# 1st class

# Mathematics

# الرياضيات

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# Chapter One

Consider an arbitrary system of equation in unknown as:

AX = B ……………………………………………………….(1)

$$\left.\begin{array}{l} a_{r1}X_1 + a_{12}X_2 + a_B X_3 + \ldots\ldots + a_{1n}X_n \\ ail\chi_1 + ai_2\chi_2 + ai_3\chi 3 + \ldots\ldots + ain\chi n = b_1 \\ a21\chi_1 + a22\chi 2 + a23\chi 3 + \ldots + a2n\chi n = b2 \\ a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + \ldots\ldots + a_{2n}X_n \\ am1\chi 1 + am2X2 + 9m3\chi 3 + \ldots + a2n\chi n = bm \\ a_{m1}X_1 + a_{m2}X_2 + a_{m3}X_3 + \ldots. + a_{mn}X_n \end{array}\right\} \ldots\ldots\ldots\ldots\ldots(2)$$

The coefficient of the variables and constant terms can be put in the form:

$$\begin{pmatrix} a_{11} & a_{12} & a_{1n} \\ a2 & a22 & a2n \\ a_{m1} & a_{m2} & a_{mn} \end{pmatrix}_{mxn} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \\ \chi n \end{pmatrix}_{nx1} \begin{pmatrix} b1 \\ b2 \\ \\ bm \end{pmatrix}_{mx1} \ldots\ldots\ldots\ldots\ldots(3)$$

Let the form

$$\begin{pmatrix} a_u a_{12} a_{1n} \\ \\ a_{21}a_{22}a_{2n} \\ \\ am_1 am_2 amn \end{pmatrix} = A = (a_{i1}) \ldots\ldots\ldots\ldots\ldots(4)$$

Is called (mxn) matrix and donated this matrix by:

[aij] i = 1, 2,……m and j = 1, 2,…..n.

We say that is an (mxn) matrix or تكملة

The matrix of order (mxn) it has m rows and n columns.

For example the first row is ($a_{11}$, a12, $a_{1n}$)

And the first column is $\begin{pmatrix} a_{12} \\ a_{11} \\ a_{21} \\ a_{m1} \end{pmatrix}$

(aij) denote the element of matrix. Lying in the i – th row and j – th column, and we call this element as the (i,j) - th element of this matrix

Also $\begin{pmatrix} \chi 1 \\ \chi_2 \\ \\ \chi_n \end{pmatrix}_{nx1}$ is (nx1) [n rows and columns]

$\begin{pmatrix} b1 \\ b2 \\ \\ bm \end{pmatrix}_{mx}$ Is (mx1) [m rows and 1 column]

**Sub – Matrix:**

Let A be matrix in (4) then the sub-matrix of A is another matrix of A denoted by deleting rows and (or) column of A.

Let A = $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

Find the sub-matrix of A with order (2×3) any sub-matrix of A denoted by deleting any row of A $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix}, \begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

Definition 1.1**:**

Tow (mxn) matrices A = [aij] (mxn) and B = [bij] (mxn) are said to be equal if and only if:

aij = bij for i = 1,2…..m and j = 1,2….n

Thus two matrices are equal if and only if:

i. They have the same dimension, and
ii. All their corresponding elements are equal for example:

$$\begin{bmatrix} 2 & 0 & -1 \\ 3 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{2} & 0(7) & -2+1 \\ 3 & \frac{20}{4} & 2 \end{bmatrix}$$

Definition 1.2

If A = [aij] mxn and B = [bij] mxn are mxn matrix their sum is the mxn matrix A+B = $[aij + bij]mxn$.

In other words if two matrices have the same dimension, they may be added by addition corresponding elements. For example if:

A = $\begin{pmatrix} 2 & -7 \\ -3 & 4 \end{pmatrix} and \ \mathbf{B} = \begin{pmatrix} -5 & 0 \\ 1 & 6 \end{pmatrix}$

Then

A+B = $\begin{pmatrix} 3+-5 & -7+0 \\ -3+1 & 4+6 \end{pmatrix} = \begin{pmatrix} -3 & -7 \\ -2 & 10 \end{pmatrix}$

Additions of matrices, like equality of matrices is defined only of matrices have same dimension.

Theorem 1.1**:**

Addition of matrices is commutative and associative, that is if A, B and C are matrices having the same dimension then:

A + B = B + A (commutative)

A + (b + C) = (A + B) + C (associative)

Definition 1.3

The product of a scalar K and an mxn matrix A = [aij] mxn is the nn,Xn matrix KA = [kaij] mXn for example:

$$6\begin{pmatrix} -1 & 0 & 7 \\ 5 & 2 & -11 \end{pmatrix} = \begin{pmatrix} 6(-1) & 6(0) & 6(7) \\ 6(5) & 6(2) & 6(-11) \end{pmatrix} = \begin{pmatrix} -6 & 0 & 2 \\ 30 & 12 & -6 \end{pmatrix}$$

**Application of Matrices**

Definition 1.4**:**

If A = [aij] mxn is mxn matrix and B = [bjk] nxp an nxp matrix, the product AB is the mxp matrix C = [cik] nxp in which

$$Cik = \sum_{j=1}^{n} aij\, bik$$

Example1: if A = $\begin{pmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \end{pmatrix}_{2x3}$ $and\ B = \begin{pmatrix} b11 \\ b21 \\ b22 \end{pmatrix}_{3\times 1}$

A B = $\begin{pmatrix} a11\, b11 + a12\, b21 + a13\, b31 \\ a21\, b11 + a22\, b21 + a23\, b31 \end{pmatrix}_{2\times 1}$

Example 2: Let A = $\begin{pmatrix} 2 & 3 \\ -1 & 4 \\ 5 & -2 \end{pmatrix}_{3\times 2}$ $and\ B = \begin{pmatrix} 3 & 1 & 4 & -5 \\ -2 & 0 & 3 & 4 \end{pmatrix}$

A B **=** $\begin{pmatrix} 0 & 2 & 17 & 2 \\ -11 & -1 & 8 & 21 \\ 19 & 5 & 14 & -33 \end{pmatrix}_{3\times 4}$

**Note 1.1:**

1 **–** in general if A and B are two matrices. Then A B may not be equal of

BA. For example A = $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} and\ B\,A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$\therefore A\,B \neq B\,A$

2 – if A B is defined then its not necessary that B A must also be defined.

For example. If A is of order (2×3) and B of order (3×1) then clearly A B is

define, but B A is not defined.

**1.3 Different Types of matrices:**

1 – Row Matrix: A matrix which has exactly one row is called row matrix.

For example (1, 2, 3, 4) is row matrix

2 – Column Matrix: A matrix which has exactly one column is called a

column matrix for example $\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$ is a column matrix.

3 – Square Matrix**:** A matrix in which the number of row is equal to the

number of columns is called a square matrix for example $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is a 2×2

square matrix.

A matrix (A) (n×n) A is said to be order  or to be an n-square matrix.

4 - Null or Zero Matrix**:** A matrix each of whose elements is zero is called null matrix or zero matrix, for example $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a (2×3) null matrix.

5 **–** Diagonal Matrix: the elements aii are called diagonal of a square matrix ($a_{11}$ $a_{22}$ – $a_{nn}$) constitute its main diagonal A square matrix whose every element other than diagonal elements is zero is called a diagonal matrix for

Example: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

6 – Scalar Matrix: A diagonal matrix, whose diagonal elements are equal, is called a scalar matrix.

For example $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ are scalar matrix

7 – Identity Matrix: A diagonal matrix whose diagonal elements are all equal to 1 (unity) is called identity matrix or (unit matrix). And denoted by in for

Example $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

**Note1.2:** if A is (mxn) matrix, it is easily to define that AIn = A and also ImA = A

**Ex:** Find AI and IA when A = $\begin{pmatrix} 3 & 7 & 2 \\ 1 & -1 & 3 \end{pmatrix}$

Solution**:** IA $\longrightarrow$ $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}_{2\times2}\begin{pmatrix}3 & 7 & 2\\ 1 & -4 & 3\end{pmatrix}_{2\times3} = \begin{pmatrix}3 & 7 & 2\\ 1 & -4 & 3\end{pmatrix}_{2\times3}$

And AI = $\begin{pmatrix}3 & 7 & 2\\ 1 & -4 & 3\end{pmatrix}_{2\times3} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}_{3\times3} = \begin{pmatrix}3 & 7 & 2\\ 1 & -4 & 3\end{pmatrix}_{2\times3}$

8 – Triangular Matrix**:** A square matrix (aij) whose element aij = 0 whenever $j \langle i$ is Called a lower triangular matrix.simillary y a square matrix (aij) whose element aij = 0 whenever is called an upper Tringular Matrix

For example: $\begin{pmatrix}1 & 0 & 0\\ 4 & 5 & 0\\ 7 & 8 & 9\end{pmatrix}$ , $\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$ are lower triangular matrix

And

$\begin{pmatrix}1 & 2 & 3\\ 0 & 4 & 5\\ 0 & 0 & 6\end{pmatrix}$ , $\begin{pmatrix}1 & 2\\ 0 & 3\end{pmatrix}$, are upper triangular

Definition 1.4:

**Transpose of matrix**

The transpose of an mxn matrix A is the nxm matrix denoted by $A^{\mathbf{T}}$ , formed by interchanging the rows and columns of A the ith rows of A is the ith columns in $A^{\mathbf{T}}$**.**

For Example**:** A = $\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & -1\end{pmatrix}_{2\times3}$ $A^T = \begin{pmatrix}1 & 0\\ 2 & 1\\ 3 & -1\end{pmatrix}_{3\times2}$

9 – Symmetric Matrix: A square matrix A such that A = $A^{\mathbf{T}}$ is called symmetric matrix i.e. A is a symmetric matrix if and only if aij = $a_{\mathbf{ji}}$ for all element.

$$\begin{pmatrix}0 & 1 & 2\\ 1 & 2 & 3\\ 2 & 3 & 4\end{pmatrix}$$

For Example: a $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, b

10 – Skew symmetric Matrix: A square matrix A such that A = $A^T$ is called that A is skew symmetric matrix. i.e A is skew matrix $\longleftrightarrow$ $a_{ji}$ = -aij for all element of A.

The following are examples of symmetric and skew – symmetric matrices respectively

$$(a)\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}, (b)\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}$$

(a) symmetric

(b) Skew – symmetric.

Note the fact that the main diagonal element of a skew – symmetric matrix must all be Zero

11 – Determinates: To every square matrix that is assigned a specific number called the determinates of the matrix.

(a) Determinates of order one: write det (A) or $|A|$ for detrimental of the matrix A. it is a number assigned to square matrix only.

The determinant of (1×1) matrix (a) is the number a itself det (a) = a.

(c) Determinants of order two: the determinant of the 2×2. matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Is denoted and defined as follows: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Theorem 1.2: determinant of a product of matrices is the product of the determinant of the matrices is the product of the determinant of the matrices det (A B) = det (A). det (B) det (A + B) # det a + det B

(C) Determinates of order three:

(i) the determinant of matrix is defined as follows:

$$\begin{vmatrix} + & - & + \\ a111 & a22 & a23 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{vmatrix} = a11\begin{vmatrix} a22 & a23 \\ a32 & a33 \end{vmatrix} - a12\begin{vmatrix} a21 & a23 \\ a31 & a33 \end{vmatrix} + a13\begin{vmatrix} a21 & a22 \\ a31 & a32 \end{vmatrix}$$

(ii) Consider the (3×3) matrix $\begin{vmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{vmatrix}$

= a11 a22 a33 + a21 a22 a31 + a13 a21 a32

- a31 a22 a13 - a32 a23 a11 -a33 a21 a12.

Show that the diagram papering below where the first two columns are rewritten to the right of the matrix.

Theorem 1.3:

A matrix is invertible if and only if its determinant is <u>not Zero</u> usually a matrix is said to be singular if determinant is zero and non singular it otherwise.

**1.5 prosperities of Determinants**

(1) det A = det $A^T$ where $A^T$ is the transpose of A.

(2) if any two rows (or two columns) of a determinates are interchanged the value of determinants is multiplied by -1.

(3) if all elements in row (or column) of a square matrix are zero.

Then det (A) = 0

(4) if two parallel column (rows) of square matrix A are equal then det (A) = 0

(5) if all the elements of one row (or one column) of a determinant are multiplied by the same factor K. the value of the new determinant is K times the given det.

Example;

$$\begin{pmatrix} 4 & 6 & 1 \\ 3 & -9 & 2 \\ -1 & 12 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 2.3 & 1 \\ 3 & -3.3 & 2 \\ -1 & 4.3 & 3 \end{pmatrix}$$

$$= 3 \begin{pmatrix} 4 & 2 & 1 \\ 3 & -3 & 2 \\ -1 & 4 & 3 \\ & & \end{pmatrix}$$

Example: $\begin{pmatrix} 1 & 0 & 4 \\ -2 & 5 & -8 \\ 3 & 6 & 12 \end{pmatrix} = 4 \begin{pmatrix} 1 & 0 & 1 \\ -2 & 5 & -2 \\ 3 & 6 & 3 \end{pmatrix} = 0$

(6) if to each element of a selected row (or column) of a square matrix = k times. The corresponding element of another selected row (or column) is added.

