Partial Differential Equations :-

1. Introduction
2. Wave equation
3. Heat Conduction "one-dimension unsteady"
4. Heat conduction "two-dimension steady" Laplace's equation

① Introduction

1.1 Equations of the form

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

Auxiliary equation $am^2 + bm + c = 0$ solutions depend on the roots of this equation.

ⓐ Real and different roots $m = m_1$ & $m = m_2$

solution, $$Y = A e^{m_1 x} + B e^{m_2 x} \quad \text{---} ①$$

ⓑ Real and equal roots $m = m_1 = m_2$

solution, $$Y = e^{m_1 x} (A + Bx) \quad \text{-----} ②$$

ⓒ Complex roots $m = \alpha \mp \beta j$

solution, $$Y = e^{\alpha x} (A \cos \beta x + B \sin \beta x) \quad \text{--} ③$$

1.2 Equations of the form $\frac{d^2y}{dx^2} \mp n^2 y = 0$

If we take the general equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c\,y = 0$$ And consider the case when $b=0$; then dividing through by a, we have $\frac{d^2y}{dx^2} + \frac{c}{a}y = 0$ which we write as $\frac{d^2y}{dx^2} \mp n^2 y = 0$ to cover separating the two cases when $\frac{c}{a}$ is positive or negative

(a) $\frac{c}{a}$ is positive $\quad \frac{d^2y}{dx^2} + n^2 y = 0$

$\Rightarrow m^2 + n^2 = 0 \Rightarrow m^2 = -n^2 \Rightarrow m = \mp nj$

solution , $y = A\cos nx + B\sin nx$ ---- (4)

(b) $\frac{c}{a}$ is negative $\quad \frac{d^2y}{dx^2} - n^2 y = 0$

$\Rightarrow m^2 - n^2 = 0 \Rightarrow m^2 = n^2 \Rightarrow m = \mp n$

solution , $y = A\cosh nx + B\sinh nx$

or $y = A e^{nx} + B e^{-nx}$

or $y = A\sinh n(x - \phi)$

In each case, A & B are arbitrary constants depending on the initial condition

A partial differential equation is a relationship between a dependent variable U and two or more independent variables $(x, y, t, ...)$ and partiad differential coefficients of U with respect to these independent variables, the solution is therefore of the form

$$U = f(x, y, t, ---)$$

1.3 Solution by direct integration :-

The simplest form of partial differential equation is such that a solution can be determined by direct partial integration.

Example: Solve the equation $\frac{\partial^2 u}{\partial x^2} = 12x^2(t+1)$ given that at

$x = 0$; $U = \cos 2t$; $\frac{\partial u}{\partial x} = \sin t$

Sol.

$\frac{\partial^2 u}{\partial x^2} = 12x^2(t+1)$ Integrat

$\Rightarrow \frac{\partial u}{\partial x} = 4x^3(t+1) + \phi t$ where ϕt = arbitrary function
θt

integrat again $\Rightarrow U = x^4(t+1) + x\phi t + \theta t$

applied initial conditions that at $x=0$

$\frac{\partial u}{\partial x} = \sin t$, $U = \cos 2t$

substituting

$\frac{\partial u}{\partial x} = 4x^3(t+1) + \phi t \Rightarrow \sin t = 0 + \phi t \Rightarrow \phi t = \sin t$

$U = x^4(t+1) + x\sin t + \theta t \Rightarrow \cos 2t = 0 + 0 + \theta t \Rightarrow \theta t = \cos 2t$

$\therefore U = x^4(t+1) + x\sin t + \cos 2t$

ex. Solve the equation $\frac{\partial^2 u}{\partial x \partial y} = \sin(x+y)$, given that at $y=0$; $\frac{\partial u}{\partial x} = 1$ and at $x=0$; $u = (y-1)^2$

sol.

$$\frac{\partial^2 u}{\partial x \partial y} = \sin(x+y) \Rightarrow \frac{\partial u}{\partial x} = -\cos(x+y) + \phi x$$

at $y = 0$; $\frac{\partial u}{\partial x} = 1 \Rightarrow 1 = -\cos x + \phi x \Rightarrow \phi x = 1 + \cos x$

$\therefore \frac{\partial u}{\partial x} = -\cos(x+y) + 1 + \cos x \Rightarrow u = -\sin(x+y) + x + \sin x + \theta y$

Put at $x = 0$, $u = (y-1)^2 \Rightarrow (y-1)^2 = -\sin y + \theta y$

$\Rightarrow \theta y = (y-1)^2 + \sin y$

$\therefore u = -\sin(x+y) + x + \sin x + \sin y + (y-1)^2$

1.4 Initial Conditions and Boundary conditions

As with any differential equation, the arbitrary constants or arbitrary functions in any particular case are determined from the additional information given concerning the variables of equation. These extra facts are called the initial conditions or more generally, the boundary conditions since they do not always refer to zero values of the independent variables.

ex. Solve the equation $\frac{\partial^2 u}{\partial x \partial y} = \sin x \sin y$ subject to the boundary conditions
that at $y = \frac{\pi}{2}$; $\frac{\partial u}{\partial x} = 2x$
at $x = \pi$, $u = 2\sin y$

