2} \right]$$

$$= \lim_{T \rightarrow \infty} \left[\frac{e^{-sT} (p \sin aT + a \cos aT)}{s^2 + a^2} + \frac{a}{s^2 + a^2} \right]$$

$$\therefore \boxed{\mathcal{L}(\sin at) = \frac{a}{s^2 + a^2}}$$

Example A: Let $f(t) = t$ when $t \geq 0$ find $\mathcal{L}(t)$

Sol: $\mathcal{L}(t) = \int_0^{\infty} e^{-st} \cdot t \cdot dt = \lim_{T \rightarrow \infty} \int_0^T t \cdot e^{-st} \cdot dt$

Let $u = t$
 $dv = e^{-st}$, $du = dt$, $v = \frac{-1}{s} e^{-st}$

$$\begin{aligned} \therefore \mathcal{L}(t) &= \lim_{T \rightarrow \infty} \left[-\frac{t}{s} e^{-st} + \frac{1}{s} \int e^{-st} \cdot dt \right]_0^T \\ &= \lim_{T \rightarrow \infty} \left[-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \right]_0^T \\ &= \lim_{T \rightarrow \infty} \left[-\frac{T}{s} e^{-sT} - \frac{1}{s^2} e^{-sT} + \frac{0}{s} e^{-s \cdot 0} + \frac{1}{s^2} e^{-s \cdot 0} \right] \end{aligned}$$

$$\Rightarrow \boxed{\mathcal{L}(t) = \frac{1}{s^2}}$$

Example B: Let $f(t) = \sinh at$ where $a = \text{constant}$ and $t \geq 0$ find $\mathcal{L}(\sinh at)$

Sol: $\mathcal{L}(\sinh at) = \mathcal{L}\left[\frac{e^{at} - e^{-at}}{2}\right] = \lim_{T \rightarrow \infty} \int_0^T e^{-st} \left[\frac{e^{at} - e^{-at}}{2}\right] dt$

$$= \frac{1}{2} \lim_{T \rightarrow \infty} \int_0^T e^{-st} \cdot e^{at} dt - \frac{1}{2} \lim_{T \rightarrow \infty} \int_0^T e^{-st} \cdot e^{-at} dt$$

equal $\int e^{at}$
equal $\int e^{-at}$

$$= \frac{1}{2} \left[\mathcal{L}(e^{at}) - \mathcal{L}(e^{-at}) \right]$$

$$= \frac{1}{2} \left[\frac{1}{s-a} - \frac{1}{s+a} \right] = \frac{1}{2} \left[\frac{s+a - s+a}{s^2 - a^2} \right] = \frac{1}{2} \cdot \frac{2a}{s^2 + a^2}$$

$$\boxed{\mathcal{L} \sinh at = \frac{a}{s^2 + a^2}}$$

A short list of some important elementary functions and their Laplace transforms is given in the following table:-

No.	$f(t)$	$\mathcal{L}(f(t))$	No.	$f(t)$	$\mathcal{L}(f(t))$
1.	1	$1/s$	6.	$\cos at$	$s/(s^2+a^2)$
2.	t	$1/s^2$	7.	$\sin at$	$a/(s^2+a^2)$
3.	t^2	$2!/s^3$	8.	$\sinh at$	$a/(s^2-a^2)$
4.	t^n	$n!/s^{(n+1)}$	9.	$\cosh at$	$s/(s^2-a^2)$
5.	e^{at}	$1/(s-a)$			

H.W ① prove that

$$a. \mathcal{L}(\cos at) = \frac{s}{s^2+a^2} \quad b. \mathcal{L}(\cosh at) = \frac{s}{s^2-a^2}$$

② Find \mathcal{L} of a. e^{-3t} b. t^2

Some important properties of Laplace transforms:

① Linearity property:-

If C_1 and C_2 are constants while $f_1(t)$ and $f_2(t)$ are functions with Laplace transformation $F_1(s)$ and $F_2(s)$, Then

$$\mathcal{L}[C_1 f_1(t) + C_2 f_2(t)] = C_1 \mathcal{L}(f_1(t)) + C_2 \mathcal{L}(f_2(t)) + \dots \\ = C_1 F_1(s) + C_2 F_2(s)$$

Example:

$$\textcircled{1} \mathcal{L}[4t^2 - 3\cos 2t + 5e^{-t}] = 4\mathcal{L}t^2 - 3\mathcal{L}\cos 2t + 5\mathcal{L}e^{-t} \\ = \frac{8}{s^3} + \frac{3s}{s^2+4} + \frac{5}{s+1}$$

$$\textcircled{2} \mathcal{L}[4e^{-5t} + 6t^3 - 3\sin 4t + 2\cos 2t] \\ = 4\mathcal{L}e^{-5t} + 6\mathcal{L}t^3 - 3\mathcal{L}\sin 4t + 2\mathcal{L}\cos 2t \\ = \frac{4}{s+5} + 6 \cdot \frac{3!}{s^4} - 3 \cdot \frac{4}{s^2+16} + 2 \cdot \frac{s}{s^2+4}$$

③

② First shifting property :-

IF $\mathcal{L}\{f(t)\} = F(s)$, Then

$$\boxed{\mathcal{L}\{e^{at} \cdot f(t)\} = F(s-a)}$$

Example:-

① $\mathcal{L}\{e^{-t} \cdot \cos 2t\} \Rightarrow \mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}, \mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$

$$\therefore \mathcal{L}\{e^{-t} \cdot \cos 2t\} = \frac{s+1}{(s+1)^2+4}$$

② $\mathcal{L}\{e^{3t} \cdot t^2\} \Rightarrow \mathcal{L}\{t^2\} = \frac{2!}{s^3}, \mathcal{L}\{e^{3t}\} = \frac{1}{s-3}$

$$\therefore \mathcal{L}\{e^{3t} \cdot t^2\} = \frac{2}{(s-3)^3}$$

③ Second shifting property :-

IF $\mathcal{L}\{f(t)\} = F(s)$ and $g(t) = \begin{cases} f(t-a) & t > a \\ 0 & t \leq a \end{cases}$

Then

$$\mathcal{L}\{g(t)\} = e^{-as} \cdot F(s)$$

or $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$

Example:-

① $g(t) = \begin{cases} (t-2)^3 & t > 2 \\ 0 & t \leq 2 \end{cases} \Rightarrow \mathcal{L}\{g(t)\} = G(s) = e^{-2s} \left(\frac{3!}{s^4} \right)$

④ Change of scale property :-

IF $\mathcal{L}\{f(at)\} = F(s)$, then

$$\boxed{\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)}$$

Examples:-

① $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$, then

$$\mathcal{L}\left\{\sin \frac{t}{3}\right\} = \frac{1}{3} \left(\frac{1}{\left(\frac{s}{3}\right)^2+1} \right) = \frac{1}{3} \left(\frac{1}{\frac{s^2+9}{9}} \right) = \frac{1}{3} \cdot \frac{9}{s^2+9} = \frac{3}{s^2+9} = \frac{3}{s^2+3^2}$$

② find $\mathcal{L}\{3t\}$ by change of scale property

$$\mathcal{L}\{3t\} = \frac{1}{3} \left[\frac{1}{\left(\frac{s}{3}\right)^2} \right] = \frac{1}{3} \left(\frac{9}{s^2} \right) = 3 \cdot \frac{1}{s^2} = 3 \cdot \mathcal{L}\{t\}$$