Example: $\begin{vmatrix} 2 & 0 & 2 \\ 1 & -1 & +1 \\ 3 & 0 & 2 \end{vmatrix} = -1 \begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} = 2$

2× row (1) + row (3) $\begin{vmatrix} 2 & 0 & 2 \\ 1 & -1 & 1 \\ 7 & 0 & 6 \end{vmatrix} = -1 \begin{vmatrix} 2 & 2 \\ 7 & 6 \end{vmatrix} = 2$

(7) if any row or column contain zero elements and only one element not zero then the determinant will reduced by elementary the row and column if the specified element indeterminate.

**1.6 Rank of Matrix:** we defined the rank of any matrix a that the order of the largest square sub-matrix of a whose determinant not zero (det of sub-matrix ‡ 0)

Example**:** Let A = $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ find the rank of A

1 × 9×5 + 2 × 6 × 7 + 3 × 4 × 8 – 3 × 5 × 7 – 1 × 6 × 8 – 2 × 4 × 9 = 0

Since $|A|$ of order 3 Rank ‡ 3

Since $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3 \neq 0$ the rank ‡ 2

1.7 Minor of matrix**:** Let A = $\begin{pmatrix} a11 & a12 & a1n \\ a21 & a22 & a2n \\ & & \\ an1 & an2 & ann \end{pmatrix}$ (4)

Is the square matrix of order n then the determinant of any square sub-matrix of a with order (n-1) obtained by deleting row and column is called the minor of A and denoted by Mij.

1.8 Cofactor of matrix: Let A be square matrix in (4) with mij which is the minors of its. Then the Cofactor of a defined by Cij = $(-1)^{i+j}$ Mij

Example: Let A = $\begin{pmatrix} -2 & 4 & 1 \\ 4 & 5 & 7 \\ -6 & 1 & 0 \end{pmatrix}$ find the minor and the cofactor of element 7.

Solution: The minor of element 7 is

M23 = det $\begin{pmatrix} -2 & 4 \\ -6 & 1 \end{pmatrix} = \begin{vmatrix} -2 & 4 \\ -6 & 1 \end{vmatrix} = 22$

i.e (denoted by take the square sub-matrix by deleting the second rows and third column in A).

the Cofactor of 7 is

$$C23=(-1)^{2+3}M_{23}=(-1)^{2+3}\begin{vmatrix}-2 & 4\\ -6 & 1\end{vmatrix}=-22$$

1.9 Adjoint of matrix: Let matrix A in (4) then the transposed of matrix of cofactor of this matrix is called adjoint of A, adjoint A = transposed matrix of Cofactor.

The inverse of matrix**:** Let A be square matrix. Then inverse of matrix {Where A is non-singular matrix} denoted by $A^{-1}$ and $A^{-1} = \frac{1}{\det A} adj(A)$

1.0 method to find the inverse of A**:** To find the inverse of matrix we must find the following:

(i) the matrix of minor of elements of A.

(ii) the Cofactor of minor of elements of A

(iii) the adjoint of A .

then $A^{-1} = \frac{1}{|A|} adjA$

Example: let A = $\begin{pmatrix}2 & 3 & -4\\ 1 & 2 & 3\\ 3 & -1 & -1\end{pmatrix}$ Find $A^{-1}$

**(1)** Minors of A is Mij = $\begin{pmatrix}1 & -10 & -7\\ -7 & 10 & -11\\ 17 & 10 & 1\end{pmatrix}$

**(2)** Cofactor of A is (-1) Mij = $\begin{pmatrix}1 & 10 & -7\\ 7 & 10 & 11\\ 17 & -10 & 1\end{pmatrix}$

**(3)** Adj of A = $\begin{pmatrix}1 & 7 & 17\\ 10 & 10 & -10\\ -7 & 11 & 1\end{pmatrix}$.

**(4)** det = 60

$$A^{-1} = \frac{1}{60}\begin{pmatrix} 1 & 7 & 17 \\ 10 & 10 & -10 \\ -7 & 11 & 1 \end{pmatrix}.$$

1.11 Properties of Matrix Multiplication:

1 – (KA) B = K (AB) = A (KB)   K is any number

2 – A (BC) = (AB) C

3 – (A + B) C = AC + BC

4 – C (A + B) = CA + CB

5 – AB ‡ BA (in general)

For example: Let A = $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and B = $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

A B = $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

B A = $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

A B = ‡ B A

6 – A B = 0 but not necessarily A = 0 or B = 0

For Example: A = $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, B = \begin{pmatrix} -1 & +1 \\ +1 & -1 \end{pmatrix}$

$$A\,B = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$A \neq 0, B \neq 0$

But

A B = 0

7 - $\begin{pmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{pmatrix} = C\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

8 – A I = I A = A   where I is identity matrix

9 – $(A\ B)^{T} = B^{T}\ A^{T}$

10 – $A^{-1}\ A = A.A^{-1} = I$

**1.12 Cramer's Rule**

Let the system of linear question as

$$\left.\begin{matrix} a_{11}\ \chi_1 + a_{12}\ \chi_2 = b1 \\ a_{21}\ \chi_1 + a_{22}\ \chi_2 = b2 \end{matrix}\right\} \rightarrow (i)$$

The system (i) can put in the form:

$$\begin{pmatrix} a_{11} & a_{12} \\ & \\ a_{11} & a_{22} \end{pmatrix} \begin{pmatrix} \chi_1 \\ \\ \chi_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \rightarrow (ii)$$

If D = $\begin{vmatrix} a11 & a12 \\ a21 & a22 \end{vmatrix} \neq 0$

Then the system (ii) has a unique solution, and Cramer's rule state that it may be found from the formulas:

$$\chi_1 = \frac{\begin{vmatrix} b1 & a12 \\ b2 & a22 \end{vmatrix}}{D}\ ,\ X_2 = \frac{\begin{vmatrix} a11 & b1 \\ a21 & b2 \end{vmatrix}}{D}$$

Example: solve the system

$3X_1 - \chi_2 = 9$

$X_1 + 2X_2 = -4$

So, the system can put in the form

$$\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}, \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, = \begin{pmatrix} 9 \\ -4 \end{pmatrix}$$

$$D = \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} = 7 \ , \ \chi_1 = \frac{\begin{vmatrix} 9 & -1 \\ -4 & 2 \end{vmatrix}}{D} = \frac{14}{7} = 2$$

$$\chi_2 = \frac{\begin{vmatrix} 3 & 9 \\ 1 & -4 \end{vmatrix}}{D} = \frac{21}{7} = 3$$

Let the following system in the unknowns:

$$\begin{aligned} a11\ \chi_1 + a11\ \chi_2 + a11\,3X_3 &= b1 \\ a21\ \chi_1 + a22\ \chi_2 + a23\,X_3 &= b2 \\ a31\ \chi_1 + a32\,X_2 + a33\ \chi_3 &= b3 \end{aligned}$$

The system (I) can be put in the form:

$$\begin{pmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} = \begin{pmatrix} b1 \\ b2 \\ b3 \end{pmatrix} \qquad \text{(II)}$$

If $D = \begin{vmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{vmatrix} \neq 0$

The system has a unique solution, given by Cramer's rule:

$$\chi_1 = \frac{1}{D}\begin{vmatrix} b1 & a12 & a13 \\ b2 & a22 & a23 \\ b3 & a32 & a33 \end{vmatrix},\ X_2 = \frac{1}{D}\begin{vmatrix} a11 & b1 & a13 \\ a21 & b2 & a23 \\ a31 & b3 & a33 \end{vmatrix}\ X_3 = \frac{1}{D}\begin{vmatrix} a11 & a12 & b1 \\ a21 & a22 & b2 \\ a31 & a32 & b3 \end{vmatrix}$$

Example: solve the system

$$\begin{aligned} X_1 + 3X_2 - 2X_3 &= 11 \\ 4X_1 - 2X_2 + X_3 &= -15 \\ 3X_1 + 4X_2 - X_3 &= 3 \end{aligned}$$

By cramer’s rule.

The system (1) become $\begin{pmatrix} 1 & 3 & -2 \\ 4 & -2 & 1 \\ 3 & 4 & -1 \end{pmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{pmatrix} 11 \\ -15 \\ 3 \end{pmatrix}$

Since D = det = $\begin{vmatrix} 1 & 3 & -2 \\ 4 & -2 & 1 \\ 3 & 4 & -1 \end{vmatrix} = -25$

Cramer’s rule gives the solution:

$$\chi_1 = \frac{\begin{vmatrix} 11 & 3 & -2 \\ -15 & -2 & 1 \\ 3 & 4 & -1 \end{vmatrix}}{-25} = \frac{50}{-25} = -2$$

$$\chi_2 = \frac{\begin{vmatrix} 1 & 11 & -2 \\ 4 & -15 & 1 \\ 3 & 3 & -1 \end{vmatrix}}{-25} = \frac{-25}{-25} = 1$$

$$\chi_3 = \frac{\begin{vmatrix} 1 & 3 & 11 \\ 4 & -2 & -15 \end{vmatrix}}{-25} = \frac{125}{-25} = -5$$

# Chapter Two

**Function**

**Numbers:**

1 – N = set of natural numbers

N = {1, 2, 3, 4……}

2 – I = set of integers

= {……., -3, -2, -1, 0, 1, 2, 3…}

Note that: NCI

3 – A = set of rational numbers

$= \left( \chi : \chi = \frac{\rho}{q} \; \rho \, and \, q \, are \, \text{int} \, egers \, q \neq 03 \right]$

Ex: $\frac{3}{2}, -\frac{4}{5}, \frac{3}{1}, \frac{-7}{1}$

Note that: ICA

4 – B = set of irrational numbers

= $\{X : X \; is \, not \, a rational \, number\}$

Ex: $\sqrt{2}, \sqrt{3}, -\sqrt{7}$

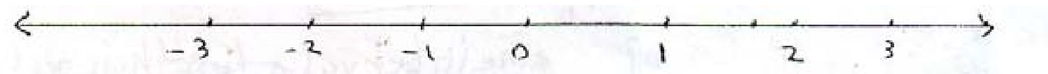
5 – R: set of real numbers

= set of all rational and irrational numbers

Note that

R = AUB

Note: the set of real numbers is represented by a line called a line of numbers:




(ii) NCR, ICR, ACR, BCR

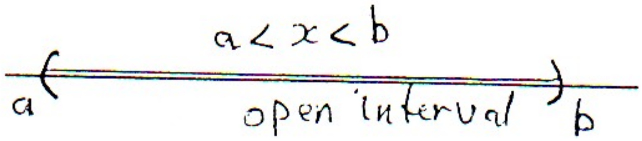
**Intervals**

The set of values that a variable $\chi$ may take on is called the domain of $\chi$. The domains of the variables in many applications of calculus are intervals like those shows below.

- **open intervals**

is the set of all real numbers that lie strictly between two fixed numbers a and b:

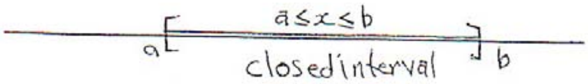
| **In symbols** | **In words** |
| --- | --- |
| $a \langle \chi \langle b \, or \, (q, b)$ | The open interval a b |




- Closed Intervals contain both endpoints:

**In symbols** **In words**

$a \le \chi \le b \quad or \quad [a,b]$ the closed interval a b

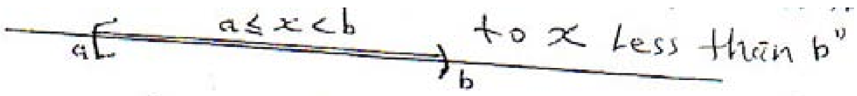



- Half – open intervals contain one but not both end points:

**In symbols:** **in wards**

$a \le \chi \langle b$ or [a,b] ‘ the interval a less than or equal

$a \le \chi c b$ To $\chi$ less than b




$a \langle \chi \le b$ or [a,b] the interval a less than $\chi$ less than or equal b

Ex. find the domain of

1 – Y = $\sqrt{1-X^2}$

The domain of $\chi$ is the closed interval

$1 - \le \chi \le 1$

2 – Y = $\frac{1}{\sqrt{1-X^2}}$

The domain for $\chi$ is open interval

$-1 \langle \chi \langle 1$ because $\frac{1}{0}$ is not defined

B - y = $\sqrt{\frac{1}{X}-1}$

$1/X - 1 \ge 0 \quad or \quad \frac{1}{X} \ge 1$

The domain for $\chi$ is the half – open $0 \langle \chi \leq 1$

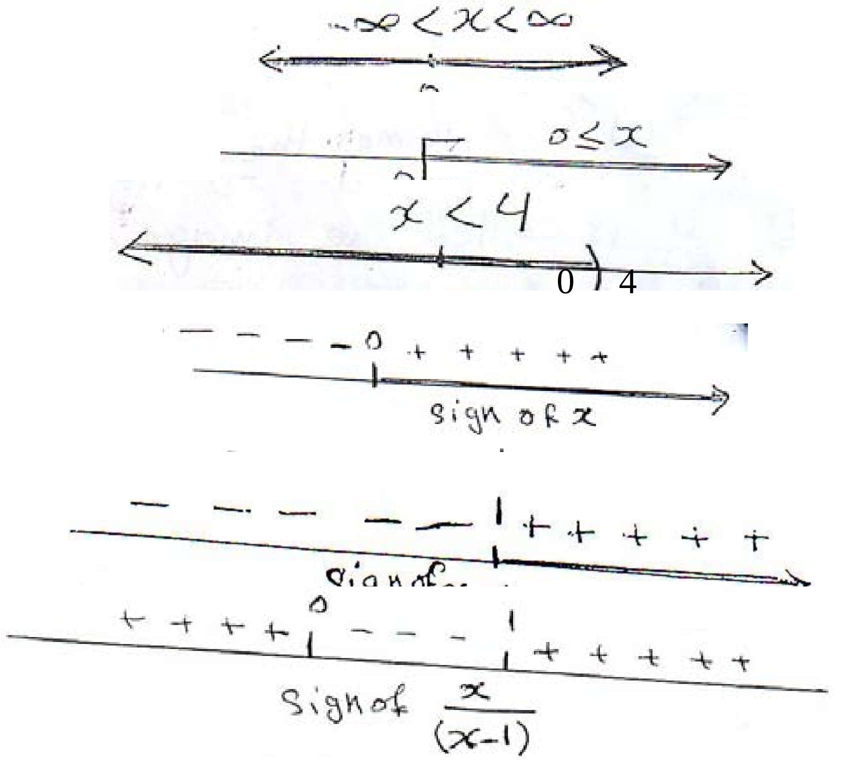
Ex: **the equation** **the domain**

Y = $\chi^2$

Y = $\sqrt{\chi}$

Y = $\dfrac{1}{\sqrt{4-X}}$

$y = \sqrt{\dfrac{\chi}{(\chi - 1)}}$




The domain for $\chi$ is $X \leq 0 U X \rangle 1$

Definition: A function, say f is a relation between the elements of two sets say A and B such that for every $\chi \in A$ there exists one and only one $Y \in B$ with Y = F(X).