Sol. $\dfrac{\partial^2 u}{\partial x \partial y} = \sin x \sin y \Rightarrow \dfrac{\partial u}{\partial x} = \sin x \overset{-\cos y}{\overset{\uparrow}{\sin y}} + \phi x$

put at $y = \frac{\pi}{2}$; $\frac{\partial u}{\partial x} = 2x \Rightarrow 2x = \sin x \cdot \sin\frac{\pi}{2} + \phi x$

$\Rightarrow 2x = \sin x + \phi x \Rightarrow \phi x = 2x - \sin x$

$\Rightarrow \dfrac{\partial u}{\partial x} = \sin x \sin y + 2x - \sin x$

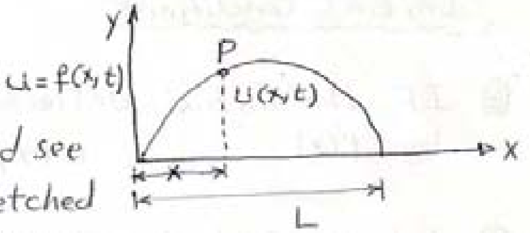
$\Rightarrow \dfrac{\partial u}{\partial x} = 2x + \sin x(\sin y - 1) \Rightarrow u = x^2 - \cos x(\sin y - 1) + \theta y$

put at $x = \pi$, $u = 2\sin y \Rightarrow 2\sin y = \pi^2 - \cos\pi(\sin y - 1) + \theta y$

$\Rightarrow 2\sin y = \pi^2 - 1 + \sin y + \theta y \Rightarrow \theta y = 1 - \pi^2 + \sin y$

$\therefore U = x^2 + \cos x(1 - \sin y) + \sin y + 1 - \pi^2$

② Wave Equation :-



In this equation we could see

* flexible elastic string stretched between two points at $x = 0$ and $x = L$ with uniform tension T
* The end points remaining fixed
* The string will vibrate
* Its displacements u at any time t can be expressed as $u = f(x, t)$

where x is its distance from the left-hand end the equation of motion is given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial^2 u}{\partial t^2}$$

where $c^2 = T/\rho$

T = tension in the string

ρ = mass per unit length of the string

2.1 Solution of the wave equation

The wave equation is $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial^2 u}{\partial t^2}$
have a solution $u = f(x,t)$ written $u(x,t)$.

Boundary Conditions

ⓐ The string is fixed at both ends

i.e $\left.\begin{array}{l} x = 0 \\ x = L \end{array}\right\}$ for all values of time $t > 0$

$u(x,t)$ becoms $\begin{array}{l} u(0,t) = 0 \\ u(L,t) = 0 \end{array}$ for $t > 0$

Initial Conditions

ⓑ If the initial deflection of P at $t=0$ is denoted by $f(x)$ $\therefore$ $u(x,0) = f(x)$ for $t=0$

ⓒ Let the initial velocity of P at $t=0$ is denoted by $g(x)$ $\left[\frac{\partial u}{\partial t}\right]_{t=0} = g(x)$

2.2 Solution by separation of variables :-

$u(x,t) = X_{(x)} \cdot T_{(t)}$ where $X(x)$ is a function of x only
$T(t)$ is a function of t only

$\therefore$ $\underline{u = X \cdot T}$

$\therefore$ $\frac{\partial u}{\partial x} = X' \cdot T \Rightarrow \frac{\partial^2 u}{\partial x^2} = X'' \cdot T$

$\frac{\partial u}{\partial t} = X \cdot T' \Rightarrow \frac{\partial^2 u}{\partial t^2} = X \cdot T''$

The wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ Can be written

$$X''.T = \frac{1}{c^2}.X.T'' \Rightarrow \boxed{\frac{X''}{X} = \frac{1}{c^2}.\frac{T''}{T}}$$

Denote this arbitrary constant by K, we have

$$\frac{X''}{X} = K \quad \text{and} \quad \frac{1}{c^2}.\frac{T''}{T} = K$$

$$\Rightarrow X'' - KX = 0 \quad , \quad T'' - c^2K.T = 0$$

Let us consider the first of these two equations for different values of K

(i) IF $K = 0$ then, $X'' = 0 \Rightarrow X' = a \Rightarrow X = ax + b$

at $x = 0$, $X = 0 \Rightarrow b = 0$

$x = L$, $X = 0 \Rightarrow a = 0$

$\therefore X = 0$ which is not oscillatory as the proplem required

(ii) IF K is positive Let $K = P^2$

$\Rightarrow X'' - KX = 0 \Rightarrow X'' - P^2X = 0$ the auxiliary equation is therefore $m^2 - P^2 = 0 \Rightarrow m^2 = P^2 \Rightarrow m = \mp P$

the solution is $X = A.e^{Px} + B.e^{-Px}$

at $x = 0 \Rightarrow X = 0 \Rightarrow 0 = A + B \Rightarrow A = -B$

$x = L \Rightarrow X = 0 \Rightarrow 0 = Ae^{PL} + Be^{-PL}$

$\Rightarrow 0 = -B(e^{PL} - e^{-PL}) \Rightarrow B = 0 = A$

Here again $X = 0$ which is oscillatory

(iii) If k is negative, let $k = -p^2$

$\therefore X'' - kX = 0 \Rightarrow X'' + p^2 X = 0$, the solution is

$\Rightarrow X = A\cos px + B\sin px$ — *

the second equation $T'' - c^2 k T = 0 \Rightarrow T'' + c^2 p^2 T = 0$

the solution is $T = C\cos cpt + D\sin cpt$ — *

$\therefore$ the general solution becoms

$U = X \cdot T$

$\Rightarrow u(x,t) = \left[A\cos px + B\sin px\right]\left[C\cos cpt + D\sin cpt\right]$

if we put $cp = \lambda \Rightarrow p = \frac{\lambda}{c}$ sub. in above eq.