⑤ Laplace transformation of derivatives :-

If $\mathcal{L} f(t) = F(s)$, then $\mathcal{L} f'(t) = s \cdot F(s) - f(0)$
and $\mathcal{L} f''(t) = s^2 \cdot F(s) - s f(0) - f'(0)$. therefore In general

$$\mathcal{L} f^{(n)}(t) = s^n \cdot F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - s f^{(n-1)}(0) - f^{(n-1)}(0)$$

Example:-

⑥ Find $\mathcal{L} \sin t$ by derivative property :-
we have

$$\frac{d}{dt} \underbrace{\cos t}_{f(t)} = \sin t = f'(t)$$

$$\begin{aligned} \therefore \mathcal{L} f'(t) &= s \cdot F(s) - f(0) \\ &= s \cdot \frac{s}{s^2+1} - 1 = \frac{s^2 - s^2 + 1}{s^2+1} = \frac{1}{s^2+1} = \mathcal{L} \sin t \end{aligned}$$

H.W Find $\mathcal{L} \cos t$ by derivative property
Note that :- $-\cos t = \frac{d}{dt} \sin t = f'(t)$

⑦ Find $\mathcal{L} e^{at}$ by derivative property

Sol. $e^{at} = \frac{1}{a} \frac{d}{dt} e^{at} = \frac{1}{a} f'(t)$

$$\begin{aligned} \Rightarrow \mathcal{L} \frac{1}{a} f'(t) &= \frac{1}{a} \mathcal{L} f'(t) = \frac{1}{a} [s \cdot F(s) - f(0)] \\ &= \frac{1}{a} \left[s \cdot \frac{1}{s-a} - 1 \right] = \frac{1}{a} \left[\frac{s - s + a}{s-a} \right] = \frac{1}{a} \cdot \frac{a}{s-a} = \frac{1}{s-a} \end{aligned}$$

⑧ prove that $\sin at = \frac{a}{s^2+a^2}$

Sol.

$$\sin at = \frac{-1}{a} \frac{d}{dt} \underbrace{\cos at}_{f(t)} = \frac{-1}{a} f'(t)$$

$$\begin{aligned} \Rightarrow \mathcal{L} f'(t) &= \frac{-1}{a} \mathcal{L} \cos at = \frac{-1}{a} [s \cdot F(s) - f(0)] = \frac{-1}{a} \left[s \cdot \frac{s}{s^2+a^2} - 1 \right] \\ &= \frac{-1}{a} \left[\frac{s^2 - s^2 - a^2}{s^2+a^2} \right] = \frac{-1}{a} \left(\frac{-a^2}{s^2+a^2} \right) \\ &= \frac{a}{s^2+a^2} \end{aligned}$$

⑥ Laplace transformation by integral:-

IF $\mathcal{L} f(t) = F(s)$, then $\mathcal{L} \left[\int_0^t f(u) \cdot du \right] = \frac{1}{s} \cdot F(s)$

Examples:-

① $\mathcal{L} \int_0^t \frac{\sin zu}{f(u)} du = \frac{F(s)}{s}$

$\therefore \mathcal{L} \sin 2t = \frac{2}{s^2 + 4} \Rightarrow \mathcal{L} \int_0^t \sin zu du = \frac{1}{s} \cdot \frac{2}{s^2 + 4}$

② Find $\mathcal{L} \int_0^t (u^2 - u + e^{-u}) du$

Sol.

where $\mathcal{L} t^2 = \frac{2!}{s^3} = \frac{2}{s^3}$, $\mathcal{L} t = \frac{1}{s^2}$, $\mathcal{L} e^{-t} = \frac{1}{s+1}$

$\therefore \mathcal{L} \int_0^t (u^2 - u + e^{-u}) du = \frac{1}{s} \left[\frac{2}{s^3} - \frac{1}{s^2} + \frac{1}{s+1} \right] = \frac{2}{s^4} - \frac{1}{s^3} + \frac{1}{s(s+1)}$

⑦ Multiplication by t^n :-

IF $\mathcal{L} f(t) = F(s)$, then $\mathcal{L} [t^n \cdot f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$

Examples:-

Find L.T for the following functions:

① $t \cdot e^{2t}$

② $t^2 \cdot \cos at$

Sol.

① $\mathcal{L} t \cdot e^{2t} \Rightarrow \mathcal{L} e^{2t} = \frac{1}{s-2}$, we have

$\mathcal{L} [t \cdot e^{2t}] = -1 \cdot \frac{d}{ds} \left(\frac{1}{s-2} \right) = \frac{-1}{(s-2)^2} = \frac{1}{(s-2)^2}$

② $\mathcal{L} t^2 \cdot \cos at \Rightarrow \mathcal{L} \cos at = \frac{s}{s^2 + a^2}$, we have

$\mathcal{L} [t^2 \cdot \cos at] = -1^2 \cdot \frac{d^2}{ds^2} \left(\frac{s}{s^2 + a^2} \right)$

$= \frac{d}{ds} \left[\frac{s^2 + a^2 - 2s^2}{(s^2 + a^2)^2} \right] = \frac{d}{ds} \left[\frac{a^2 - s^2}{(s^2 + a^2)^2} \right]$
 $= \frac{-2s(s^2 + a^2)^2 - (a^2 - s^2) \cdot 4s(s^2 + a^2)}{(s^2 + a^2)^4} = \frac{-2s[3a^2 - s^2]}{(s^2 + a^2)^3}$

⑧ Division by t :-

If $\mathcal{L} f(t) = F(s)$, then $\mathcal{L} \left[\frac{f(t)}{t} \right] = \int_s^\infty F(u) \cdot du$

ex. ② Find $\mathcal{L} \frac{\sin t}{t}$, sol. since $\mathcal{L} \sin t = \frac{1}{s^2+1}$

$$\therefore \mathcal{L} \frac{\sin t}{t} = \int_s^\infty \frac{1}{u^2+1} du = \tan^{-1} u \Big|_s^\infty = \tan^{-1} \infty - \tan^{-1} s = \frac{\pi}{2} - \tan^{-1} s$$

③ show that $\mathcal{L} \left[\frac{e^{-at} - e^{-bt}}{t} \right] = \ln \frac{s+b}{s+a}$

Sol. by linearity property $\Rightarrow \mathcal{L} \frac{e^{-at}}{t} - \mathcal{L} \frac{e^{-bt}}{t}$

$$= \int_s^\infty \frac{1}{u+a} du - \int_s^\infty \frac{1}{u+b} du = \ln(u+a) \Big|_s^\infty - \ln(u+b) \Big|_s^\infty$$