The set A which contain the values of $\chi$ is called the domain of function F.

The set B which contains the values of Y corresponding to the values of $\chi$ is called the range of the function F. $\chi$ is called the independer variable of the function F, while Y is called the dependant variable of F.

**Note:**

1 – Some times the domain is denoted by DF and the range by RF.

2 – Y is called the image of $\chi$.

Example: Let the domain of $\chi$ be the set {0,1,2,3,4}. Assign to each value of $\chi$ the number Y = $\chi^2$. The function so defined is the set of pairs, { (0,0), (1,1), (2,4), (3,9), (4,16) }.

Example: Let the domain of $\chi$ be the closed interval $-2 \le \chi \le 2$. Assign to each value of $\chi$ the number y = $\chi^2$.

The set of order pairs $(\chi, y)$ such that $-2 \le \chi \le 2$

And y = $\chi^2$ is a function.

**Note**: Now can describe function by two things:

1 – the domain of the first variable $\chi$.

2 – the rule or condition that the pairs $(\chi, y)$ must satistfy to belong to the function.

**Example**:

The function that pairs with each value of $\chi$ diffrent from 2 the number

$$\frac{\chi}{\chi - 2}$$

$$y = f(\chi) = \frac{\chi}{\chi - 2} \quad \chi \ne 2$$

Note 2: Let $f(\chi)$ and $g(\chi)$ be two function.

1 - $(f \pm g)(\chi) = f(\chi) \pm g(\chi)$

2 - $(f.g)(\chi) = f(\chi) \cdot g(\chi)$

3 - $(\frac{f}{g})(\chi) = \frac{f(\chi)}{g(\chi)}$ if g $(\chi) \ne 0$

Example: Let $f(\chi) = \chi + 2, g(\chi) = \sqrt{\chi - 3}$ evaluate

$$f \pm g, f.g \; and \; \frac{f}{g}$$

So: $(f \pm g)(\chi) = f(\chi) \pm g(\chi) = \chi + 2 \pm (\sqrt{\chi - 3}$

$(f.g)(\chi) = f(\chi) \cdot g(\chi) = (\chi + 2)(\sqrt{\chi - 3}$

$(\frac{f}{g})(\chi) = \frac{f(\chi)}{g(\chi)} = \frac{\chi + 2}{\sqrt{\chi - 3}} \quad \{X : X \rangle 3\}$

**Composition of Function:**

Let $f(\chi)$ and $g(\chi)$ be two functions

We define: $(fog)(\chi) = f(g(\chi))$

Example: Let $f(\chi) = \chi^2, g(\chi) = \chi - 7$ evaluate fog and gof

So: $(fog)(\chi) = f[g(x)] = f(\chi - 7) = (\chi - 7)^2$

$(gof)(\chi) = g[f(\chi)] = g(\chi^2) = \chi^2 - 7$

$\therefore fog \neq gof$

**Inverse Function**

Given a function F with domain A and the range B.

The inverse function of f written f, is a function with domain B and range A such that for every $y \in B$ there exists only $\chi \in A$ with $\chi = f^{-1}(y)$.

Note that: $f^{-1} \neq \frac{1}{f}$

**Polynomials**: A polynomial of degree n with independent variable, written $\mathbf{f_n(x)}$ or simply $f(\chi)$ is an expression of the form:

$fn(\chi) = q_o + a_1\chi + a_2X^2 + \ldots\ldots\ldots + anX^n \ldots\ldots\ldots.(*)$

Where $q_0, a1, \ldots\ldots., an$ are constant (numbers).

The degree of polynomial in equation ( * ) is n ( the highest power of equation)

Example:

(i) f $(\chi)$ = 5X polynomial of degree one.

(ii) f $(\chi) = 3X^5 - 2X + 7$ polynomial of degree five.

(iii) F $(\chi)$ = 8 polynomial of degree Zero.

**Notes:**

The value of $\chi$ which make the polynomial f $(\chi)$ = 0 are called the roots of the equation $\left(f(\chi) = 0\right)$

Example: $(\chi)$ = 2 is the root of the polynomial

F $(\chi)$ = $\chi^2 - \chi - 2$

Since f (2) = 0

Example: F $(\chi)$ Linear function if

F $(\chi)$ = $a\chi + b$.

**Even Function**:

F $(\chi)$ is even if f (-x) = F (x)

Example: 1 - F $(\chi)$ = $(\chi)^2$ is even since f$(-\chi)$ = $(-\chi)^2$ = $(\chi)^2$ = f $(\chi)$

2 - F $(\chi)$ = cos $(\chi)$ is even because f$(-\chi)$ = cos $(-\chi)$ = cos $(\chi)$ = f $(\chi)$

**Odd Function:**

If $f(-\chi) = -f(\chi)$ the function is called odd.

Example: 1 - $f(\chi) = \chi^3$ is odd since $f(-\chi) = -\chi 3 = -f(\chi)$

2 - $f(\chi) = Sin(-\chi) = -Sin\,X = -f(\chi)$.

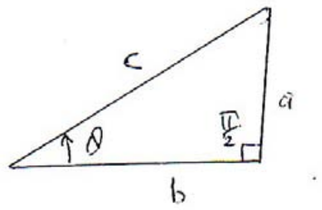
**Trigonometric Function:**

1 – Sin $\varphi$ = $\frac{a}{c}$

2 – Cos $\varphi = \frac{b}{c}$

3 – tan $\varphi = \frac{a}{b}$

4 – Cotan $\varphi = \frac{1}{\tan\varphi} = \frac{b}{a}$

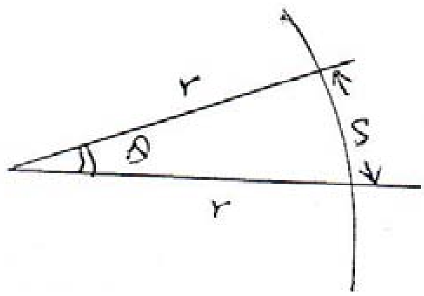



5 – Sec $\varphi = \frac{1}{\cos\varphi} = \frac{c}{b}$

6- CSC $\varphi = \frac{1}{Sin\varphi} = \frac{c}{a}$

**Relation ships between degrees and radians**

$\varphi$ In radius **=** $\frac{s}{r}$




$360^{\mathbf{o}} = \frac{2\pi r}{r}$
$= 2\pi\, radius$

$1^{\mathbf{o}} = \frac{\pi}{180}$ radius = 0.0174 radian

1 radian = $\frac{180}{\pi} \deg ree = 57.29578^{o}$

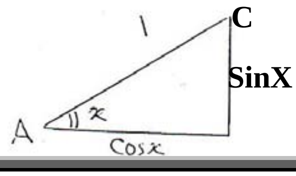
$\left(\frac{360}{2\pi}\right) = 1 radian = 57^{o}.18$

$180^{\mathbf{o}} = \pi$ radians = 3.14159 – radians

$1^{\mathbf{o}} = \frac{2\pi}{360} = \frac{\pi}{180} \approx 0.001754$ radians

$\tan\chi = \frac{\sin\chi}{\cos\chi}$

$Cot\chi = \frac{Cos\chi}{Sin\chi} = \frac{1}{\tan\chi}$



$$Sec\,\chi = \frac{1}{\cos\chi}$$ **B**

$$Csc\,\chi = \frac{1}{\sin\chi}$$

$$Cos^2\chi + Sin^2\chi = 1$$

$$\tan^2\chi + 1 = Sec^2\chi$$

$$Cot^2\chi + 1 = Csc^2 X$$

$$Sin(\chi \pm y) = Sin \times Cosy \; \mu \; Cos \times Sin\, y$$

$$Cos(\chi \pm y) = Cos \times Cosy \pm Sin \times Siny$$

$$\tan(\chi \pm y) = \frac{\tan x \pm \tan y}{1 \,\mu \tan x \tan y}$$

1 – $SinA + Sin B = 2\,Sin\,\frac{A+B}{2}\,Cos\,\frac{A-B}{2}$

2 - $Sin\,A - Sin\,B = 2\,Cos\,\frac{A+B}{2}\,Sin\,\frac{A-B}{2}$

3 – $Cos\,A + Cos\,b = 2\,Cos\,\frac{A+B}{2}\,Cos\,\frac{A-B}{2}$

4 - $Cos\,A - Cos\,B = 2\,Sin\,\frac{A+B}{2}\,Sin\,\frac{A-B}{2}$

$$Sin\,2X = 2\,Sin\,X\,CosX$$

$$Cos^2 = Cos^2 X - Sin^2 X$$

$$= 1 - 2\,Sin^2 X$$

$$= 2Cos^2 X - 1$$

$$Cos^2 x = \frac{1 + Cos^2 x}{2}$$

$$Sin^2 x = \frac{1 - Cos^2 x}{2}$$

$$Sin(\varphi + 2\pi) = Sin\,\varphi$$

$$Cos(\varphi + 2\pi) = Cos\,\varphi$$

$$\tan(\varphi + \pi) = \tan\varphi$$

| Degree | $O^{o}$ | $30^{o}$ | $45^{o}$ | $60^{o}$ | $90^{o}$ | $180^{o}$ | $270^{o}$ | $360^{o}$ |
|---|---|---|---|---|---|---|---|---|
| $\theta$ radius | O | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
| Sin $\theta$ | O | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\sqrt{\frac{3}{2}}$ | 1 | O | -1 | O |
| Cos $\theta$ | 1 | $\sqrt{\frac{3}{2}}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | O | -1 | O | 1 |
| tan $\theta$ | O | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | | | | |

$Cos(\varphi + 2n\pi) = Cos\varphi$

$Sin(\varphi + 2n\pi) = Sin\varphi$

$Cos(-\varphi) = Cos\varphi$

$Sin(-\varphi) = -Sin\varphi$

$Cos(\frac{\pi}{2} + \varphi) = Cos\varphi$

$\tan(\pi - \varphi) = -\tan\varphi$

$\tan(\frac{\pi}{2} + \varphi) = -Cot\varphi$

**Graphs:**

The set of points in the plane whose coordinate pairs are also the ordered pairs of function is called the graph of function.

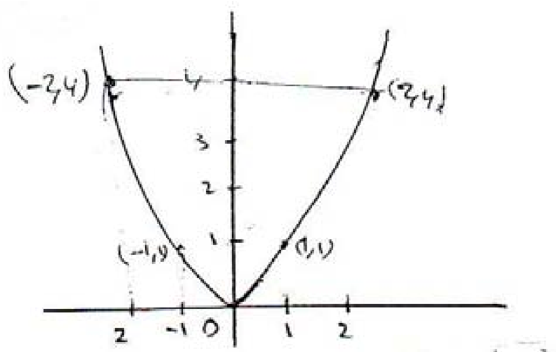
Example: Graph a function we carry out three steps $y = \chi^2, -2 \le x \le 2$

1 – Make a table of pairs from the function as

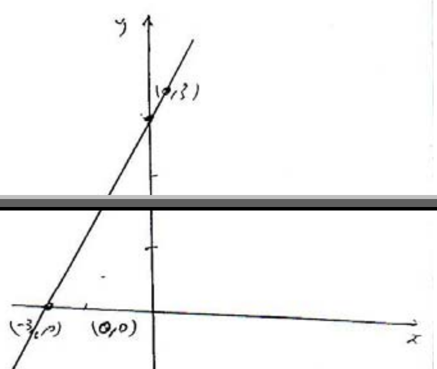
| $\chi$ | $y = \chi^2$ | $(\chi, y)$ |
|---|---|---|
| -2 | 4 | (2,4) |
| -1 | 1 | (-1,1) |
| 0 | 0 | (0,0) |
| +1 | 1 | (1,1) |
| 2 | 4 | (2,4) |

2 – Plot enough of the corresponding points to learn the shape of the graph. Add more pairs to the table if necessary.

3 – Complete the sketch by connecting the points.