$\Rightarrow u(x,t) = \left[A\cos\frac{\lambda}{c}x + B\sin\frac{\lambda}{c}x\right]\left[C\cos\lambda t + D\sin\lambda t\right]$ --- * (A)

where A, B, C, and D are arbitrary constants.
the results of Course must be satisfy the set of boundary conditions which we now turn to.

(a) $u(0,t) = 0$
$u(L,t) = 0$ for $t > 0$

Then
at $x=0$, $u=0$ sub. in eq. (A) we get

$u(x,t) = \left[A\cos\frac{\lambda}{c}x + B\sin\frac{\lambda}{c}x\right]\left[C\cos\lambda t + D\sin\lambda t\right]$

$\Rightarrow 0 = \left[A \times 1 + B \times 0\right]\left[C \cdot \cos\lambda t + D\sin\lambda t\right] \Rightarrow A = 0$

sub. in eq. (A)

$\Rightarrow u(x,t) = B \cdot \sin\frac{\lambda}{c}x\left[C\cos\lambda t + D\sin\lambda t\right]$ --- eq. (B)

at $x = L$, $u = 0$ Then eq. (B) becoms

$$0 = B \sin \frac{\lambda L}{c} \left[C \cos \lambda t + D \sin \lambda t \right]$$

Now

$B \neq 0$, $\therefore \sin \frac{\lambda L}{c} = 0 \Rightarrow \frac{\lambda L}{c} = n\pi$ where $n = 1, 2, \ldots$

$$\Rightarrow \lambda = \frac{n \cdot c \pi}{L} \quad \text{for } n = 1, 2, 3, \ldots$$

there is an infinite set of values of λ and each separate value gives aparticular solution of $u(x,t)$
The values of λ are called the Eigen values and each corresponding solution the Eigen function
putting $n = 1, 2, 3, \ldots$ we have that solution.

	Eigen values	Eigen function
n	$\lambda = \frac{n \cdot c \cdot \pi}{L}$	$u(x,t) = B \sin \frac{\lambda x}{c} \left[C \cos \lambda t + D \sin \lambda t \right]$
1	$\lambda_1 = \frac{1 \cdot c \cdot \pi}{L}$	$u_1 = \sin \frac{\pi x}{L} \left[C_1 \cos \frac{c \pi t}{L} + D_1 \sin \frac{c \pi t}{L} \right]$
2	$\lambda_2 = \frac{2 \cdot c \cdot \pi}{L}$	$u_2 = \sin \frac{2\pi x}{L} \left[C_2 \cos \frac{2c\pi t}{L} + D_2 \sin \frac{2c\pi t}{L} \right]$
3	$\lambda_3 = \frac{3 \cdot c \cdot \pi}{L}$	$u_3 = \sin \frac{3\pi x}{L} \left[C_3 \cos \frac{3c\pi t}{L} + D_3 \sin \frac{3c\pi t}{L} \right]$
$\vdots$	$\vdots$	
r	$\lambda_r = \frac{r \cdot c \pi}{L}$	$u_r = \sin \frac{r\pi x}{L} \left[C_r \cos \frac{rc\pi t}{L} + D_r \sin \frac{rc\pi t}{L} \right]$

where $C_1, C_2, C_3, ----$ and $D_1, D_2, D_3, ----$ are arbitrary constants.

$\therefore \; u = u_1 + u_2 + u_3 + \cdots$

the more general solution is therefore

$$u(x,t) = \sum_{r=1}^{\infty} u_r$$

$$\Rightarrow \; u(x,t) = \sum_{r=1}^{\infty} \sin\frac{r\pi x}{L}\left[C_r \cos\frac{rc\pi t}{L} + D_r \sin\frac{rc\pi t}{L}\right] - *$$

We use the initial conditions

Ⓒ at $t=0$; $u(x,0) = f(x)$ for $0 \leqslant x \leqslant L$

sub. in eq. $*$

$$\Rightarrow \; u(x,0) = f(x) = \sum_{r=1}^{\infty} C_r \cdot \sin\frac{r\pi x}{L}$$

Ⓓ Also at $t=0$; $\left[\frac{\partial u}{\partial t}\right]_{t=0} = g(x)$ for $0 \leqslant x \leqslant L$

from eq. $*$

$$u(x,t) = \sum_{r=1}^{\infty} \sin\frac{r\pi x}{L}\left[C_r \cdot \cos\frac{rc\pi t}{L} + D_r \cdot \sin\frac{rc\pi t}{L}\right]$$

for

$$\frac{\partial u}{\partial t} = \sum_{r=1}^{\infty} \sin\frac{r\pi x}{L}\left[-C_r \cdot \frac{rc\pi}{L} \sin\frac{rc\pi t}{L} + D_r \cdot \frac{rc\pi}{L} \cos\frac{rc\pi t}{L}\right]$$