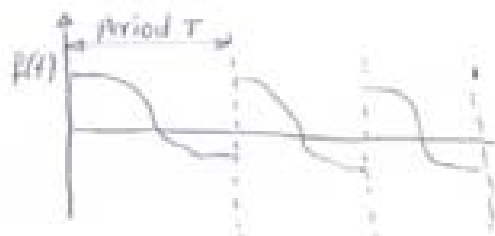
$$= \ln(\infty+a) - \ln(s+a) - \ln(\infty+b) + \ln(s+b) =$$

$$= \ln(s+b) - \ln(s+a) = \ln \frac{s+b}{s+a}$$

⑨ Periodic Function :-

Let $f(t)$ have period $T > 0$, so that $f(t+T) = f(t)$, then

$$\mathcal{L} f(t) = \frac{\int_0^T e^{-st} \cdot f(t) \cdot dt}{1 - e^{-sT}}$$



ex. Graph the function

$$f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases} \quad \text{then find } \mathcal{L} f(t)$$



$$\begin{aligned}
 \mathcal{L} f(t) &= \frac{1}{1 - e^{-2\pi p}} \left[\int_0^{\pi} \sin t \cdot e^{-pt} \cdot dt + \int_{\pi}^{2\pi} 0 \cdot e^{-pt} \cdot dt \right] \\
 &= \frac{1}{1 - e^{-2\pi p}} \int_0^{\pi} \sin t \cdot e^{-pt} \cdot dt \quad \Rightarrow \text{by u dv integral} \\
 &= \frac{1}{1 - e^{-2\pi p}} \left[\frac{e^{-pt} (-p \sin t - \cos t)}{p^2 + 1} \right]_0^{\pi} \\
 &= \frac{1}{1 - e^{-2\pi p}} \left[\frac{e^{-p\pi} (0+1)}{p^2 + 1} - \frac{e^{-p \cdot 0} (0-1)}{p^2 + 1} \right] \\
 &= \frac{1}{1 - e^{-2\pi p}} \left[\frac{1 + e^{-p\pi}}{p^2 + 1} \right] = \frac{1}{(1 - e^{-2\pi p})(1 + e^{-p\pi})} \cdot \frac{1 + e^{-p\pi}}{p^2 + 1}
 \end{aligned}$$

$$\Rightarrow \mathcal{L} f(t) = \frac{1}{(1 - e^{-2\pi p})(p^2 + 1)}$$

H.W

(a) Graph the function

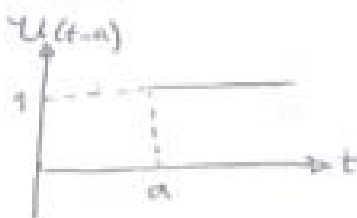
$$f(t) = \begin{cases} \cos t & 0 \leq t < \pi \\ 0 & \pi \leq t < 2\pi \end{cases}$$

(b) Find $\mathcal{L} f(t)$

(i) The unit step function:-

The unit step function, also called Heaviside's unit step function, is defined as:

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$$



$$\therefore \mathcal{L} u(t-a) = \frac{e^{-as}}{s} \quad s > 0$$

Examples:- (i) Prove that $\mathcal{L} u(t-a) = \frac{e^{-as}}{s}$ if $s > 0$.

Sol.

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases} \text{ so that}$$

$$\begin{aligned} \mathcal{L} u(t-a) &= \int_0^a e^{-st} \cdot (0) \cdot dt + \int_a^\infty e^{-st} \cdot (1) \cdot dt \\ &= 0 + \left. \frac{e^{-st}}{-s} \right|_a^\infty = \frac{e^{-as}}{s} \quad \text{if } s > 0 \end{aligned}$$

(ii) Express the function $f(t) = \begin{cases} 8 & t < 2 \\ 6 & t > 2 \end{cases}$ in terms of unit step function and thus obtain its Laplace transform.

Sol.

$$f(t) = 8 + \begin{cases} 0 & t < 2 \\ -2 & t > 2 \end{cases} = 8 - 2 \begin{cases} 0 & t < 2 \\ 1 & t > 2 \end{cases}$$

$$f(t) = 8 - 2u(t-2)$$

$$\therefore \mathcal{L} f(t) = \mathcal{L} [8 - 2u(t-2)]$$

$$= \frac{8}{s} - 2 \cdot \frac{e^{-2s}}{s} = \frac{8 - 2e^{-2s}}{s}$$

H.W Find

① $\mathcal{L} [2\mathcal{U}(t-1) + 3\mathcal{U}(t-2)]$

② $\mathcal{L} [t \cdot \mathcal{U}(t-3)]$

③ Express the function

$$f(t) = \begin{cases} \cos t & t > \pi/2 \\ 3 & t < \pi/2 \end{cases} \quad \text{in terms of the}$$

unit step function and find its Laplace transformation

Example :- Find $\int^{-1} \frac{s+1}{s^2+3s+2}$

Sol.

$$\frac{s+1}{s^2-3s+2} = \frac{s+1}{(s-1)(s-2)} = \frac{A}{(s-1)} + \frac{B}{(s-2)} = \frac{A(s-2) + B(s-1)}{(s-1)(s-2)}$$

$$\Rightarrow s+1 = A(s-2) + B(s-1) \quad \text{--- *}$$

Then $1 = A+B$ --- ①
 $1 = -2A+B$ --- ② \Rightarrow We can be solved, then $A = -2$, $B = 3$

$$\Rightarrow \int^{-1} \frac{s+1}{(s-1)(s-2)} = \int^{-1} \frac{-2}{s-1} + \int^{-1} \frac{3}{s-2}$$

$$= -2 \frac{e^t}{e} + 3 \frac{e^t}{e}$$

* Factors $(as+b)^2$ give P.F. $\frac{A}{as+b} + \frac{B}{(as+b)^2}$

ex. Find $\int^{-1} \frac{s^2}{(s+1)(s-1)^2}$

Sol.

$$\frac{s^2}{(s+1)(s-1)^2} = \frac{A}{(s+1)} + \frac{B}{(s-1)} + \frac{C}{(s-1)^2}$$

$$= \frac{A(s-1)^2 + B(s-1)(s+1) + C(s+1)}{(s+1)(s-1)^2}$$

$$\therefore s^2 = A(s-1)^2 + B(s-1)(s+1) + C(s+1)$$

$$s^2 = A(s^2 - 2s + 1) + B(s^2 - 1) + C(s+1)$$

$$\therefore \text{For } s^2 \Rightarrow 1 = A + B \Rightarrow B = 1 - A \quad \text{--- ①}$$

$$s \Rightarrow 0 = -2A + C \Rightarrow C = 2A \quad \text{--- ②}$$

$$\text{At } s = -1 \Rightarrow 0 = A - B + C \Rightarrow 0 = A - 1 + A + 2A \Rightarrow A = \frac{1}{4}$$

And $C = \frac{1}{2}$, $B = \frac{3}{4}$

$$\therefore \int^{-1} \frac{s^2}{(s+1)(s-1)^2} = \int^{-1} \frac{1}{4} \cdot \frac{1}{s+1} + \int^{-1} \frac{3}{4} \cdot \frac{1}{s-1} + \int^{-1} \frac{1}{2} \cdot \frac{1}{(s-1)^2}$$

$$= \frac{1}{4} \frac{e^t}{e} + \frac{3}{4} \frac{e^t}{e} + \frac{1}{2} t \cdot \frac{e^t}{e}$$

* Factors $(ax+b)^3$ gives P.F. $\frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3}$

ex. Find $\mathcal{L}^{-1} \frac{s^2+1}{(s+1)^3}$

$$\frac{s^2+1}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

$$\therefore s^2+1 = A(s+1)^2 + B(s+1) + C$$

Let $s+1=0 \Rightarrow s=-1$

$$5 = A(0) + B(0) + C \Rightarrow C=5$$

from

$$s^2: 1 = A$$

$$s: \Rightarrow C=5$$

$$\text{مقابلتي: } 1 = A + 2B + C \Rightarrow B = -4$$

$$\begin{aligned} \therefore \mathcal{L}^{-1} \frac{s^2+1}{(s+1)^3} &= \mathcal{L}^{-1} \frac{1}{s+1} + \mathcal{L}^{-1} \frac{-4}{(s+1)^2} + \mathcal{L}^{-1} \frac{5}{(s+1)^3} \\ &= e^{-st} - 4e^{-st} \cdot t + \frac{5}{2} e^{-st} \cdot t^2 \end{aligned}$$

$$\text{In general } \frac{P(s)}{(Q(s))^n} = \sum_{i=1}^n \frac{A_i}{(as+b)^i}$$

* A quadratic factor (as^2+bs+c) gives P.F. $\frac{As^2+B}{as^2+bs+c}$

ex. Find $\mathcal{L}^{-1} \frac{s^3}{(s-2)(s^2+1)}$

$$\frac{s^3}{(s-2)(s^2+1)} = \frac{A}{(s-2)} + \frac{Bs+C}{(s^2+1)} = \frac{A(s^2+1) + (Bs+C)(s-2)}{(s-2)(s^2+1)}$$

$$\therefore s^3 = A(s^2+1) + (Bs+C)(s-2)$$

$$\text{let } s-2=0 \Rightarrow s=2$$

$$\text{Then } 4 = 5A + (Bs+C) \cdot 0 \Rightarrow A = 4/5$$

by equating coeff.