Example: $y = 2\chi + 3$



| X | $y = 2X + 3$ | (X,y) |
|---|---|---|
| O | 3 | (0,3) |
| $-\frac{3}{2}$ | O | $(\frac{-3}{2},0)$ |

**Absolute Value:**

We define the absolute value function $y = |\chi|$, the function assign every negative number to non-negative, which corresponding points.

The absolute values of X:

$$|X| = \sqrt{X^2} = \begin{cases} \chi & if X \geq 0 \\ -\chi & if X \langle 0 \end{cases}$$

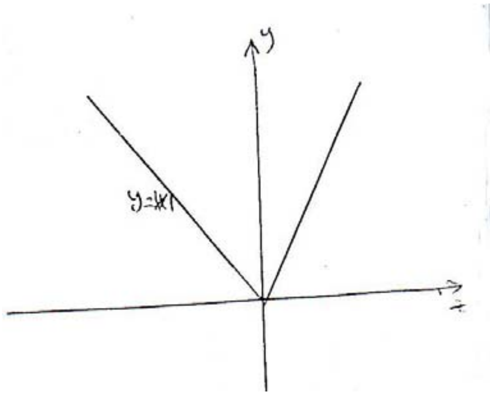
Then:

1 - $|a.b| = |a|,|b|$

2 - $|a + b| \leq |a| + |b|$

3 - $|a| \leq C \Leftrightarrow -C \leq a \leq C$

$y = f(\chi) = \chi$



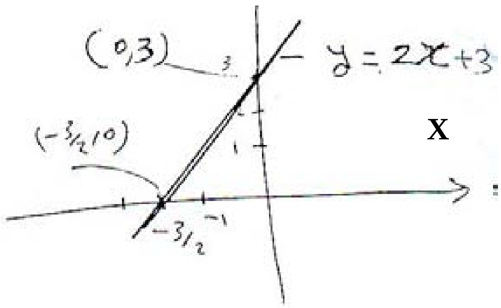

| X | y | (X,y) |
|---|---|---|
| -1 | 1 | ( -1,1) |
| 0 | 0 | (0,0) |
| 1 | 1 | (1,1) |

**Example 2:**

$y = f(\chi) = a\chi + b$

$y = f(\chi) = 2\chi + 3$

| X | y | (X,y) |
|---|---|---|
| O | 3 | (0,3) |
| $-\frac{3}{2}$ | 0 | $(-\frac{3}{2}, 0)$ |



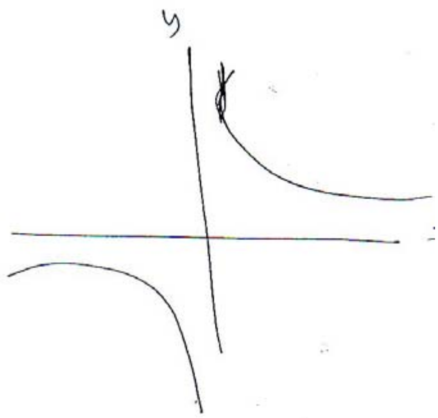

**Example 3:**

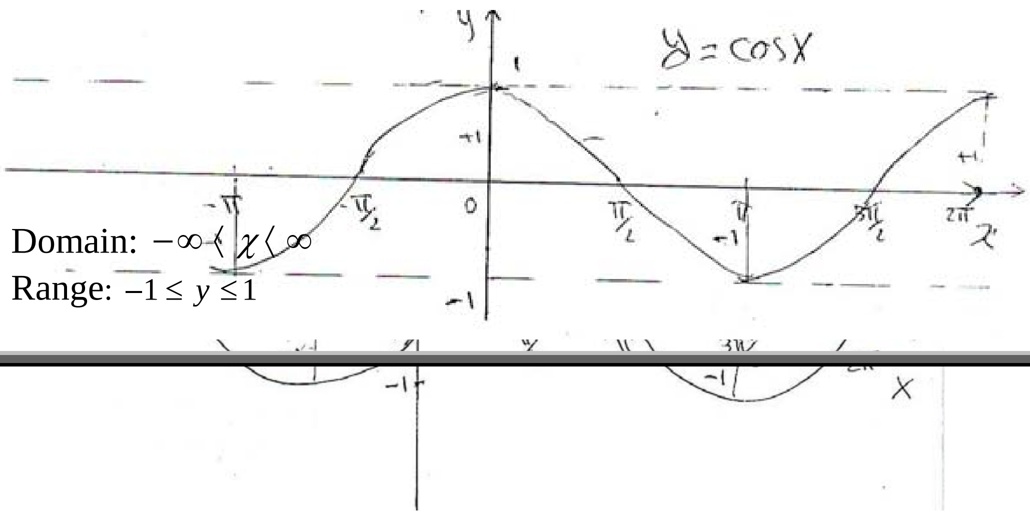
$y = f(\chi) = \frac{1}{\chi}$

| X | y | (X,y) |
|---|---|---|
| 1 | 1 | (1,1) |
| -1 | -1 | (-1,1) |
| $\frac{1}{2}$ | 2 | $(\frac{1}{2},2)$ |
| $\frac{1}{3}$ | 3 | $(\frac{1}{3},3)$ |
| 2 | $\frac{1}{2}$ | $(2, \frac{1}{2})$ |
| 3 | $\frac{1}{3}$ | $(3, \frac{1}{3})$ |

Domain: $-\infty \langle \chi \langle \infty$

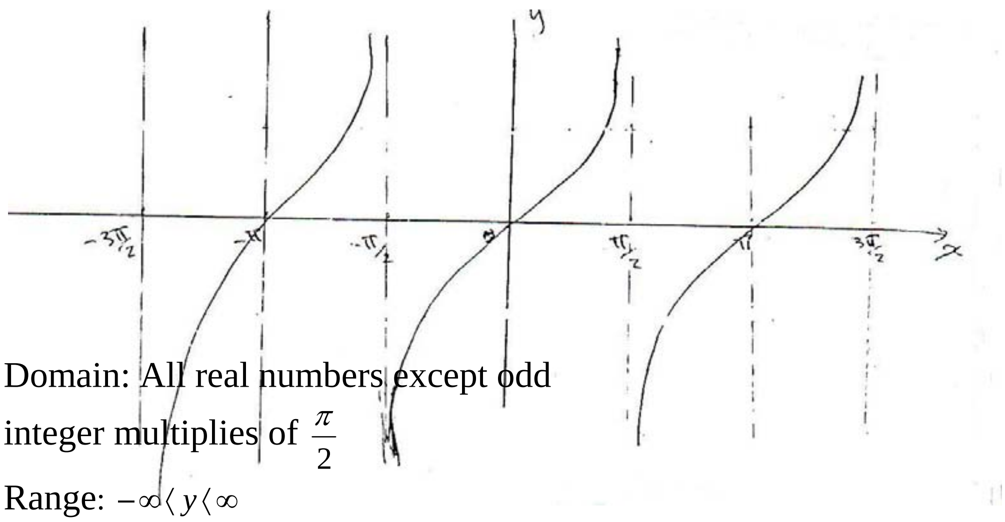
Range: $-1 \le y \le 1$



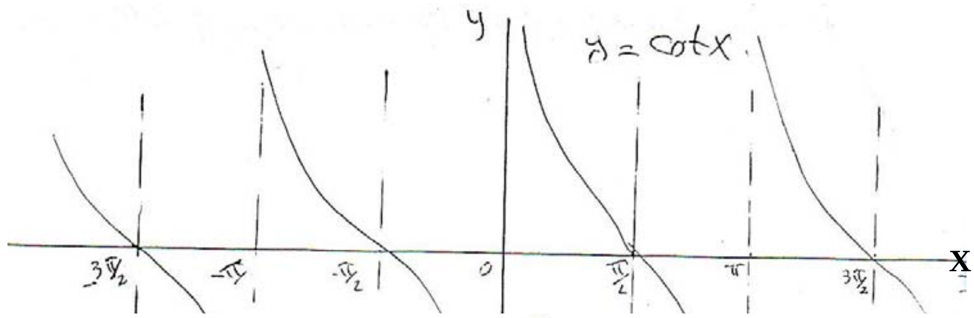

Domain: $-\infty \langle \chi \langle \infty$

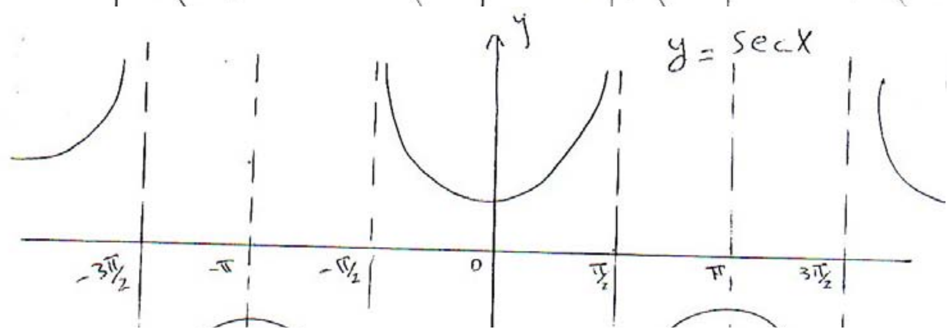
Range: $-1 \le y \le 1$

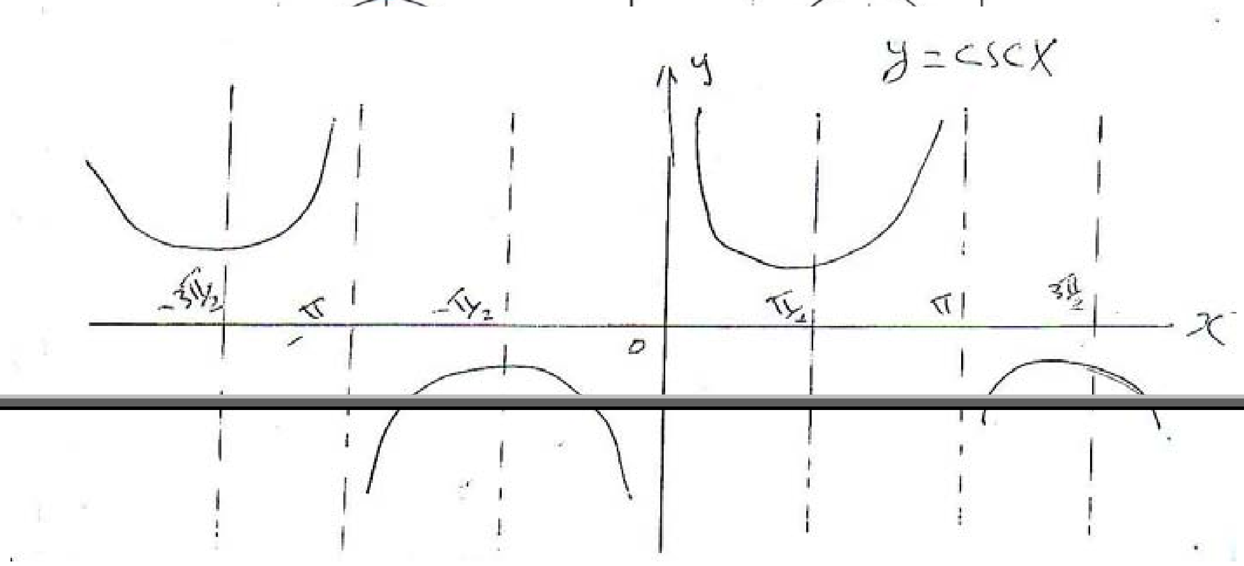


Domain: All real numbers except odd integer multiplies of $\frac{\pi}{2}$

Range: $-\infty \langle\, y \,\langle \infty$



**Limits**:

We say that L is a right hand limit for $f(\chi)$ when X approaches C for the right, written

$$\lim_{X \to C^{+}} f(\chi) = L.$$

Similary, L is the left – hand limit for $f(\chi)$ when X approaches C for the left, written

$$\lim_{X \to C^{-}} = f(\chi) = L,.$$

Then $\lim_{X \to C} = f(\chi) = L,$

If and.only if $\lim_{\chi \to C^{+}} f(X) = \lim_{\chi \to C^{-}} f(x)$

Example:

$$\lim_{\chi \to 1} \frac{\chi \pm 1}{\chi - 1} = \lim_{\chi \to 1} \frac{(\chi-1)(\chi+1)}{(\chi-1)}$$

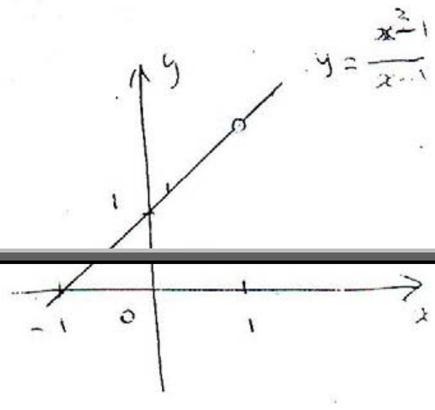
$$\lim_{\chi \to 1} \times + 1 = 1 + 1 = 2$$

Theorem 1

If $\lim_{\chi \to c} f(\chi) = L_1, \lim_{\chi \to c} g(\chi) = L_2$

Then

1 - $\lim_{\chi \to c} [\, f(\chi) \pm g(\chi)] = L1 \pm L2$



2 – Lim $[f(\chi)g(\chi)] = L1 \cdot L2$

3 - $Lim\left[\frac{f(\chi)}{g(\chi)}\right] = \frac{L1}{L2}$ $if\ L2 \neq 0$

4 - $\underset{\chi \to c}{Lim}\ [K\ f(\chi)] = KL_1$ Where K is a constant

Theorem 2

1 - $Lim\, K = K$, K is constant

2 - $\underset{\chi \to c}{Lim}[a0 + a1\chi_1 + a2\chi^2 + ......+ an\chi^n] = a_0 + a_1c + a_2c^2 .......anc^n$