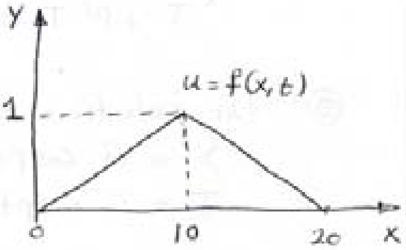
with $t=0$, $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$

$$\Rightarrow \; g(x) = \sum_{r=1}^{\infty} D_r \cdot \frac{rc\pi}{L} \sin\frac{r\pi x}{L}$$

$$\therefore \; g(x) = \frac{c\pi}{L} \sum_{r=1}^{\infty} D_r \cdot r \cdot \sin\frac{r\pi x}{L}$$

Example:- A stretched string of length 20 cm is set oscillating by displacing its mid-point a distance 1 cm from its rest position and releasing it with zero initial velocity. Solve the wave equation where $c^2 = 1$ to determine the resulting motion, $u(x,t)$.

Sol. ① to find the boundary conditions from the data given in the question



$$\left.\begin{array}{l} u(0,t) = 0 \\ u(20,t) = 0 \end{array}\right\} \text{fixed end points}$$

$$u(x,0) = f(x) = \begin{cases} \dfrac{x}{10} & \text{for } 0 \leqslant x \leqslant 10 \\[2ex] \dfrac{20-x}{10} & \text{for } 10 \leqslant x \leqslant 20 \end{cases}$$

$$\left.\frac{\partial u}{\partial t}\right]_{t=0} = 0 \quad (\text{zero initial velocity})$$

② where $c^2 = 1 \Rightarrow$ the equation $\Rightarrow \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$

for $u = X \cdot T$

$$\therefore \frac{\partial u}{\partial x} = X' \cdot T \quad , \quad \frac{\partial u}{\partial t} = X \cdot T'$$

$$\frac{\partial^2 u}{\partial x^2} = X'' \cdot T \quad , \quad \frac{\partial^2 u}{\partial t^2} = X \cdot T''$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \Rightarrow X'' \cdot T = X \cdot T''$$

③ We rearrange the equation to separate the varables

$$\frac{X''}{X} = \frac{T''}{T}$$

④ Since the two sides are equal for a values of the variables, each must be equal to constant K and to give an oscillatory solution we put $K = -P^2$

$$X'' + P^2 X = 0$$
$$T'' + P^2 T = 0$$

⑤ the solution of the above eq. are

$$X = A \cos px + B \sin px$$
$$T = C \cos pt + D \sin pt$$

$$\therefore u(x,t) = \left[A \cos px + B \sin px\right]\left[C \cdot \cos pt + D \sin pt\right]$$

⑥ put $cp = \lambda$, in this case $c = 1 \Rightarrow p = \lambda$

$$u(x,t) = \left[A \cos \lambda x + B \sin \lambda x\right]\left[C \cdot \cos \lambda t + D \sin \lambda t\right]$$

⑦ Now we determine A & B from B.C

(i)- $u(0,t) = 0$

$$u(x,t) = \left[A \cos \lambda x + B \sin \lambda x\right]\left[C \cos \lambda t + D \sin \lambda t\right]$$
$$0 = \left[A \cdot 1 + B \cdot 0\right]\left[C \cdot \cos \lambda t + D \sin \lambda t\right]$$
$$\Rightarrow A = 0$$
$$\therefore u(x,t) = B \sin \lambda x \left[C \cdot \cos \lambda t + D \cdot \sin \lambda t\right]$$

(ii) $u(20,t) = 0$

$$u(x,t) = B \sin \lambda x \left[C \cdot \cos \lambda t + D \cdot \sin \lambda t\right]$$
$$0 = B \sin 20\lambda \left[C \cdot \cos \lambda t + D \sin \lambda t\right]$$
$$\Rightarrow B \neq 0$$
$$\therefore \sin 20\lambda = 0 \Rightarrow 20\lambda = n\pi \Rightarrow \lambda = \frac{n\pi}{20}$$

$$\therefore u(x,t) = B \sin \lambda x \left[C \cdot \cos \lambda t + D \cdot \sin \lambda t\right]$$ Let $B \cdot C = G$, $B \cdot D = \varphi$

$$\therefore u(x,t) = \sin \frac{n\pi}{20} x \left[G \cos \frac{n\pi}{20} t + \varphi \sin \frac{n\pi}{20} t\right]$$