$$s^2 : 1 = A + B \Rightarrow B = 1/5$$

$$\text{C.T.} : 0 = A - 2C \Rightarrow C = 2/5$$

$$\begin{aligned} \therefore \int^{-1} \frac{s^2}{(s-2)(s^2+1)} &= \int^{-1} \frac{4/5}{s-2} + \int^{-1} \frac{\frac{1}{5}s + \frac{2}{5}}{s^2+1} \\ &= \frac{4}{5} \int^{-1} \frac{1}{s-2} + \int^{-1} \frac{1}{5} \frac{s}{s^2+1} + \int^{-1} \frac{2}{5} \frac{1}{s^2+1} \\ &= \frac{4}{5} e^{2t} + \frac{1}{5} \cos t + \frac{2}{5} \sin t \end{aligned}$$

* Factors of quadratic function $(as^2+bs+c)^2$ has P.F

$$\frac{As+B}{as^2+bs+c} + \frac{Cs+D}{(as^2+bs+c)^2} \quad \text{In general}$$

$$\frac{P(s)}{(Q(s))^n} = \sum_{i=1}^n \frac{(As+B)_i}{(as^2+bs+c)^i} \quad \text{where } Q(s) \text{ is a quadratic function}$$

ex. find $\int^{-1} \frac{s^4+2s+4}{(s-1)(s^2+1)^2}$

sol.

$$\begin{aligned} \frac{s^4+2s+4}{(s-1)(s^2+1)^2} &= \frac{A}{s-1} + \frac{Bs+C}{s^2+1} + \frac{Ds+E}{(s^2+1)^2} \\ &= \frac{A(s^2+1)^2 + (Bs+C)(s-1)(s^2+1) + (Ds+E)(s-1)}{(s-1)(s^2+1)^2} \end{aligned}$$

$$\therefore s^4+2s+4 = A(s^2+1)^2 + (Bs+C)(s-1)(s^2+1) + (Ds+E)(s-1)$$

$$\text{let } s-1=0 \Rightarrow s=1 \quad \text{then}$$

$$7 = 4A + (Bs+C)(0) + (Ds+E)(0) \Rightarrow A = 7/4$$

by equating coeff.

$$s^4 : 1 = A + B \Rightarrow B = -3/4$$

$$s^2 : 0 = 2A + B + C + D \Rightarrow D = -7/2$$

$$s^3 : 0 = C - B \Rightarrow C = -3/4$$

$$\text{C.T.} : 4 = A - C - E \Rightarrow E = -3/2$$

$$\therefore \mathcal{L}^{-1} \frac{s^4 + 2s + 4}{(s-1)(s^2+1)^2} = \frac{7}{4} \mathcal{L}^{-1} \frac{1}{s-1} - \frac{3}{4} \mathcal{L}^{-1} \frac{s+1}{s^2+1} - \frac{1}{2} \mathcal{L}^{-1} \frac{7s+3}{(s^2+1)^2}$$

where $\mathcal{L}^{-1} \frac{1}{s-1} = e^t$

$$\mathcal{L}^{-1} \frac{s+1}{s^2+1} = \mathcal{L}^{-1} \frac{s}{s^2+1} + \mathcal{L}^{-1} \frac{1}{s^2+1} = \cos t + \sin t$$

$$\begin{aligned} \mathcal{L}^{-1} \frac{7s+3}{(s^2+1)^2} &= \mathcal{L}^{-1} \frac{7s}{(s^2+1)^2} + \mathcal{L}^{-1} \frac{3}{(s^2+1)^2} = \frac{7}{2} \mathcal{L}^{-1} \frac{2s}{(s^2+1)^2} + \frac{3}{2} \mathcal{L}^{-1} \frac{1}{(s^2+1)^2} \\ &= \frac{7}{2} \cdot t \cdot \cos t + \frac{3}{2} t \cdot \sin t \end{aligned}$$

Then

$$\mathcal{L}^{-1} \frac{s^4 + 2s + 4}{(s-1)(s^2+1)^2} = \frac{7}{2} e^t - \frac{3}{4} (\cos t + \sin t) - \frac{1}{2} \left[\frac{7}{2} t \cdot \cos t + \frac{3}{2} t \sin t \right]$$

⑥ Long division

IF degree of $P(s)$ equal or more than that of $Q(s)$, can be written as the sum of rational functions by long division

ex. Find $\mathcal{L}^{-1} \frac{s^2 + s + 1}{s^2 - 5s + 6} \Rightarrow \begin{matrix} \text{degree of} \\ P(s) \end{matrix} = \begin{matrix} \text{degree of} \\ Q(s) \end{matrix}$

sol. by long division

$$\therefore \frac{s^2 + s + 1}{s^2 - 5s + 6} = 1 + \frac{6s - 5}{s^2 - 5s + 6}$$

$$\begin{array}{r} s^2 - 5s + 6 \overline{) s^2 + s + 1} \\ \underline{+ s^2 - 5s + 6} \\ 6s - 5 \end{array}$$

where $\frac{6s-5}{s^2-5s+6} = \frac{6s-5}{(s-2)(s-3)} = \frac{A}{s-2} + \frac{B}{s-3}$ by partial fraction

$$\therefore 6s - 5 = A(s-3) + B(s-2)$$

let $s-2=0 \Rightarrow s=2$, then $\Rightarrow 7 = -A + B(0) \Rightarrow A = -7$

let $s-3=0 \Rightarrow s=3$, then $\Rightarrow 13 = A(0) + B \Rightarrow B = 13$

$$\therefore \mathcal{L}^{-1} \frac{6s-5}{s^2-5s+6} = \mathcal{L}^{-1} \frac{7}{s-2} + \mathcal{L}^{-1} \frac{13}{s-3}$$

⑦

Then,

$$= -7 e^{2t} + 13 e^{3t}$$

$$\therefore \mathcal{L}^{-1} \frac{s^2 + s + 1}{s^2 - 5s + 6} = \mathcal{L}^{-1} \left[1 + \frac{6s - 5}{s^2 - 5s + 6} \right]$$

$$= 0 - 7 e^{2t} + 13 e^{3t}$$

H.W. Find $\mathcal{L}^{-1} \frac{2s^4 - 5s^3 + 6s^2 - 5s + 3}{s^3 - 3s^2 + 3s - 1}$

Hint, @ by long division

$$\mathcal{L}^{-1} \left[\frac{2s^4 - 5s^3 + 6s^2 - 5s + 3}{s^3 - 3s^2 + 3s - 1} \right] = \mathcal{L}^{-1} \left[2s + 1 + \frac{3s^2 + 4}{(s-1)^3} \right]$$

$$② \mathcal{L}^{-1}(2s-1) = 0$$

② Application of Laplace transformation :-

The Laplace transform is useful in the following applications:

- (i) solution of ordinary differential equations with const. coeff.
- (ii) solution of ordinary differential equations with variable coeff.