3 - $\underset{\chi \to 0}{Lim}\, smx = 0$

4 - $\underset{\chi \to 0}{Lim}\ Cos\,\chi = 1$

5 - $\underset{\chi \to 0}{Lim}\frac{Sin\,\chi}{X} = 1$

Example: Evaluate

1 - $\underset{\chi \to 2}{Lim}(4\chi^2) = 4\underset{\chi \to 2}{Lim}\,\chi^2 = 4(2)^2 = 16$

2 - $\underset{x \to 2}{Lim}(\chi^2 - 9) = 4 - 9 = 5$

$3 - Lim\frac{\chi^3 - 8}{\chi^2 - 4} = \frac{(\chi - 2)(\chi^2 + 2\chi + 4)}{(\chi - 2)(\chi + 2)} = \frac{\chi^2 + 2\chi + 4}{\chi + 2}$

$= \frac{12}{4} = 3$

$4 - \underset{\chi \to 0}{Lim}\frac{Sin\chi}{\chi} = 3\underset{3\chi \to 0}{Lim}\frac{Sin3\chi}{3\chi} = 3(1) = 3$

$$5-\underset{\chi\to 0}{Lim}\frac{\tan x}{\chi}=\frac{\frac{Sin\,\chi}{Cos\,\chi}}{\chi}=\underset{X\to 0}{Lim}\left[\frac{Sin\,\chi}{Cos}.\frac{1}{\chi}\right]$$

$$=\underset{X\to 0}{Lim}\left[\frac{Sin\,\chi}{\chi}.\frac{1}{Cos\,\chi}\right]$$

$$=\left[\underset{\chi\to 0}{Lim}\frac{Sin\,\chi}{\chi}\right].\left[\underset{\chi\to 0}{Lim}\frac{1}{Cos\,\chi}\right]=(1)\ (1)=1$$

**Infinity as Limits**

Evaluate:

1 - $\underset{X\to 0^+}{Lim}\ \frac{1}{X}=\infty$ (2) $\underset{\chi\to 0^-}{Lim}\ \frac{1}{X}=-\infty$

3 - $\underset{x\to\infty}{Lim}\frac{1}{X}=0$

4 - $\underset{x\to\infty}{Lim}\ \frac{2\chi 2-\chi+3}{3\chi^2-5}=\underset{x\to\infty}{Lim}\frac{\frac{2x2}{X2}}{\frac{3\chi^2}{\chi^2}-\frac{3}{X^2}}$

Theorem

If $f(\chi)\le g(\chi)\le h(X)$ and $Lim\, f(\chi)=Lim\, h(X)=L$ then

L is the limit of g(x)

Example: Evaluate

1 - $\underset{X\to\infty}{Lim}\frac{Sin\,X}{X}$

2 - $\underset{X \to \infty}{Lim}\, \chi\, Sin(\frac{1}{X})$

**Continuity**

Definition: A function f is said to be continuous at $\chi = C$ provided the following conditions are satisfied:

1 $f(C)$ is defined

2 $\underset{\chi \to C}{Lim}\, f(x)$ exists

3 $\underset{x \to C}{Lim}\, f(x) = f(C)$

Theorem

Any Polynomial

1 $P(\chi) = a_0 + a_1\chi + a_2 X^2 + ........ + an\, X^4$ (an $\neq 0$)

Is continuous for all $\chi$

2 $R(\chi) = \frac{a_0 + a1X + a_2 X^2 + ...... + anX^4}{bo + b_1 X + b_2 X^2 ....... + bnX^4}$ $(an \neq 0, bn \neq 0)$

Is continuous at every point of its domain of definition that is at every point where its denominator isd not zero

3 Each of the igonometric function SinX, CosX, tanX, CotX SecX, and CscX, is continuous at every point of its domain of definition.

Example1

$\underset{X \to \pi}{Lim}(Cos^2 X + Cos\, X + 1)$

Solution

$Lim(Cos^2 X + CosX + 1) = (Cos^2 \pi + Cos\, \pi + 1)$

$= (-1)^2 - 1 + 1) = 1$

Example2

$f(\chi) = \frac{|X|}{X}$ Discontinuous at $\chi = 0$

$$\lim_{X \rangle 0} \frac{|X|}{X} = 1$$

$$\lim_{\chi \langle 0} \frac{|X|}{X} \frac{+1}{-1} = -1$$

$Lim \frac{|X|}{X}$ does not exist

$f(\chi)$ discontinuous

Example 3: check the continuity of the function $\chi = 3$

$$f(\chi) = \begin{cases} \chi - 2 & \chi \neq 3 \\ 1 & \chi = 3 \end{cases}$$

SOL

f (3) = 1

$$\lim_{\chi \to 3} (\chi - 2) = 3 - 2 = 1$$

$$f(3) = \lim_{\chi \to 3} f(\chi)$$

The function continuous at $\chi = 3$

**Problems**

Q1// find Domain, range and sketch each of the following:

1 - $y = \chi^2$ 2 - $y = \sqrt{X}$

3 - $y = 1\chi 1$ 4- $y = |\chi + 2|$

5 - $y = \frac{|\chi|}{\chi}$ 6 - $y = \frac{1}{\chi}$

7 - $y = \dfrac{\chi+1}{\chi-1}$

8 - $y = 2\,Sin\,\chi$

9 - $y = -2\,Sin\,\chi$

10 - $y = 2 + Cos\chi$

Q2 // Evaluate each of the following limits:

1 - $\lim\limits_{y\to 2} \dfrac{t+3}{t+2}$

2 - $\lim\limits_{\chi\to 1} \dfrac{\chi^2-1}{\chi-1}$

3 - $\lim\limits_{y\to 2} \dfrac{y2+5y+b}{y+2}$

4 - $\lim\limits_{y\to 2} \dfrac{y^2-5y+6}{y-2}$

5 - $\lim\limits_{\chi\to -3} \dfrac{\chi 2+4\chi+3}{\chi+3}$

6 - $\lim\limits_{t\to\infty} \dfrac{t+1}{t^2+1}$

7 - $\lim\limits_{t\to\infty} \dfrac{t^2-2t}{2t^2+5t-3}$

8 - $\lim\limits_{\theta\to 0} \dfrac{\tan\theta}{\theta}$

9 - $\lim\limits_{\theta\to 0} \dfrac{Sin\,2\theta}{\theta}$

10- $\lim\limits_{\chi\to 0} \dfrac{Sin\chi}{3\chi}$

11 - $\lim\limits_{\chi\to 0} \dfrac{Sin\,5\chi}{Sin\,3\chi}$

12 - $\lim\limits_{\chi\to\infty} \chi\,Sin\dfrac{1}{\chi}$

13 - $\lim\limits_{\chi\to 0} \dfrac{Sin^2\chi}{\chi}$

14 - $\lim\limits_{\chi\to 0} \dfrac{Sin^2\chi}{2\chi^2+\chi}$

15 - $\lim\limits_{\chi\to 0} \tan 2\chi\, Csc\, 4\chi$

## Chapter Three

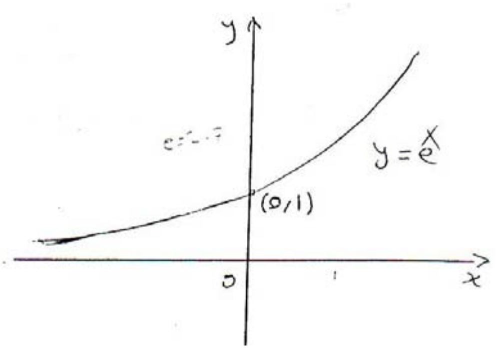
**Special Function**

1 - Exponential function

(i) $y = e^{X}$ , $e = 2.7$

Domain: R

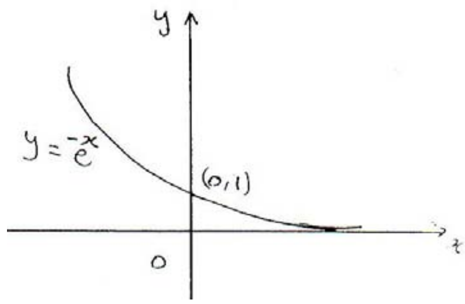
Range: R $(0,\infty)$




(ii) $y = e^{-X}$

Domain: R

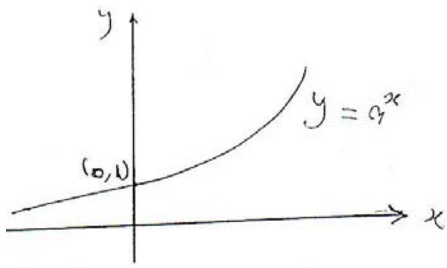
Range: R: $(0,\infty)$




(iii) $y = a^{X}$ , $a \rangle 0$

Domain: R

Range: $(0,\infty)$



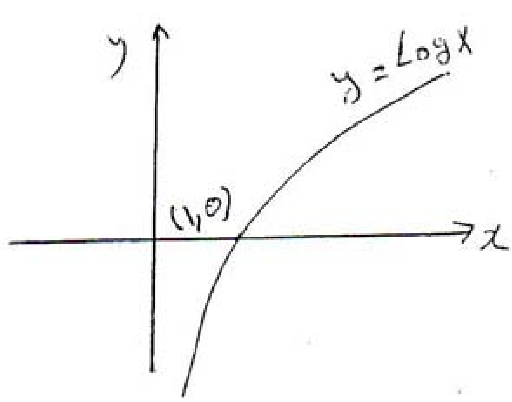

2 – Logarithmic Function:

(i) Common logarithmic function (Log X)

$y = Log_{10} X \Leftrightarrow \chi = 10^{Y}$

Domain: $(0,\infty)$

Range: R

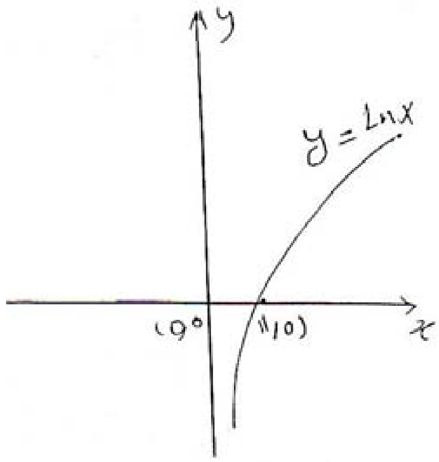


(ii) Natural Logarithmic function (Ln x)

$y = Ln\,x \Leftrightarrow X = e^{X}$ , $e = 2.7$

Domain: $(0,\infty)$

Range: R




**Theorem:**

If a and X are positive numbers and n is any rational number, then

(i) Ln1 = 0

(ii) Lne = 1

(iii) LnaX = Lna + Lnx

(iv) $Ln(\frac{X}{a}) = LnX - Lna$

(v) $Ln\,X^{n} = n\,LnX$ .

**Note:**

(1) $e^{LnX} = X \quad \chi\rangle 0$

(2) $Ln\,e^{X} = \chi$

(3) $e^{X}e^{Y} = e^{X+y}$

(4) $\lim_{\chi\to\infty} Ln\chi = \infty$

(5) $\lim_{\chi\to-\infty} e^{X} = 0$

(6) $Ln\chi^{a} = a\,Ln\chi$

**Hyperbolic Functions:**

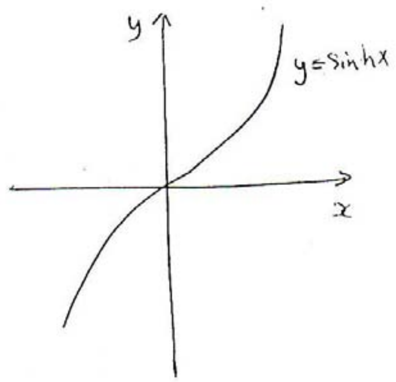
The Hyperbolic Functions are certain combinations of the exponential functions $e^{x}$ and $e^{-x}$ they are:

(i) Hyperbolic Sine (Sinh):

$$y = SinhX \; , SinhX = \frac{e^{x} - e^{-x}}{2}$$

Domain: R

Range: R

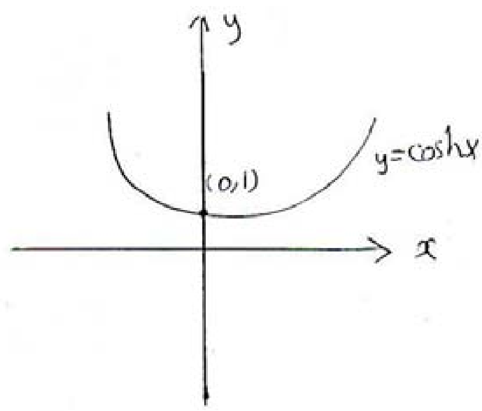



(ii) Hyperbolic Cosin (cosh)

$$y = Coshx \; , \; CoshX = \frac{e^{x} + e^{-x}}{2}$$

Domain: R

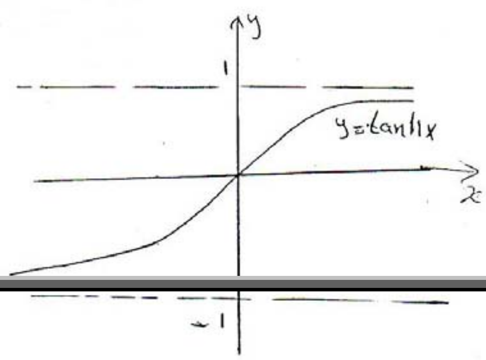
Range: $(1, \infty)$




(iii) Hyperbolic tangent (tanh)