⑧ to find the eigen values and eigen functions

	Eigen values	Eigen functions
n	$\lambda = \frac{n\pi}{20}$	$u(x,t) = \sin \lambda x \left[G \cos \lambda t + \varphi \sin \lambda t \right]$
1	$\lambda_1 = \frac{\pi}{20}$	$u_1 = \sin \frac{\pi x}{20} \left[G_1 \cos \frac{\pi t}{20} + \varphi_1 \sin \frac{\pi t}{20} \right]$
2	$\lambda_2 = \frac{2\pi}{20}$	$u_2 = \sin \frac{2\pi x}{20} \left[G_2 \cos \frac{2\pi t}{20} + \varphi_2 \sin \frac{2\pi t}{20} \right]$
3	$\lambda_3 = \frac{3\pi}{20}$	$u_3 = \sin \frac{3\pi x}{20} \left[G_3 \cos \frac{3\pi t}{20} + \varphi_3 \sin \frac{3\pi t}{20} \right]$
$\vdots$	$\vdots$	$\vdots$
r	$\lambda_r = \frac{r\pi}{20}$	$u_r = \sin \frac{r\pi x}{20} \left[G_r \cdot \cos \frac{r\pi t}{20} + Q_r \cdot \sin \frac{r\pi t}{20} \right]$

where $u = u_1 + u_2 + u_3 + \cdots$

$$\Rightarrow u(x,t) = \sum_{r=1}^{\infty} \sin \frac{r\pi x}{20} \left[G_r \cos \frac{r\pi t}{20} + \varphi_r \cdot \sin \frac{r\pi t}{20} \right]$$

⑨ Now we apply the remaining initial conditions

① $u(x,0) = f(x) = \frac{x}{10}$ for $0 \leq x \leq 10$

$= \frac{20-x}{10}$ for $10 \leq x \leq 20$

$\therefore G_r = 2 \times$ mean value of $f(x) \sin \frac{r\pi x}{20}$

$$\Rightarrow G_r = \frac{2}{20} \int_0^{20} f(x) \sin \frac{r\pi x}{20}$$

$$\Rightarrow 10\, G_r = \underbrace{\int_0^{10} \frac{x}{10} \sin \frac{r\pi x}{20}\, dx}_{I_1} + \underbrace{\int_{10}^{20} \frac{20-x}{10} \sin \frac{r\pi x}{20}\, dx}_{I_2}$$

Note

$f(x) = \sum G_r \sin \lambda x \Big] \times \sin \lambda x\, dx$

$f(x) \cdot \sin \lambda x\, dx = \sum G_r \int \sin^2 \lambda x\, dx$

$f(x) \cdot \sin \lambda x\, dx = G_r \cdot \underbrace{\int_0^L \sin^2 \lambda x\, dx}_{\frac{L}{2}}$

$$I_1 = \int_0^{10} \frac{x}{10} \sin \frac{r\pi x}{20}\, dx \quad \text{integrating by parts}$$

$$I_1 = -\frac{20}{r\pi} \cos \frac{r\pi}{2} + \frac{40}{r^2\pi^2} \sin \frac{r\pi}{2}$$

similarly $\Rightarrow I_2 = \int_{10}^{20} \frac{20-x}{10} \sin \frac{r\pi x}{20}\, dx$

$$\Rightarrow I_2 = \frac{20}{r\pi} \cos \frac{r\pi}{2} - \frac{40}{r^2\pi^2} \sin r\pi$$

then

$$10\, G_r = \cancel{-\frac{20}{r\pi} \cos \frac{r\pi}{2}} + \frac{40}{r^2\pi^2} \sin \frac{r\pi}{2} + \cancel{\frac{20}{r\pi} \cos \frac{r\pi}{2}} - \frac{40}{r^2\pi^2} \sin r\pi$$

for $r = 1, 2, 3, \ldots \quad \Rightarrow G_r = \frac{4}{r^2\pi^2} \sin \frac{r\pi}{2}$

$$\therefore u(x,t) = \sum_{r=1}^{\infty} \sin \frac{r\pi x}{20} \left[\frac{4}{r^2\pi^2} \sin \frac{r\pi}{2} + \varphi_r \sin \frac{r\pi t}{20} \right]$$

(ii) Also at $t=0$; $\frac{\partial u}{\partial t} = 0$

$$\Rightarrow \frac{\partial u}{\partial t} = \sum_{r=1}^{\infty} \sin \frac{r\pi x}{20} \left[\left(\frac{4}{r^2\pi^2} \sin \frac{r\pi}{2}\right)\left(-\frac{r\pi}{20} \sin \frac{r\pi t}{20}\right) + \varphi_r \cdot \frac{r\pi}{20} \cos \frac{r\pi t}{20} \right]$$

at $t = 0$

$$\Rightarrow 0 = \sum_{r=1}^{\infty} \sin \frac{r\pi x}{20} \times \varphi_r \cdot \frac{r\pi}{20} \times 1 \quad \Rightarrow \varphi_r = 0$$

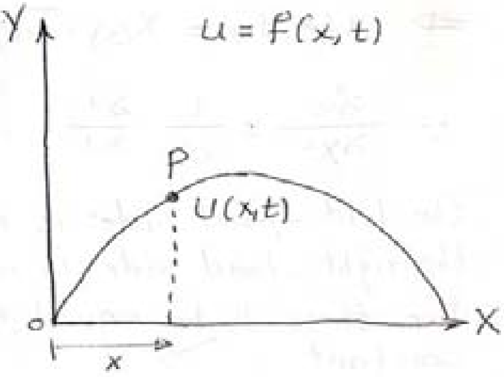
$$\therefore \boxed{u(x,t) = \frac{4}{\pi^2} \sum_{r=1}^{\infty} \frac{1}{r^2} \sin \frac{r\pi x}{20} \sin \frac{r\pi}{2} \cos \frac{r\pi t}{20}}$$