The L.T is useful in solving linear-ordinary differential equations with constant coefficients by the following steps:

- ① Taking the Laplace transform of both sides of O.D.E.
- ② Using initial or boundary conditions to obtain an algebraic equation for determination of: $\mathcal{L} f(t) = F(s)$
- ③ Find the inverse Laplace transform of $F(s)$ to obtain the required solution $f(t)$.

ex-① solve $y'' + y = t$, $y(0) = 1$, $y'(0) = -2$

s.l. $\mathcal{L} y'' + \mathcal{L} y = \mathcal{L} t$ where $\mathcal{L} y'' = s^2 y(s) - s y(0) - y'(0)$
 $= s^2 y(s) - s \cdot 1 + 2$

$$\Rightarrow s^2 y(s) - s + 2 + y(s) = \frac{1}{s^2}$$

$$\Rightarrow y(s) [s^2 + 1] = \frac{1}{s^2} + s - 2 \Rightarrow y(s) (s^2 + 1) = \frac{-2s^2 + s^3 + 1}{s^2}$$

$$\Rightarrow y(s) = \frac{-2s^2 + s^3 + 1}{s^2(s^2 + 1)} = \frac{1 + s^2(s - 2)}{s^2(s^2 + 1)} = \frac{1}{s^2(s^2 + 1)} + \frac{s^2(s - 2)}{s^2(s^2 + 1)}$$

$$\Rightarrow \boxed{\mathcal{L}^{-1} y(s) = y(t) = \mathcal{L}^{-1} \frac{1}{s^2(s^2 + 1)} + \mathcal{L}^{-1} \frac{s - 2}{s^2 + 1}}$$

$$\text{then } \mathcal{L}^{-1} \frac{1}{s^2(s^2 + 1)} \Rightarrow \mathcal{L}^{-1} \frac{1}{s^2(s^2 + 1)} = \int_0^t \sin u \, du = -\cos u \Big|_0^t = 1 - \cos t$$

$$\text{and } \mathcal{L}^{-1} \frac{1}{s^2(s^2 + 1)} = \int_0^t (1 - \cos u) \, du = u - \sin u \Big|_0^t = t - \sin t$$

$$\mathcal{L}^{-1} \frac{s - 2}{s^2 + 1} = \mathcal{L}^{-1} \frac{s}{s^2 + 1} - \mathcal{L}^{-1} \frac{2}{s^2 + 1} \\ = \cos t - 2 \sin t$$

$$\therefore y(t) = t - \sin t + \cos t - 2 \sin t$$

OR by Partial fraction for

$$\mathcal{L}^{-1} \frac{1}{s^2(s^2+1)} \Rightarrow \frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$$

$$= \frac{As(s^2+1) + B(s^2+1) + (Cs+D)s^2}{s^2(s^2+1)}$$

by equating coeff. $\Rightarrow 1 = As(s^2+1) + B(s^2+1) + (Cs+D)s^2$

$$s^3 : 0 = A + C$$

$$s^2 : 0 = B + D$$

$$s^1 : 0 = C$$

$$C.T : 1 = B \Rightarrow D = -1, C = 0$$

$$\mathcal{L}^{-1} \frac{1}{s^2(s^2+1)} = \mathcal{L}^{-1} \frac{1}{s^2} + \mathcal{L}^{-1} \frac{-1}{s^2+1} = t - \sin t$$

Ex. 2 Solve $\ddot{y} - 3\dot{y} + 2y = 4e^{2t}$, $y(0) = -3$, $\dot{y}(0) = 5$

Sol:

$$\mathcal{L} \ddot{y} - 3\mathcal{L} \dot{y} + 2\mathcal{L} y = 4\mathcal{L} e^{2t} \quad \text{where } \mathcal{L} \ddot{y} = s^2 y(s) - s y(0) - \dot{y}(0)$$

$$\mathcal{L} \dot{y} = s y(s) - y(0)$$

$$\Rightarrow s^2 y(s) - s(-3) - 5 - 3[s y(s) - (-3)] + 2 y(s) = 4 \cdot \frac{1}{s-2}$$

$$\Rightarrow \underbrace{s^2 y(s)} + 3s - 5 - \underbrace{3s y(s)} - 9 + \underbrace{2 y(s)} = \frac{4}{s-2}$$

$$y(s) [s^2 - 3s + 2] = \frac{4}{s-2} - 3s + 4 = \frac{4 - (3s - 14)(s-2)}{(s-2)}$$

$$\Rightarrow y(s) = \frac{4 - 3s^2 + 6s + 14s - 28}{(s-2)(s^2 - 3s + 2)} = \frac{-3s^2 + 20s - 24}{(s-2)^2(s-1)}$$

by partial fraction

$$\frac{-3s^2 + 20s - 24}{(s-2)^2(s-1)} = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-2)^2}$$

$$= \frac{A(s-2)^2 + B(s-1)(s-2) + C(s-1)}{(s-1)(s-2)^2}$$

$$\therefore -3p^2 + 20p - 24 = A(p-2)^2 + B(p-1)(p-2) + C(p-1)$$

$$\text{Let } p-1=0 \Rightarrow p=1$$

$$\text{then } -3(1)^2 + 20(1) - 24 = A + B(0) + C(0) \Rightarrow A = -7$$

$$\text{Let } p-2=0 \Rightarrow p=2$$

$$-3(2)^2 + 20(2) - 24 = A(0) + B(0) + C \Rightarrow C = 4$$

const. term

$$\text{C.T : } -24 = 4A + 2B + C \Rightarrow B = 4$$

$$\Rightarrow \int^{-1} \frac{(-3p^2 + 20p - 24)}{(p-1)(p-2)^2} = \int^{-1} \left[\frac{-7}{(p-1)} + \frac{4}{(p-2)} + \frac{4}{(p-2)^2} \right]$$

$$\begin{aligned} \int^{-1} y(p) &= \int^{-1} \frac{-7}{p-1} + \int^{-1} \frac{4}{p-2} + \int^{-1} \frac{4}{(p-2)^2} \\ \Rightarrow y(t) &= -7e^t + 4e^t + 4 \cdot t \cdot e^t \end{aligned}$$

Ex 6 solve $ty + 2\dot{y} + 5y = e^t \cdot \sin t$; $y(0)=0, \dot{y}(0)=1$

sol.

$$\mathcal{L} \dot{y} + 2 \mathcal{L} \dot{y} + 5 \mathcal{L} y = \mathcal{L} [e^t \sin t]$$

$$\left[p^2 y(p) - p y(0) - \dot{y}(0) \right] + 2 \left[p y(p) - y(0) \right] + 5 y(p) = \frac{1}{(p+1)^2 + 1}$$

$$\underline{p^2 y(p)} - 1 + \underline{2p y(p)} + \underline{5 y(p)} = \frac{1}{(p+1)^2 + 1}$$

$$\Rightarrow y(p) [p^2 + 2p + 5] = \frac{1}{(p+1)^2 + 1} + 1 = \frac{1}{p^2 + 2p + 2} + 1$$

$$\Rightarrow y(p) = \frac{1}{p^2 + 2p + 5} + \frac{1}{(p^2 + 2p + 5)(p^2 + 2p + 2)} = \frac{p^2 + 2p + 2 + 1}{(p^2 + 2p + 5)(p^2 + 2p + 2)}$$

$$\Rightarrow y(p) = \frac{p^2 + 2p + 3}{(p^2 + 2p + 2)(p^2 + 2p + 5)} \quad \text{by partial fraction}$$

$$\frac{p^2 + 2p + 3}{(p^2 + 2p + 2)(p^2 + 2p + 5)} = \frac{Ap + B}{p^2 + 2p + 2} + \frac{Cp + D}{p^2 + 2p + 5}$$

$$\begin{aligned} \therefore p^2 + 2p + 3 &= (Ap+B)(p^2+2p+5) + (Cp+D)(p^2+2p+2) \\ &= p^3(A+C) + p^2(2A+B+2C+D) + p(5A+2B+2C+2D) \\ &\quad + (5B+2D) \end{aligned}$$