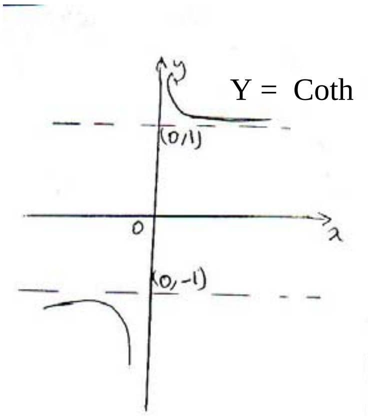
$$y = \tanh x \quad , \quad \tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

Domain: R



Range: ( -1,1)

(iv) Hyperbolic cotangent (coth)

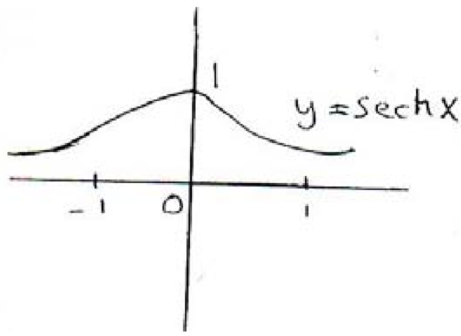



$y = \coth\chi$ , $\coth\chi = \dfrac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$

Domain: R - $\{0\}$

Range: $\{y : y\langle -1 \text{ or } y\rangle 1\}$

(v) Hyperbolic Secant (Sech)

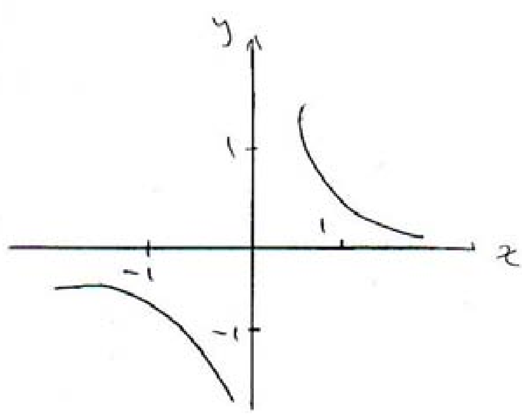



$y = Sech\chi$ , $Sech\chi = \dfrac{2}{e^{x} + e^{-x}}$

Domain: R

Range: (o,1)

(vi) Hyperbolic cosecant (Csch)




$y = Csch\chi$ , $Csch\chi = \dfrac{2}{e^{x} - e^{-x}}$

Domain: R - $\{0\}$

Range: R - $\{0\}$

**Relationships among**

Hyperbolic Function

1 - $Cosh^2\chi - Sinh2\chi = 1$

2 - $Sech^2\chi + \tanh^2\chi = 1$

3 - $Coth^2\chi - Csch^2\chi = 1$

Functions of negative arguments

1 - $Sinh(-\chi) = -Sinh\chi$

2 - $Cosh(-\chi) = Cosh\chi$

3 - $\tanh(-\chi) = -\tanh\chi$

4 - $Coth(-\chi) = -Coth\chi$

5 - $Sech(-\chi) = Sech\chi$

6 - $Csch(-\chi) = -Csh\chi$

Addition Formula:

1 - $Sinh(\chi \pm y) = Sinh\chi\ Coshy\ \mu\ Coshy\ Sinhy$

2 - $Cosh(\chi \pm y) = Cosh\chi\ Coshy \pm Sinh\chi\ Sinhy$

Double angle formula:

1 - $Sinh2\chi = 2Sinh\chi\ Cosh\chi$

2 - $$\begin{aligned} Cosh2\chi &= Cosh^2\chi + Sinh^2\chi \\ &= 1 + 2Sinh^2\chi \\ &= 2\ Cosh^2\chi - 1 \end{aligned}$$

Inverse Trigonometric Function

1 – Inverse Sine ($\text{Sin}^{-1}$)

$$y = Sin^{-1}\chi = arc\ Sine \Leftrightarrow \chi = Siny.$$

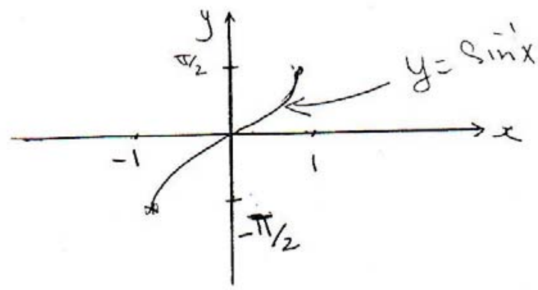
Y is the angle whose Sine is $\chi$

Example:

$$45^{o} = Sin^{-1}\frac{1}{\sqrt{2}}, \frac{\pi}{2} = Sin^{-1}1$$

Domain: $[-1,1]$

Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ Principle values of y.




2 – Inverse Cosine $(Cos^{-1})$

$$y = Cos^{-1}\chi = arc\,Cos\chi \rightarrow \chi = Cosy$$

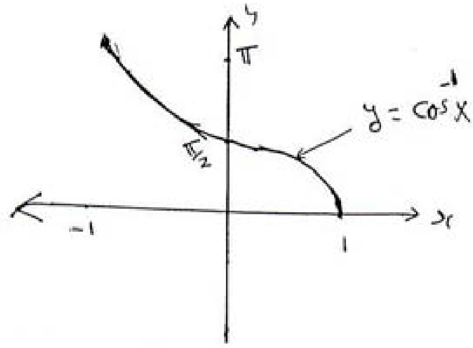
y is the angle whose cosin is $\chi$

Example:

$$30^{o} = Cos-1\sqrt{3/2} \rightarrow \frac{\sqrt{3}}{2} = Cos\,30^{o}$$

Domain: $[-1,1]$

Range: $[0,\pi]$




3 – Inverse of tangent (tan $^{-1}$)

$$y = \tan -1\chi = arc\tan\chi \Leftrightarrow \chi = \tan y$$

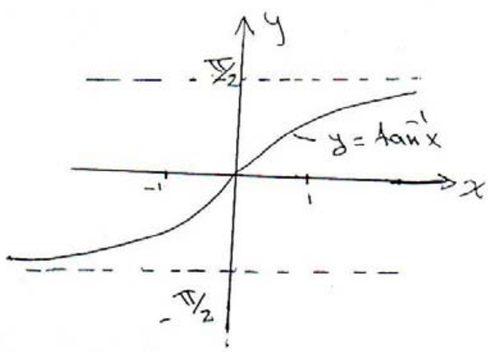
y = is the angle whose cosin is $\chi$

Example:

$$45^{o} = \tan^{-1}1 \Leftrightarrow 1 = \tan 45$$

Domain: R

Range: $(-\frac{\pi}{2}, \frac{\pi}{2})$

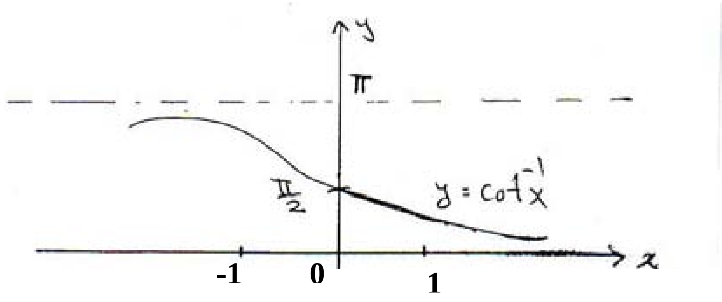


4 – Inverse of cotangent (Cot $^{-1}$)

$$y = Cot - 1\chi = arc\,Cot\,\chi \Leftrightarrow \chi = Coty$$

Domain: R

Range: $(0\ ,\ \pi)$

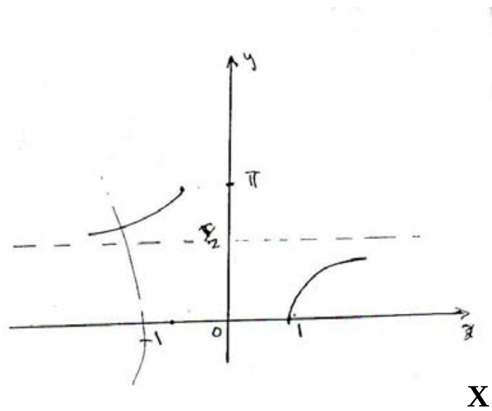



5 – Inverse secant (Sec $^{-1}$)

$$y = Sec - 1\chi = arc\,Sec\chi \rightarrow\ \chi = Secy$$

Domain: $(-\infty\ ,\ -1]\ \mathrm{Y}\ [1, +\infty)$

Range: $\left[0\ ,\ \frac{\pi}{2}\right)\ \mathrm{Y}\left(\frac{\pi}{2}\ \pi\right]$

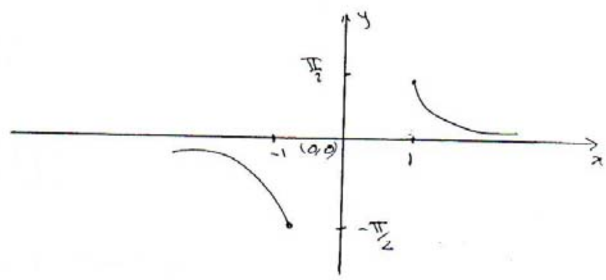



6 – Inverse Csc (Csc -1)

$$y = Csc^{-1}\,\chi = arc\,Csc\chi \Leftrightarrow X\ = Cscy$$

Domain: $(-\infty\ ,\ -1]\ \mathrm{Y}\ [1\ ,\ \infty)$

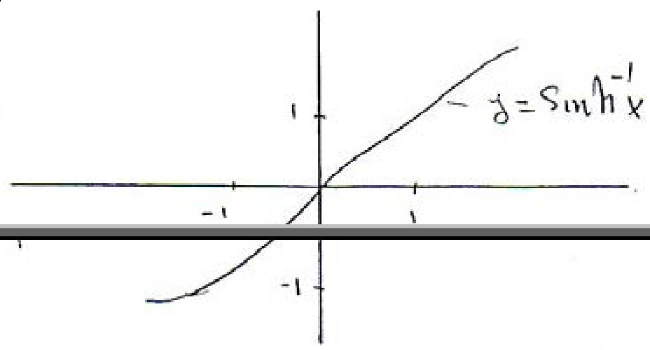
Range: $\left[-\frac{\pi}{2}\ ,\ 0\right)\ \mathrm{Y}\left(0\ ,\ \frac{\pi}{2}\right]$




**Inverse hyperbolic Functions**

1 – Inverse hyperbolic sine ( Sinh -1)

$$y = Sinh - 1\chi \Leftrightarrow \chi = Sinhy$$



Domain: R

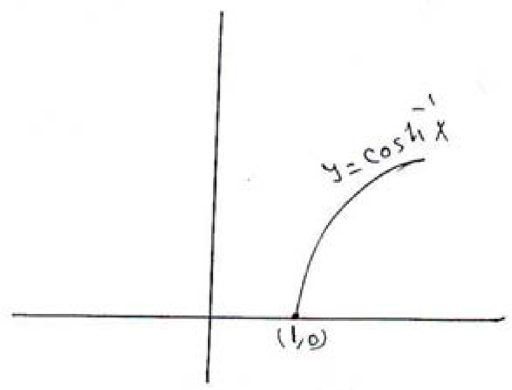
Range:R

2 - Inverse hyperbolic cosin (Cosh $^{-1}$)

$y = Cosh - 1\chi \Leftrightarrow = Coshy$

Domain: $[1 , \infty)$

Range: $[0 , \infty)$

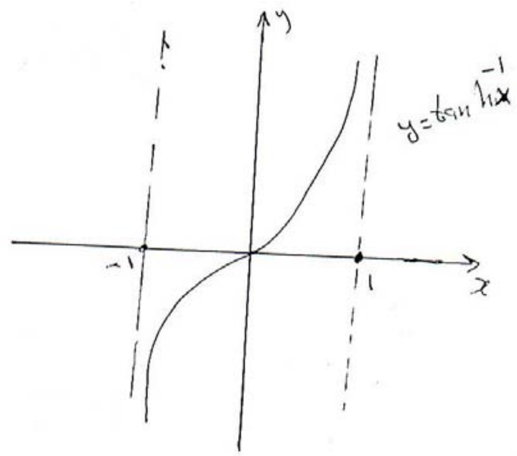



3 – Inverse hyperbolic tangent (tanh -1)

$y = \tanh - 1\chi \Leftrightarrow \chi = \tanh y$

Domain: (-1, 1)