③ Heat conduction Equation for a uniform finite bar :-

the one-dimensional heat equation is then of the form

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial u}{\partial t}$$

where $c^2 = \frac{k}{\sigma \cdot \rho}$



K = thermal conductivity of the material

σ = specific heat of the material

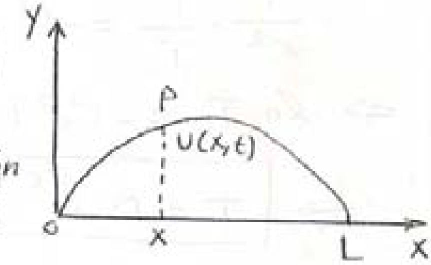
ρ = mass per unit length of the bar

3.1 Solutions of the heat conduction equation :-

ⓐ the bar extends from $x=0$ to $x=L$

ⓑ the temperature of the ends of the bar is maintained at zero

ⓒ the initial temperature distribution a long the bar is defined by $f(x)$



the boundary conditions can be expressed

$$\left.\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}\right\} \text{ for all } t \geq 0$$

$$u(x,0) = f(x) \quad \text{for } 0 \leq x \leq L \quad \text{at } t=0$$

the solution of the form $u(x,t)$

$\Rightarrow u(x,t) = X_{(x)} \cdot T_{(t)}$ where X is a function of x only
T is a function of t only

$$\because \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial u}{\partial t} \quad \Rightarrow \boxed{\frac{X''}{X} = \frac{1}{c^2} \cdot \frac{T'}{T}}$$

the left - hand side is a function of x only
the right - hand side is a function of t only
for these to be equal each side must be equal the same constant

$$\therefore \frac{X''}{X} = -p^2 \quad \Rightarrow X'' + p^2 X = 0 \quad \Rightarrow \text{giving the solution}$$

$$\boxed{X = A \cos px + B \sin px}$$

And

$$\frac{1}{c^2} \cdot \frac{T'}{T} = -p^2 \quad \Rightarrow T' + p^2 c^2 T = 0 \Rightarrow \frac{T'}{T} = -p^2 c^2$$

$$\Rightarrow \ln T = -p^2 c^2 t + c_1 \Rightarrow T = e^{-p^2c^2 t + c_1} = e^{-p^2c^2 \cdot t} \cdot \underbrace{e^{c_1}}_{D}$$

$$\Rightarrow \boxed{T = D\, e^{-p^2c^2 t}}$$

$$\Rightarrow \quad u(x,t) = X \cdot T$$

$$u(x,t) = \left[A \cos px + B \sin px\right] D\, e^{-p^2c^2 t}$$

$$u(x,t) = \left[G \cos px + Q \sin px\right] e^{-p^2c^2 t}$$

Now put $\lambda = p.c \Rightarrow p = \frac{\lambda}{c}$

$$\therefore \boxed{u(x,t) = e^{-\lambda^2 t}\left[G \cos \frac{\lambda}{c} x + Q \sin \frac{\lambda}{c} x\right]}$$

Applying the boundary condition

$u(0,t)=0$

$$\Rightarrow u(x,t) = e^{-\lambda^2 t}\left[G \cos\frac{\lambda}{c}x + \varphi \sin\frac{\lambda}{c}x\right]$$

$$\Rightarrow 0 = e^{-\lambda^2 t}\left[G \cdot 1 + \varphi \cdot 0\right] \Rightarrow G = 0$$

$$\therefore \boxed{u(x,t) = \varphi\, e^{-\lambda^2 t} \sin\frac{\lambda}{c}x}$$

Also $u(L,t)=0$

$$\Rightarrow 0 = \varphi\, e^{-\lambda^2 t} \cdot \sin\frac{\lambda}{c}L \Rightarrow \varphi \neq 0$$

$$\therefore \sin\frac{\lambda}{c}L = 0 \Rightarrow \frac{\lambda}{c}L = n\pi \quad, n=1,2,3,\cdots \Rightarrow \lambda = \frac{nc\pi}{L}$$

n	$\lambda = \frac{nc\pi}{L}$	$u(x,t) = \varphi\, e^{-\lambda^2 t} \sin\frac{n\pi x}{L}$
1	$\lambda_1 = \frac{c\pi}{L}$	$u_1 = \varphi_1 e^{-\lambda_1^2 t} \cdot \sin\frac{\pi x}{L}$
2	$\lambda_2 = \frac{2c\pi}{L}$	$u_2 = \varphi_2 e^{-\lambda_2^2 t} \cdot \sin\frac{2\pi x}{L}$
3	$\lambda_3 = \frac{3c\pi}{L}$	$u_3 = \varphi_3 \cdot e^{-\lambda_3^2 t} \cdot \sin\frac{3\pi x}{L}$
$\vdots$	$\vdots$	$\vdots$
r	$\lambda_r = \frac{rc\pi}{L}$	$U_r = \varphi_r \cdot e^{-\lambda_r^2 t} \cdot \sin\frac{r\pi x}{L}$

$$\Rightarrow u = u_1 + u_2 + u_3 + \cdots$$

$$\boxed{u(x,t) = \sum_{r=1}^{\infty} \varphi_r\, e^{-\lambda_r^2 t} \sin\frac{r\pi x}{L}}$$