Then

$$p^3: 0 = A+C \quad \text{--- (1)}$$

$$p^2: 1 = 2A+B+2C+D \quad \text{--- (2)}$$

$$p: 2 = 5A+2B+2C+2D \quad \text{--- (3)}$$

$$C.T: 3 = 5B+2D \quad \text{--- (4)}$$

$$\text{From eq. (1)} \Rightarrow A = -C \quad \text{--- (5)}$$

$$\text{Sub. (5) in (4)} \Rightarrow B = \frac{1}{5}(3-2D) \quad \text{--- (6)}$$

Sub. (5) & (6) in eq. (2)

$$\Rightarrow -2C + \frac{1}{5}(3-2D) + 2C + D = 1 \Rightarrow D = 2/3$$

From eq. (6) $\Rightarrow B = 1/3$, sub. the values of D and B in eq. (3)

$$2 = 5A + \frac{2}{3} + 2C + 2 \cdot \frac{2}{3} \Rightarrow 5A + 2C = 0 \quad \text{--- (7)}$$

Sub. eq. (7) in eq. (5) $\Rightarrow C = 0, A = 0$

Then

$$y(p) = \frac{1/3}{p^2+2p+2} + \frac{2/3}{p^2+2p+5}$$

$$\mathcal{L}^{-1} y(p) = \mathcal{L}^{-1} \frac{1/3}{p^2+2p+2} + \mathcal{L}^{-1} \frac{2/3}{p^2+2p+5}$$

$$\text{where } \mathcal{L}^{-1} \frac{1/3}{p^2+2p+2} = \frac{1}{3} \mathcal{L}^{-1} \frac{1}{p^2+2p+1+1} = \frac{1}{3} \mathcal{L}^{-1} \frac{1}{(p+1)^2+1} = \frac{1}{3} e^{-t} \sin t$$

$$\mathcal{L}^{-1} \frac{2/3}{p^2+2p+5} = \frac{2}{3} \mathcal{L}^{-1} \frac{1}{p^2+2p+1+4} = \frac{2}{3} \mathcal{L}^{-1} \frac{2/2}{(p+1)^2+2^2} = \frac{1}{3} e^{-t} \sin 2t$$

Then

$$y(t) = \frac{1}{3} e^{-t} [\sin t + \sin 2t]$$

H.W solve . Find y(p) only

$$\textcircled{1} \quad y' + 5y' + 2y = e^{-t} \cdot \sin 2t$$

$$\textcircled{2} \quad y - y = e^t$$

Note: The Laplace transform of t^n by gamma function
 when $n > -1$ and $s > 0$ is :-

$$\Rightarrow \boxed{\mathcal{L} t^n = \frac{\Gamma(n+1)}{s^{n+1}}}; \text{ and } \mathcal{L}^{-1} \frac{1}{s^{n+1}} = \frac{t^n}{\Gamma(n+1)}$$

ex-6 $\mathcal{L} t^{-1/2}$,

$$n = -1/2, n+1 = 1/2, \mathcal{L} t^{-1/2} = \frac{\Gamma(1/2)}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$$

③ $\mathcal{L} t^{1/2}$,

$$n = 1/2, n+1 = 3/2 \Rightarrow \mathcal{L} t^{1/2} = \frac{\Gamma(3/2)}{s^{3/2}} \Rightarrow \Gamma(3/2) = \frac{1}{2} \Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

$$\therefore \mathcal{L} t^{1/2} = \frac{\sqrt{\pi}}{2 s^{3/2}} = \frac{1}{2} \sqrt{\frac{\pi}{s^3}}$$

⑤ $\mathcal{L} t^{3/2}$,

$$n = 3/2, n+1 = 5/2$$

$$\therefore \mathcal{L} t^{3/2} = \frac{\Gamma(5/2)}{s^{5/2}}$$

$$\text{where } \Gamma(5/2) = \frac{3}{2} \Gamma(3/2) = \frac{3}{2} \cdot \frac{\sqrt{\pi}}{2} = \frac{3\sqrt{\pi}}{4}$$

$$\Gamma(3/2) = \frac{1}{2} \Gamma(1/2)$$

$$\mathcal{L} t^{3/2} = \frac{3\sqrt{\pi}}{4 s^{5/2}} = \frac{3}{4} \sqrt{\frac{\pi}{s^5}}$$

④ $\mathcal{L} t^{-3/2}$, $\Rightarrow n = -3/2, n+1 = -1/2 \Rightarrow \mathcal{L} t^{-3/2} = \frac{\Gamma(-1/2)}{s^{-1/2}} = \frac{-2\sqrt{\pi}}{s^{1/2}}$

$$\mathcal{L} t^{-3/2} = -2\sqrt{\pi/s}$$

⑤ $\mathcal{L}^{-1} \frac{1}{s^{3/2}}$, we have $\mathcal{L}^{-1} \frac{1}{s^{n+1}} = \frac{t^n}{\Gamma(n+1)}$, $n+1 = 3/2$

$$\therefore \mathcal{L}^{-1} \frac{1}{s^{3/2}} = \frac{t^{1/2}}{\Gamma(3/2)} = \frac{\sqrt{t}}{\frac{1}{2}\sqrt{\pi}} = \frac{\sqrt{t}}{\frac{1}{2}\sqrt{\pi}} = 2\sqrt{\frac{t}{\pi}}$$