also apply the remaining boundary condition

$$u(x,t) = f(x) \quad \text{at} \quad t=0$$
$$u(x,0) = f(x)$$

$$\Rightarrow \quad f(x) = \sum_{r=1}^{\infty} Q_r \cdot \sin \frac{r\pi x}{L}$$

where $\quad Q_r = \frac{2}{L} \cdot$ mean value of $f(x) \sin \frac{r\pi x}{L}$

$$\Rightarrow \quad Q_r = \frac{2}{L} \int_0^L f(x) \sin \frac{r\pi x}{L} \, dx \quad \text{for } L=1$$

$$Q_r = 2 \int_0^1 f(x) \cdot \sin \frac{r\pi x}{1} \, dx$$

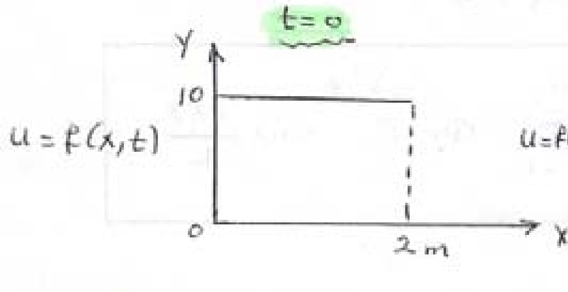
substitution Q_r in the equation $u(x,t)$ for $L=1$

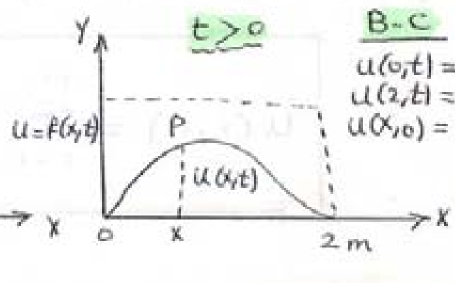
$$u(x,t) = 2 \sum_{r=1}^{\infty} \left[\int_0^1 f(x) \sin r\pi x \, .dx \right] e^{-\lambda_r^2 t} \cdot \sin r\pi x$$

where $\quad \lambda_r = \frac{rc\pi}{L} = rc\pi \quad ; r = 1, 2, 3, \ldots$

Example :- A bar of length 2 m is fully insulated along its sides. It is initially at a uniform temperature of $10^\circ C$ at $t=0$. the ends are plunged into ice and maintained at a temperature of $0\, C^\circ$. Determine an expression for temperature of a point P at a distance x from one end at any subsequent time t seconds after $t=0$.

Sol.





B.C

$u(0,t) = 0$

$u(2,t) = 0$

$u(x,0) = 10$

the solution is $u(x,t) = e^{-\lambda^2 t}\left[G \cos \frac{\lambda}{c}x + Q \sin \frac{\lambda}{c}x\right]$

where $X = A \cos px + B \sin px$

$T = D\, e^{-p^2c^2t}$ $\Rightarrow$ Let $D.A = G$, $D.B = Q$ and $pc = \lambda$; $p = \frac{\lambda}{c}$

$$\Rightarrow u(x,t) = e^{-\lambda^2 t}\left[G \cos \frac{\lambda}{c}x + Q \sin \frac{\lambda}{c}x\right]$$

Applying B.C

for $u(0,t) = 0 \quad \Rightarrow 0 = e^{-\lambda^2 t}\left[G \times 1 + Q \times 0\right] \Rightarrow G = 0$

$$\Rightarrow u(x,t) = e^{-\lambda^2 t} \cdot Q \sin \frac{\lambda}{c}x$$

Also $u(2,t) = 0$

$$\Rightarrow 0 = e^{-\lambda^2 t} \cdot Q \sin \frac{2\lambda}{c} \Rightarrow Q \neq 0 \Rightarrow \therefore \sin \frac{2\lambda}{c} = 0$$

$$\therefore \frac{2\lambda}{c} = n\pi \Rightarrow \lambda = \frac{nc\pi}{2} \quad ; n = 1, 2, 3, \cdots$$

$$\therefore u(x,t) = e^{-\lambda^2 t} \cdot Q \cdot \sin \frac{n\pi x}{2}$$

when $t = 0$; $u(x,0) = 10 = f(x)$

$$\Rightarrow 10 = \sum_{r=1}^{\infty} Q_r \cdot \sin \frac{r\pi x}{2}$$ where $Q_r = 2 \times$ mean value of $10 \sin \frac{r\pi x}{2}$ from 0 to 2

$$\Rightarrow Q_r = \frac{2}{2}\int_0^2 10 \sin \frac{r\pi x}{2}\, dx$$

$$\Rightarrow Q_r = 10 \int_0^2 \sin \frac{r\pi x}{2}\, dx = \frac{-20}{r\pi}\left[\cos \frac{r\pi x}{2}\right]_0^2 = \frac{20}{r\pi}\left[1 - \cos r\pi\right]$$

$$\Rightarrow Q_r = \begin{cases} 0 & \text{for } r = \text{even} \\ \frac{40}{r\pi} & \text{for } r = \text{odd} \end{cases}$$

$$\Rightarrow \therefore u(x,t) = \frac{40}{\pi} \sum_{r=1,3,5,7,\cdots \text{odd}} \frac{1}{r} \sin \frac{r\pi x}{2} \cdot e^{-\lambda^2 t}$$ where $\lambda = \frac{rc\pi}{2}$

④ Heat conduction "two-dimension ; Laplace equation"

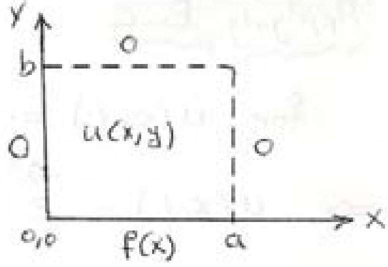
the solution of the Laplace equation two-dimension equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad u = f(x,y)$$

4.1 Solution of the Laplace eq.

to determine solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



for the rectangle bounded by the lines $x = 0$, $x = a$

$y = 0$, $y = b$

Boundary Condition

① at $x = 0$, $u = 0$ for $0 \leq y \leq b$
② at $x = a$, $u = 0$ for $0 \leq y \leq b$
③ at $y = b$, $u = 0$ for $0 \leq x \leq a$
④ at $y = 0$, $u = f(x)$ for $0 \leq x \leq a$

i.e $u(0,y) = 0$, $u(a,y) = 0$ for $0 \leq y \leq b$
$u(x,b) = 0$, $u(x,0) = f(x)$ for $0 \leq x \leq a$

the solution $u = f(x,y)$ will give the potential at any point within the rectangle domain , we start off , as used by assuming a solution of the form

$$u(x,y) = X_{(x)} \cdot Y_{(y)} \quad \text{where}$$

X is a function of x only
Y is a function of y only

The equation in terms of X and Y is separate the variables to give

for $\quad U = X \cdot Y \quad$ where $\dfrac{\partial u}{\partial x} = X' \cdot Y$ & $\dfrac{\partial^2 u}{\partial x^2} = X'' \cdot Y$

then, $\qquad\qquad\qquad\qquad\qquad\quad\ \ \dfrac{\partial u}{\partial y} = X \cdot Y'$ & $\dfrac{\partial^2 u}{\partial y^2} = X \cdot Y''$

$$X'' \cdot Y = - X \cdot Y''$$

$$\Rightarrow \frac{X''}{X} = - \frac{Y''}{Y}$$ putting each side equal to a constant $-p^2$ gives two equations

$\Rightarrow X'' + p^2 X = 0 \quad \Rightarrow$ has a solution $\quad X = A \cos px + B \sin px$

$Y'' - p^2 Y = 0 \quad \Rightarrow$ " " " $\quad Y = C \cosh py + D \sinh py$

which can also be expressed as (For Y equation)

$$Y = E \sinh p(y + \phi)$$

$$\therefore u(x,y) = \left[A \cos px + B \sin px\right] \cdot E \sinh p(y+\phi)$$

$$\therefore \boxed{u(x,y) = \left[G \cos px + \varphi \sin px\right] \cdot \sinh p(y+\phi)}$$

Now we apply the first of the boundary conditions

$u(0,y) = 0$

$$\Rightarrow 0 = \left[G \cdot 1 + \varphi \cdot 0\right] \sinh p(y+\phi) \Rightarrow G = 0$$

$$\therefore u(x,y) = \varphi \sin px \cdot \sinh p(y+\phi)$$

from the second boundary condition

$u(a,y) = 0$

$$\Rightarrow 0 = \varphi \sin pa \cdot \sinh p(y+\phi) \Rightarrow \varphi \neq 0$$

$$\therefore \sin pa = 0 \Rightarrow pa = n\pi$$

Let $\lambda = P \Rightarrow \lambda = \frac{n\pi}{a}$

$\therefore u(x,y) = \varphi \sin\lambda x \cdot \sinh\lambda(y+\phi)$

from the third condition

$u(x,b) = 0$

$$\Rightarrow 0 = \varphi \sin\lambda x \cdot \sinh\lambda(b+\phi)$$

$\Rightarrow \sinh\lambda(b+\phi) = 0 \quad \Rightarrow \phi = -b$

$\therefore u(x,y) = \varphi \sin\lambda x \cdot \sinh\lambda(y-b)$

for λ_r , $u_r = u_1 + u_2 + u_3 + \cdots$

$$\Rightarrow u(x,y) = \sum_{r=1}^{\infty} \varphi_r \cdot \sin\lambda_r x \cdot \sinh\lambda_r(y-b)$$

from fourth B.C

$u(x,0) = f(x) \quad \Rightarrow f(x) = \sum_{r=1}^{\infty} \varphi_r \cdot \sin\lambda_r x \cdot \sinh\lambda_r b$

$$\Rightarrow \varphi_r \cdot \sinh\lambda_r b = \frac{2}{a}\int_0^a - f(x)\sin\lambda x \cdot dx$$

then

$$\boxed{u(x,y) = \sum_{r=1}^{\infty}\left[\frac{-2}{a}\int_0^a f(x)\sin\lambda_r x\, dx\right] \cdot \sin\lambda_r x \cdot \frac{\sinh\lambda_r(y-b)}{\sinh\lambda_r b}}$$