Laplace Transformation (\mathcal{L})			Inverse Laplace Transformation (\mathcal{L}^{-1})		
Name	Case	Example	Name	Case	Example
1. Linearly property	$\mathcal{H}:$ $\mathcal{L}(t) = c_1 f_1(t) + c_2 f_2(t) + \dots$ then: $\mathcal{L} f(t) = c_1 \mathcal{L} f_1(t) + c_2 \mathcal{L} f_2(t) + \dots$	$\mathcal{H}:$ $f(t) = 2t^2 + 5 \sinh 4t$, then: $\mathcal{L} f(t) = \frac{2 \cdot 2!}{s^3} + \frac{5 \cdot 4}{s^2 + 16}$	1. Linearity property	$\mathcal{H}:$ $F(s) = c_1 F_1(s) + c_2 F_2(s) + \dots$, then: $\mathcal{L}^{-1} F(s) = c_1 \mathcal{L}^{-1} F_1(s) + c_2 \mathcal{L}^{-1} F_2(s) + \dots$	$\mathcal{L}^{-1} \left\{ \frac{2}{s} + \frac{7}{s-3} \right\} = 2 + 7e^{3t}$
2. First Shifting property	$\mathcal{H}:$ $\mathcal{L} f(t) = F(s)$ $\mathcal{L} \{e^{at} f(t)\} = F(s-a)$	$\mathcal{H}:$ $f(t) = \cos 4t$, then: $\mathcal{L} \{e^{-t} \cos 4t\} = \frac{s+1}{(s+1)^2 + 16}$	2. First Shifting property	$\mathcal{H}:$ $\mathcal{L}^{-1} F(s) = f(t)$, then: $\mathcal{L}^{-1} F(s-a) = e^{at} f(t)$	$\mathcal{H}:$ $F(s) = \frac{s+1}{(s+1)^2 + 16}$ then: $\mathcal{L}^{-1} F(s) = e^{-t} \cos 4t$
3. Second Shifting property	$\mathcal{H}:$ $\mathcal{L} f(t) = F(s)$, and use Heaviside $g(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$, then: $\mathcal{L} g(t) = e^{-as} F(s)$	$\mathcal{H}:$ $g(t) = \begin{cases} 4(t-2)^2 & t > 2 \\ 0 & t < 2 \end{cases}$, then: $\mathcal{L} g(t) = e^{-2s} \frac{4 \cdot 2!}{s^3}$	3. Second Shifting property	$\mathcal{H}:$ $\mathcal{L}^{-1} F(s) = f(t)$, then: $\mathcal{L}^{-1} e^{-as} F(s) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$	$\mathcal{H}:$ $\mathcal{L}^{-1} e^{-2s} \frac{1}{s-2} = \begin{cases} e^{2(t-2)} & t > 2 \\ 0 & t < 2 \end{cases}$
4. Change of Scale	$\mathcal{H}:$ $\mathcal{L} f(t) = F(s)$, then: $\mathcal{L} f(at) = \frac{1}{a} F\left(\frac{s}{a}\right)$	$\mathcal{H}:$ $f(t) = \cosh 2t$, then: $\mathcal{L} \cosh 2t = \frac{1}{2} \frac{\frac{s}{2}}{\left(\frac{s}{2}\right)^2 - 4} = \frac{1}{4} \frac{\frac{s}{2}}{\frac{s^2}{4} - 4} = \frac{s}{s^2 - 4}$	4. Change of Scale	$\mathcal{H}:$ $\mathcal{L}^{-1} F(s) = f(t)$ $\mathcal{L}^{-1} F(as) = \frac{1}{a} f\left(\frac{t}{a}\right)$	$F(s) = \frac{1}{3s+7}$, $a=3$ then: $\mathcal{L}^{-1} F(s) = \frac{1}{3} e^{-\frac{7}{3}t}$

$$\therefore 6s-5 = A(s-3) + B(s-2)$$

$$\text{Let } s-2=0 \Rightarrow s=2$$

$$7 = -A + B(0) \Rightarrow \boxed{A = -7}$$

$$\text{Let } s-3=0 \Rightarrow s=3$$

$$13 = A(0) + B \Rightarrow \boxed{B = 13}$$

$$\begin{aligned} \therefore \mathcal{L}^{-1} \frac{6s-5}{s^2-5s+6} &= \mathcal{L}^{-1} \frac{-7}{(s-2)} + \mathcal{L}^{-1} \frac{13}{(s-3)} \\ &= -7e^{2t} + 13e^{3t} \end{aligned}$$

$$\begin{aligned} \therefore \mathcal{L}^{-1} \frac{s^2+s+1}{s^2-5s+6} &= \mathcal{L}^{-1} \left(1 + \frac{6s-5}{s^2-5s+6} \right) \\ &= 0 - 7e^{2t} + 13e^{3t} \\ &= \underline{\underline{13e^{3t} - 7e^{2t}}} \end{aligned}$$

H.W.:-

$$\text{Find } \mathcal{L}^{-1} \frac{2s^4 - 5s^3 + 6s^2 - 5s + 3}{s^3 - 3s^2 + 3s - 1}$$

Hint:- by long division:-

$$\mathcal{L}^{-1} \frac{2s^4 - 5s^3 + 6s^2 - 5s + 3}{s^3 - 3s^2 + 3s - 1} = \mathcal{L}^{-1} \left(2s + 1 + \frac{3s^2 + 4}{(s-1)^3} \right)$$

$$\textcircled{2} \mathcal{L}^{-1}(2s-1) = 0$$

5.	L.T. of derivative	$\mathcal{L}\{f(t)\} = F(s)$, $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ $\mathcal{L}\{f''(t)\} = s^2F(s) - sf'(0) - f(0)$	$\mathcal{L}\{f(t) = \sin t\}$, then: $\mathcal{L}\{\sin t\} = -\mathcal{L}\left\{\frac{d}{dt} \cos t\right\} = -\mathcal{L}\{f'(t)\} = -[sF(s) - 1] = -\left[\frac{s^2 - 1}{s^2 + 1}\right] = \frac{1}{s^2 + 1} = \mathcal{L}\{\sin t\}$	5.	$\mathcal{L}^{-1}\{F(s)\} = f(t)$, $\mathcal{L}^{-1}\{sF(s)\} = \mathcal{L}^{-1}\left\{\frac{d}{dt} f(t)\right\} = -tf'(t)$ $\mathcal{L}^{-1}\{s^2F(s)\} = \mathcal{L}^{-1}\left\{\frac{d^2}{dt^2} f(t)\right\} = f(t) - tf'(t) - t^2f''(t)$	$\mathcal{L}^{-1}\left\{\frac{-2s}{(s^2 + 1)^2}\right\} = \mathcal{L}^{-1}\left[\frac{d}{ds} \frac{1}{s^2 + 1}\right] = -t \sin t$
6.	L.T. of Integral	$\mathcal{L}\{f(t)\} = F(s)$, $\mathcal{L}\left\{\int_0^t f(u) du\right\} = \frac{F(s)}{s}$	$\mathcal{L}\left\{\int_0^t \sin 2u du\right\} = \frac{\frac{1}{2} \frac{s^2 - 4}{s^2 + 4}}{s} = \frac{1}{s(s^2 + 4)}$	6.	$\mathcal{L}^{-1}\{F(s)\} = f(t)$, $\mathcal{L}^{-1}\left\{\int_0^s F(u) du\right\} = \frac{f(t)}{t}$	$\mathcal{L}^{-1}\left\{\frac{1}{s} \int_0^s \frac{1}{u^2} du\right\} = \mathcal{L}^{-1}\left\{1 - \frac{1}{s}\right\} = \frac{1}{t} - 1$
7.	L.T. of Multiplication by t^n	$\mathcal{L}\{f(t)\} = F(s)$, $\mathcal{L}\{tf(t)\} = (-1)^n \frac{d}{ds} F(s)$ $\mathcal{L}\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2} F(s)$	$\mathcal{L}\left\{t^2 e^{2t}\right\} = \frac{d^2}{ds^2} \frac{1}{s-2} = \frac{-1}{(s-2)^3} = \frac{1}{(2-s)^3}$	7.	$\mathcal{L}^{-1}\{F(s)\} = f(t)$, $\mathcal{L}^{-1}\{sF(s)\} = f'(t)$ $\mathcal{L}^{-1}\{s^2F(s)\} = f''(t)$	$\mathcal{L}^{-1}\left\{\frac{s}{(s^2 + 4)}\right\} = \frac{1}{2} \frac{d}{dt} \sin 2t = \cos 2t$
8.	L.T. of Division by t	$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u) du$	$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{u^2 + 1} du = \tan^{-1} \frac{1}{s} = \tan^{-1} \frac{1}{s} = \tan^{-1} \frac{1}{s}$	8.	$\mathcal{L}^{-1}\{F(s)\} = f(t)$, $\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u) du$	$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = \int_0^t 1 du = u \Big _0^t = t - 0 = t$

9.	L.T. of Periodic Function	$f(t) \text{ have period } T > 0, \text{ so that:}$ $L f(t) = \frac{\int_0^T e^{-st} f(t) dt}{1 - e^{-sT}}$	9.	Show, by using the Laplace Transform,		
10.	L.T. of Unit Step Function	If: $L f(t) = F(s), \text{ and we have}$ $u(t-a) = \begin{cases} 1 & t > a \\ 0 & t < a \end{cases}$ then: $L u(t-a) = \frac{e^{-as}}{s}$	10.	Show, by using the Laplace Transform,		

13 The Gamma function:-

If $n > 0$, we define the gamma function

$$\Gamma(n) = \int_0^{\infty} u^{n-1} e^{-u} du$$

the following are some important properties of the gamma function

- ① $\Gamma(n+1) = n \Gamma(n)$, $n > 0$
- ② $\Gamma(1) = 1$
- ③ $\Gamma(1/2) = \sqrt{\pi}$
- ④ for $n < 0$ we define $\Gamma(n)$ by $\Gamma(n) = \frac{\Gamma(n+1)}{n}$

Examples:- Find

$$\textcircled{1} \Gamma(2) \Rightarrow n+1=2 \Rightarrow n=1 > 0 \Rightarrow \therefore \Gamma(2) = 1 \cdot \Gamma(1) = 1 \cdot 1 = 1!$$

$$\textcircled{2} \Gamma(3) \Rightarrow n+1=3 \Rightarrow n=2 > 0 \Rightarrow \therefore \Gamma(3) = 2 \cdot \Gamma(2) = 2!$$

$$\textcircled{3} \Gamma(4) \Rightarrow n+1=4 \Rightarrow n=3 > 0 \Rightarrow \therefore \Gamma(4) = 3 \cdot \Gamma(3) = 3! = 6!$$

and in general $\Gamma(n+1) = n \Gamma(n) = n!$ if $n = 1, 2, 3, \dots$

$$\textcircled{4} \Gamma(-1/2) \text{ by using } \Gamma(n) = \frac{\Gamma(n+1)}{n} \text{ for } n < 0$$

$$n = -1/2 \text{ and } n+1 = 1/2 \Rightarrow \Gamma(-1/2) = \frac{\Gamma(1/2)}{-1/2} = -2\sqrt{\pi}$$

$$\textcircled{5} \Gamma(-3/2) \Rightarrow n = -3/2, n+1 = -1/2 \Rightarrow \Gamma(-3/2) = \frac{\Gamma(-1/2)}{-3/2} = \frac{-2\sqrt{\pi}}{-3/2} = \frac{4\sqrt{\pi}}{3}$$

$$\textcircled{6} \Gamma(-5/2) \Rightarrow n = -5/2, n+1 = -3/2 \Rightarrow \Gamma(-5/2) = \frac{\Gamma(-3/2)}{-5/2} = \frac{4\sqrt{\pi}/3}{-5/2} = -\frac{8}{15}\sqrt{\pi}$$

$$\textcircled{7} \Gamma(0) \Rightarrow n=0, n+1=1 \Rightarrow \Gamma(0) = \frac{\Gamma(1)}{0} = \frac{1}{0} = \infty$$

$$\textcircled{8} \Gamma(-1) \Rightarrow n=-1, n+1=0 \Rightarrow \Gamma(-1) = \frac{\Gamma(0)}{-1} = \frac{\infty}{-1} = \infty$$

$$\textcircled{9} \Gamma(-2) \Rightarrow n=-2, n+1=-1 \Rightarrow \Gamma(-2) = \frac{\Gamma(-1)}{-2} = \frac{\infty}{-2} = \infty$$

In general $\Gamma(-p) = \infty$ if p any positive const. integer

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ex. 4) Find the general solution of the diff. eq.

$$6y'' - 3y' + 3y - y = t^2 \cdot e^t, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2$$

Sol.

$$\mathcal{L}\{6y'' - 3y' + 3y - y\} = \mathcal{L}\{t^2 \cdot e^t\}$$

$$6\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} - \mathcal{L}\{y\} = \mathcal{L}\{t^2 \cdot e^t\}$$

$$6\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{t^2 \cdot e^t\}$$

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Then

$$6s^2 \cdot \mathcal{L}\{y\} - 3s \cdot \mathcal{L}\{y\} + 2\mathcal{L}\{y\} = \frac{2}{(s-1)^3}$$

$$\mathcal{L}\{y\} [6s^2 - 3s + 2] = \frac{2}{(s-1)^3}$$

by simplifying

$$\mathcal{L}\{y\} = \frac{2}{(s-1)^6} + \frac{(s^2 - 3s + 1)}{(s-1)^3}$$

$$\begin{aligned} &= \frac{s^2 - 3s + 1}{(s-1)^3} \\ &= \frac{s^2 - 2s + 1 - 2s + 1}{(s-1)^3} \\ &= \frac{(s-1)^2 - 2s + 1}{(s-1)^3} \\ &= \frac{(s-1)^2 - 2(s-1) - 1}{(s-1)^3} \end{aligned}$$

$$\mathcal{L}\{y\} = \frac{2}{(s-1)^6} + \frac{(s-1)^2 - 2(s-1) - 1}{(s-1)^3} = \frac{2}{(s-1)^6} + \frac{1}{(s-1)^3} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3}$$

$$\Rightarrow \mathcal{L}^{-1} \mathcal{L}\{y\} = y(t) = \frac{1}{60} t^5 \cdot e^t + e^t - t \cdot e^t - t^2 \cdot e^t / 2$$

ex. 4) Find the general solution of the diff. eq.

$$y'' - 3y' + 3y - y = t^2 \cdot e^t, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2$$

Sol.

$$\mathcal{L}\{y'' - 3y' + 3y - y\} = \mathcal{L}\{t^2 \cdot e^t\}$$

$$\mathcal{L}\{y''\} = s^2 \cdot y(s) - s \cdot y(0) - y'(0)$$

$$\mathcal{L}\{y'\} = s \cdot y(s) - y(0)$$

$$\mathcal{L}\{y\} = y(s)$$

$$\mathcal{L}\{y'' - 3y' + 3y - y\} = \mathcal{L}\{t^2 \cdot e^t\}$$

$$\mathcal{L}\{y'' - 3y' + 3y - y\} = \mathcal{L}\{t^2 \cdot e^t\}$$

Then

$$s^2 \cdot y(s) - s \cdot y(0) - y'(0) - 3[s \cdot y(s) - y(0)] + 3y(s) - y(s) = \frac{2}{(s-1)^3}$$

$$y(s) [s^2 - 3s^2 + 3s - 1] = \frac{2}{(s-1)^3} + s - 3s + 1$$

by simplifying

$$y(s) = \frac{2}{(s-1)^6} + \frac{(s^2 - 3s + 1)}{(s-1)^3}$$

$$\begin{aligned} s^2 - 3s + 1 &= s^2 - 2s + 1 - s^2 + 3s - 1 + 1 \\ &= (s-1)^2 - s^2 + 3s - 1 + 1 \\ &= (s-1)^2 \cdot (s-1) - 1 \end{aligned}$$

$$y(s) = \frac{2}{(s-1)^6} + \frac{(s-1)^2 - (s-1) - 1}{(s-1)^3} = \frac{2}{(s-1)^6} + \frac{1}{(s-1)^5} - \frac{1}{(s-1)^4} - \frac{1}{(s-1)^3}$$

$$\Rightarrow \mathcal{L}^{-1} y(s) = y(t) = \frac{1}{60} t^5 \cdot e^t + e^t - t \cdot e^t - t^2 \cdot e^t / 2$$