Range: R

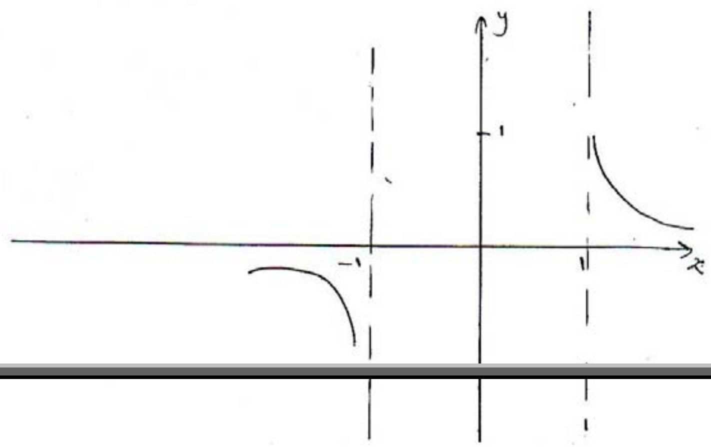



4 – Inverse hyperbolic cotangent (Coth $^{-1}$)

$y = Coth - 1\chi \Leftrightarrow \chi = Cothy$

Domain: $\{X \rangle 1 \cup X \langle -1\}$

Range: R / $\{0\}$

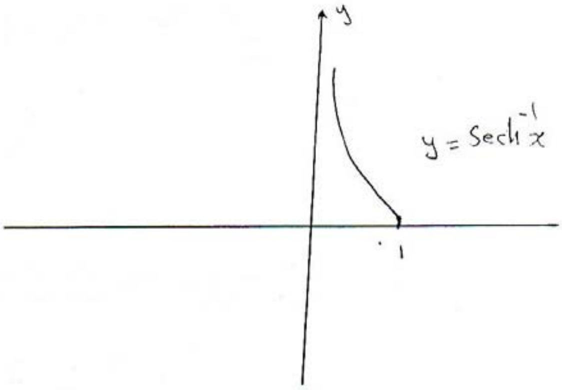


5 – Inverse hyperbolic secant (Sech -1)

$y = \sec h - 1\chi \Leftrightarrow \chi = \sec hy$

Domain: $(0\ ,1]$

Range: $y \geq 0$




6 – Inverse hyperbolic cosecant (Csch -1)

$y = Csch - 1\chi \Leftrightarrow \chi = Cschy$

Logarithmic Form of Inverse hyperbolic Functions

Theorem: the following relationships hold for all $\chi$ in the domain of the stated inverse hyperbolic.

Functions:

1 - $Sinh^{-1} = Ln(\chi + \sqrt{\chi 2 + 1})$

2 - $Cosh - 1\chi = Ln\ (\chi + \sqrt{\chi 2 - 1})$

3 - $\tanh -1\chi = \frac{1}{2} \quad Ln\ (\frac{1+\chi)}{(1-\chi)})$

4 - $Coth -1\chi = \frac{1}{2}\ Ln\ \frac{(\chi + 1)}{(\chi - 1)}$

5 - $Sech -1\chi = Ln\ (\frac{1+\sqrt{1-\chi^2}}{\chi})$

6 - $Csch -1\chi = Ln\ (\frac{1}{\chi} + \frac{\sqrt{1+\chi^2}}{|\chi|})$

Prove that:

* $Sinh -1\chi = Ln(\chi + \sqrt{\chi 2 + 1})$

Sol

Let $y = Sinh -1\chi$

* $\chi = Sinhy$ Since $Sinhy = \frac{e^{y} - e^{-y}}{z}$

$X = \frac{e^{y} - e^{-y}}{z} \rightarrow 2X = e^{y} - e^{-y}$

$\left(e^{y} - z\chi - e^{-y} = 0\right) e^{y} \rightarrow e^{2y} - 2\chi e^{y} - 1 = 0$

$e^{y} = \frac{2\chi \pm \sqrt{4\chi^2 + 4}}{2} = \chi \pm \sqrt{\chi^2 + 1}$

Since $e^{y} \rangle\ 0$ then

$e^{y} = \chi + \sqrt{\chi^2 + 1} \rightarrow y = Ln(\chi + \sqrt{\chi 2 + 1})$ or

$Sinh -1\chi = Ln(\chi + \sqrt{\chi 2 + 1})$

Example:

$Sinh^{-1}\ 1 = Ln(1 + \sqrt{1+1})$

$= Ln(1 + \sqrt{2})$

$= 0.88$

$*\ \tanh-1\chi = \frac{1}{2}\, Ln(\frac{1-\chi}{1+\chi})$

Sol

Let $y = \tanh-1\chi$

$\therefore\ \chi = \tanh y = \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}} \quad \rightarrow \left[(e^{2y}+1)\ \chi = e^{y} - e^{-y}\right] e^{y}$

$(e^{2y}+1)\ \chi = e^{2y} - 1 \rightarrow \chi\, e^{2y} + \chi = e^{2y} - 1$

$\chi + 1 = e^{2y} - \chi\, e^{2y} \rightarrow e^{2y}\ (1-\chi) = 1+\chi$

$e^{2y} = \frac{1+\chi}{1-\chi} \rightarrow 2y = \frac{1}{2}\, Ln(\frac{1+\chi}{1-\chi})$

$\tanh-1 = \frac{1}{2}\, Ln(\frac{1+\chi}{1-\chi})$

Example:

$\tanh-1(\frac{1}{2}) = \frac{1}{2}\, Ln(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}) = \frac{1}{2}\, Ln 3 = 0.5493$

$= 0.55$

**Problems:**

Q1: find domain, range and sketch each of following:

1 - $y = Sin-1\chi \quad , \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

2 - $y = Cot-1\chi \quad , \quad 0 \langle\ y\ \langle\ \pi$

3 - $y = \tan-1\chi \quad , \quad -\frac{\pi}{2} \langle\ y\ \langle\ \frac{\pi}{2}$

4 - $y = \csc - 1\chi \quad , \quad -\frac{\pi}{2} \langle \; y \; \langle \; \frac{\pi}{2}$

5 - $y = Cot - 1\chi \quad , \quad \partial \langle \; y \; \langle \; \pi$

6 - $y = y = Sec - 1\chi, \partial \le y \le \pi$

7 - $y = 3Ln\chi$

8 - $y = -3Ln\chi$

9 - $y = 2 e^{3X}$

10 - $y = 2 e^{-3X}$

11 - $y = Sinh^{-1}\chi$

12 - $y = Cosh - 1\chi$

13 - $y = Cosh - 1\chi$

14 - $y = \tanh^{-1} \chi$

15 - $y = Coth^{-1}\chi$

16 - $y = Coth^{-1}\chi$

17 - $y = Sech^{-1}\chi$

18 - $y = \csc h^{-1}\chi$

Q2: Prove that:

1 - $Sinh - 1\chi = Ln(\chi + \sqrt{\chi^2 + 1}) \quad , \quad -\infty \langle \chi \langle \infty$

2 - $Coch - 1\chi = Ln(\chi + \sqrt{\chi^2 - 1}) \quad , \quad \chi \ge 1$

3 - $\tanh - 1\chi = \frac{1}{2} Ln(\frac{1+\chi}{1-\chi}) \quad , \quad |\chi| \langle 1$

4 - $Coth - 1\chi = \frac{1}{2} Ln(\frac{\chi + 1}{\chi - 1}) \quad , \quad |\chi| \rangle 1$

5 - $Sech - 1\chi = Ln(\frac{1 + \sqrt{1 - \chi^2}}{\chi}) \quad , \; 0 \langle \chi \le 1$

6 - $Csch^{-1}\chi = Ln(\frac{1}{\chi} + \frac{\sqrt{1+\chi^2}}{|\chi|}) \qquad , \chi \neq 0$

Q3// Discuss the Continuity of the following functions at the given points:

1 - $f(\chi) = |\chi + 4| \quad at \quad \chi = -4$

2 - $f(\chi) = \begin{cases} \sqrt{1-\chi^2} & if \ 0 \leq \chi \langle 1 \\ 1 & if \ 1 \leq \chi \langle 2 \\ 2 & if \quad \chi = 2 \end{cases}$ $\quad at \quad \chi = 0$

3 - $f(\chi) = \frac{|\chi|}{\chi} \ at \ \chi = 0$

4 - $f(\chi) = \begin{cases} \frac{1 - Cos\chi}{Sin2\chi} & for \ \chi \neq 0 \\ \frac{1}{2} & for \ \chi = 0 \end{cases}$ $\quad at \ \chi = 0$

5 - $f(\chi) = \begin{cases} \chi - 2 & for \ \chi \neq 3 \\ 1 & for \ \chi = 3 \end{cases}$ $\quad at \ \chi = 3$

6 - $f(\chi) = \begin{cases} \frac{\chi^3 - 8}{\chi 2 - 4} & for \ \chi \neq 2 \\ 0 & for \ \chi = 2 \end{cases}$ $\quad at \quad \chi = 2$

7 - $f(\chi) = \begin{cases} Sin\,\pi\chi & 0 \langle \ \chi \ \langle 1 \\ Lnx & 1 \langle \ \chi \ \langle 2 \end{cases}$ $\quad at \ \chi = 1$

8 - $f(\chi) = \dfrac{\chi - |\chi|}{\chi}$ at $\chi = 2$

9 - $f(\chi) = \begin{cases} \dfrac{|\chi - 3|}{\chi - 3} & for\ \chi \neq 3 \\ 0 & for\ \chi = 3 \end{cases}$ at $\chi = 3$

Q4 // Simplify each of the following:

1 - $e^{LnX}$

2 - $Ln\,(e^{X})$

3 - $e^{-Ln(X2)}$

4 - $Ln\,(e^{-X2})$

5 - $Ln\,(e^{\frac{1}{X}})$

6 - $Ln\,(e^{\frac{1}{X}})$

7 - $e^{Ln(\frac{1}{X})}$

8 - $e^{-Ln(\frac{1}{X})}$

9 - $e^{Ln2 + LnX}$

10 - $e^{2LnX}$

11 - $Ln\,(e^{X - X2})$

12 - $Ln\,(\chi^2\ ,\ e^{-2X})$

13 - $e^{X} + LnX$

14 - $e^{LnX} - 2Lny$

# Chapter Four

# The derivative

1 – Derivative of a function:

Let = $y = f(X)$ and let P(X1, y1) be fixed point on the curve, and Q $(X1 + \Delta X, y1 + \Delta y)$ is another point on the curve as see in the figure

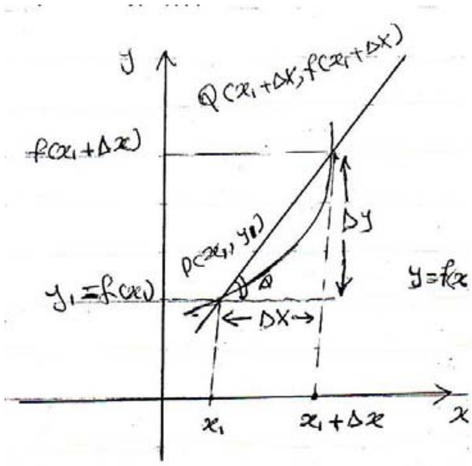
$y1 = f(X1)$, and

$y1 + \Delta y = f(X1 + \Delta X)$

$\Delta y = f(X1 + \Delta X) - y1$

Divided by $\Delta X$

$$\frac{\Delta y}{\Delta X} = \frac{f(X_1 + \Delta X) - f(X_1)}{\Delta X}$$




The

slope of the curve $f(X)$ is

M = $\tan\phi = \frac{\Delta y}{\Delta X}$

$$\therefore M = \frac{f(X + \Delta X) - f(X)}{\Delta X}$$

We define the limit may exist for some value of $X_1$ .

At each point $\chi_1$ where limit does exist, then f is said to have a derivative or to be differentiable.

**Rules of Derivations:**

$y = f(X) = C$ C constant

$y^1 = f^1(X) = \frac{dy}{dX} = 0$

$- f(X) = X^n$

$f^1(X) = nX^{n-1}$

$- f(X) = CnX^{n-1}$

$- f(X) = U \pm V$ n Positive integer

$f^1(X) = \frac{dy}{dX} = \frac{du}{dx} + \frac{dv}{dX}$

$- f(X) = UV$

$f^1(X) = U\frac{dv}{dX} + V\frac{du}{dX}$

$- f(X) = \frac{u}{v}$

$f^1(X) = \frac{vu^1 - uv^1}{V^2}$ , $Where\ U^1 = \frac{du}{dX}$

$- f(X) = [U]^n$

$f^1(X) = n[U]^{n-1}\frac{du}{dX}$

$- f(X) = e^u$

$f^1(X) = e^u \frac{du}{dX}$

$- f(X) = C^U$ $U$ $C$ $Cons\tan t$

$f^1(X) = C^U.LnC .\frac{du}{dx}$

Derivative of trigonometric functions:

1) $(\sin u)' = \cos u \, du$
2) $(\cos u)' = - \sin u \, du$
3) $(\tan u)' = \sec^2 u \, du$
4) $(\cot u)' = - \csc^2 u \, du$
5) $(\sec u)' = \sec u \tan u \, du$
6) $(\csc u)' = - \csc u \cot u \, du$

Derivative of hyperbolic functions:

1) $\sinh u = \cosh u \, du$
2) $\cosh u = \sinh u \, du$
3) $\tanh u = \mathrm{sech}^2 u \, du$
4) $\cot u = - \mathrm{csch}^2 u \, du$
5) $\mathrm{sech}\, u = - \mathrm{sech}\, u \tanh u \, du$
6) $\mathrm{csch}\, u = - \mathrm{csch}\, u \coth u \, du$

derivative of the inverse trigonometric functions:

1) $(\sin^{-1} u)' = du/(1-u^2)^{1/2}$
2) $(\cos^{-1} u)' = - du/(1-u^2)^{1/2}$
3) $(\tan^{-1} u)' = du/1+u^2$
4) $(\cot^{-1} u)' = - du/1+u^2$
5) $(\sec^{-1} u)' = du/ u(u^2-1)^{1/2}$
6) $(\csc^{-1} u)' = - du/ u(u^2-1)^{1/2}$

derivative of the inverse of hyperbolic functions:

1) $(\sinh^{-1} u)' = du/(1+ u^2)^{1/2}$
2) $(\cosh^{-1} u)' = du/ ( u^2-1)^{1/2}$
3) $(\coth^{-1} u)' = du/1-u^2$ if $|u|>1$
4) $(\tanh^{-1} u) = du/1-u^2$ if $|u|<1$

5) $(\text{sech}^{-1}u)' = - du/ u(1-u^2)^{1/2}$

6) $(\text{csch}^{-1}u)' = - du/ u(1+u^2)^{1/2}$

ex: find $y'$ of

(1) $y = [\ln (3x+ 1)]^3$ (2) $y = 4^x$

Sol:

(1) $y' = 3[\ln (3x+ 1)]^2 [3/(3x+1)] = 9[\ln (3x+ 1)]^2/ (3x+1)$

(2) $y' = 4^x \ln4$

{Applications of derivative}

Velocity and acceleration

Ex: find velocity and acceleration at time t to a moving body as

$S= 2t^3 – 5t^2 + 4t -3.$

Sol:

$V= ds/dt = 6t^2 – 10t +4$

$A= dv/dt = 12t-10$

<u>Theorem:</u>

Prove that:

$D(\sin^{-1}u) = 1/(1- u^2)^{1/2} (du/dx)$

<u>Proof</u>

Let $y = \sin^{-1} \rightarrow \sin y = u$ $u=[-1,1] \rightarrow y = [–\Pi/2 , \Pi/2]$

$\text{Cos } y\ dy/dx = du/dx \rightarrow dy/dx = 1/\cos y\ du/dx$

Since $\cos^2y + \sin^2y = 1$ this implies that

$\text{Cos } y = (1 - \sin^2y)^{1/2} \rightarrow \text{Cos } y = \pm (1 - u^2)^{1/2}$

Cos y is positive between $–\Pi/2$ and $\Pi/2$

$Dy/dx = 1/(1- u^2)^{1/2} (du/dx) = D(\sin^{-1}u)$

Ex: find dy/dx for the following functions:

(1) $y = \tan (3x^2)$

(2) $y = x \sin^{-1} x + (1 - x^2)^{1/2}$

(3) $y = \cosh^{-1}(\sec x)$

sol:

(1) $y' = \sec^2(3x^2)\, 6x = 6x \sec^2(3x^2)$

(2) $y' = x/(1 - x^2)^{1/2} + \sin^{-1} x - x/(1 - x^2)^{1/2} = \sin^{-1} x$

(3) $y' = [1/(\sec^2 x - 1)^{1/2}] \sec x \tan x = \sec x \tan x/(\sec^2 x - 1)^{1/2}$

<u>Implicit relations:</u>

Ex: find dy/dx if

$x^5 + 4x y^3 - 3y^5 = 2$

sol:

$5x^4 + 4x\, 3y^2(dy/dx) + 4y^3 - 15y^4(dy/dx) = 0$

$(12x y^2 - 15y^4)\, dy/dx = - 5x^4 - 4y^3$

$dy/dx = ( - 5x^4 - 4y^3)/(12x y^2 - 15y^4)$

## **<u>Chain Rule</u>**

1- If y = f (x), and x = x (t), then

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial x}\ \frac{\partial x}{\partial t}$$

2- If y = f (t), and x = x (t), then

$$\frac{\partial y}{\partial x} = \frac{\frac{\partial y}{\partial t}}{\frac{\partial x}{\partial t}}$$

Ex: find $\frac{dy}{dt}, \frac{dx}{dt}$ and $\frac{dy}{dx}$ of $x = 3t + 1$ and $y = t^2$

Sol:

$$\frac{dx}{dt} = 3.\ \frac{\mathrm{dy}}{\mathrm{dt}} = 2t,\ \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3}$$

### **<u>LHopital's Rule</u>**

Let f and g be two functions which are differentiable in an open interval I containing the point c and let $g^{'}(x) \neq 0$. if

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{0}{0},\ \frac{\infty}{\infty},\ 0.\infty,\ \infty.\infty,\ \infty.\infty,\ \frac{\infty}{0}, \infty^{0}, \text{ then}$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f^{'}(x)}{g^{'}(x)}.$$

Ex: evaluate (1) $\lim_{x \to 0} \frac{x^2 - \sin x}{x^2}$, (2) $\lim_{x \to 0} x \ln x$

Sol:

(1) $\lim_{x \to 0} \frac{x^2 - \sin x}{x^2} = \frac{0 - 0}{0} = \frac{0}{0}$

$$\lim_{x \to 0} \frac{x^2 - \sin x}{x^2} = \lim_{x \to 0} \frac{2x - \cos x}{2x} = \frac{-1}{0} = \infty$$

(2) $\lim_{x \to 0} x \ln x = 0.\infty$

$$\lim_{x \to 0} x \ln x = \lim_{x \to 0} \frac{\ln x}{\frac{1}{x}} = \frac{\infty}{\infty}$$

$$\lim_{x \to 0} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x \to 0} x = 0$$

## <u>Series</u>

(Power series): If $\{a_n\}$ is a sequence of constants, the expression:

$$a_0 + a_1 + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots = \sum_{n=0}^{\infty} a_n x^n$$

is called power series in x.

(Taylor's series): If a function f can be represented by a power series in (x-b) called Taylor's series and has the form:

$$f(x) = f(b) + f^{'}(b)(x-b) + \frac{f^{''}(b)(x-b)^2}{2!} + ..... + \frac{f^{n}(b)(x-b)^n}{n!} + ....$$

<u>Example:</u>

Find Taylor series expansion of cos x about a point a=2$\pi$

Sol:

f(x) = cos x

f(2$\pi$) = cos ( 2$\pi$) = 1

$f^{'}(x) = -\sin x,\ \ f^{'}(2\pi) = -\sin 2\pi = 0$

$f^{''}(x) = -\cos x,\ \ f^{''}(2\pi) = -\cos 2\pi = -1$

$f^{'''}(x) = \sin x,\ \ f^{'''}(2\pi) = \sin 2\pi = 0$

$f^{iv}(x) = \cos x,\ \ f^{iv}(2\pi) = \cos 2\pi = 1$

$$\cos x = 1 - \frac{(x-2\pi)^2}{2!} + \frac{(x-2\pi)^4}{4!} - \frac{(x-2\pi)^6}{6!} + ...$$

(Maclaurin series): when b = 0, Taylor series called Maclaurin series.

Example:

Find Maclaurin series for the function f(x) = $e^x$

Sol:

f(x) = $e^x \rightarrow$ f(0) = $e^0$ = 1

$f^{'}(x) = e^x \rightarrow f^{'}(0) = e^0 = 1$

$f^{''}(x) = e^x \rightarrow f^{''}(0) = e^0 = 1$

$f^{'''}(x) = e^x \rightarrow f^{'''}(0) = e^0 = 1$

$e^x = 1 + x + (x^2/2!) + (x^3/3!) + ….$

# Chapter five

(INTEGRALS)

The process of finding the function whose derivative is given is called integration, it's the inverse of differentiation.

Definition:(indefinite integral)

A function $y=F(x)$ is called a solution of $dy/dx=f(x)$ if $dF(x)/dx=f(x)$.

We say that $F(x)$ is an integral of $f(x)$ with respect to x and $F(x) + c$ is also an integral of $f(x)$ with a constant c s.t

$D(F(x) +c)=f(x)$.

Formulas of Integration:

1) $\int dx=x + c$.
2) $\int a\, dx=a \int dx$
3) $\int(du \pm dv)= \int du \pm \int dv$.
4) $\int x^n\, dx=(x^{n+1}/n+1)+ c$
5) $\int(u)^n\, du=(u^{n+1}/n+1)+ c$
6) $\int e^u\, du= e^u+ c$
7) $\int a^u\, du=(a^u/\ln a) + c$
8) $\int du/u= \ln u + c$.

Example1:

Solve the differential equation: $dy/dx=3x^2$.

Sol:

$dy=3x^2\, dx$

since $d(x^3)=3x^2\, dx$, then we have:

$\int dy= \int 3x^2\, dx= \int d(x^3)\, dx$

$y= x^3+ c$.

9 methods for finding integrals:

1"Integral of trigonometric functions":

1) $\int \cos u\, du = \sin u + c$
2) $\int \sin u\, du = -\cos u + c$
3) $\int \sec^2 u\, du = \tan u + c$
4) $\int \csc^2 u\, du = -\cot u + c$
5) $\int \sec u \tan u\, du = \sec u + c$
6) $\int \csc u \cot u\, du = -\csc u + c$

2"Integral of hyperbolic functions":

1) $\cosh u\, du = \sinh u + c$
2) $\int \sinh u\, du = \cosh u + c$
3) $\int \operatorname{sech}^2 u\, du = \tanh u + c$
4) $\int \operatorname{csch}^2 u\, du = -\cot u + c$
5) $\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + c$
6) $\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + c$

Integral of the inverse trigonometric functions:

1) $\int du/(1-u^2)^{1/2} = \{\sin^{-1} u + c \quad \text{or} \quad -\cos^{-1} u + c\ \}$
2) $\int du/1+u^2 = \{\tan^{-1} u + c \quad \text{or} \quad -\cot^{-1} u + c\}$
3) $\int du/\ u(u^2-1)^{1/2} = \sec^{-1} u + c \quad \text{or} \quad -\csc^{-1} u + c\}$

Integral of the inverse of hyperbolic functions:

1) $\int du/(1+ u^2)^{1/2} = \sinh^{-1} u + c$
2) $\int du/\ (\ u^2-1)^{1/2} = \cosh^{-1} u + c$
3) $\int du/1-u^2 = \tanh^{-1} u + c$ if $|u|<1$ and $\int du/1-u^2 = \coth^{-1} + c$ if $|u|>1$
4) $\int du/\ u(1-u^2)^{1/2} = -\operatorname{sech}^{-1} u + c$
5) $\int du/\ u(1+u^2)^{1/2} = -\operatorname{csch}^{-1} u + c$

ex:

evaluate:

1) $\int (5x^4-6x^2+2/\ x^2)dx$

2) $\int \cos 2x\ dx$

3) $\int \cos^2 x\ dx$

sol:

1) $\int (5x^4-6x^2+2/\ x^2)dx = 5\int x^4 dx-6\int x^2 dx+2\int x^{-2}dx$

$X^5 -2x^3-2/x+ c$

2) $\int \cos 2x\ dx = \sin 2x/2 + c$

3) $\int \cos^2 x\ dx = 1/2 \int(1+ \cos 2x)\ dx = 1/2[\int dx + \int \cos 2x\ dx]$

$1/2[x + \sin 2x/2] + c = 1/2x + 1/4 \sin 2x + c$

<u>3- "Integration by parts"</u>

Let u and v be functions of x and $d(uv) = u\ dv + v\ du$

By integration both sides of this equation (w.r.t x)

$\int d(uv)=\int u\ dv + \int v\ du$ this implies $(uv)=\int u\ dv + \int v\ du$

$\int u\ dv= (uv) - \int v\ du$

<u>ex:</u> find $\int x\ e^x\ dx$

<u>sol:</u>

let $u=x \rightarrow du = dx$ and let $dv = e^x\ dx$

from $\int u\ dv= (uv) - \int v\ du \rightarrow \int x\ e^x\ dx= x\ e^x- e^x +c$

<u>4- "Integrals involving $(a^2 - u^2)^{1/2}$, $(a^2 + u^2)^{1/2}$, $(u^2 - a^2)^{1/2}$, $a^2 - u^2$, $a^2 + u^2$, $u^2 - a^2$ "</u>

(A) $u= a \sin \Phi$ replaces $a^2 - u^2 = a^2 - a^2 \sin^2 \Phi = a^2(1- \sin^2 \Phi) = a^2\cos^2 \Phi$

(B) $u= a \tan \Phi$ replaces $a^2 + u^2 = a^2 + a^2 \tan^2 \Phi = a^2 \sec^2 \Phi$

(C) $u= a \sec \Phi$ replaces $u^2 - a^2 = a^2 \sec^2 \Phi - a^2 = a^2 \tan^2 \Phi$

Ex: find $\int dx/x^2(4 - x^2)^{1/2}$

Sol:

Let $x=2\sin \Phi \rightarrow dx = 2\cos \Phi \, d\Phi$

$\int dx/x^2(4 - x^2)^{1/2} = \int 2\cos \Phi \, d\Phi/4 \sin^2 \Phi(4- 4 \sin^2 \Phi)^{1/2} =$

$\int 2\cos \Phi \, d\Phi/4 \sin^2 \Phi(2\cos \Phi ) = \int d\Phi/4 \sin^2 \Phi = 1/4\int \csc^2 \Phi \, d\Phi = -1/4\cot \Phi + c$

now

from $x= 2\sin \Phi \rightarrow \sin \Phi = x/2 \rightarrow \cos \Phi = (1- x^2/4)^{1/2} = 1/2(4 - x^2)^{1/2}$

$-1/4\cot \Phi + c = (-1/4)(4 - x^2)^{1/2}/x$

5-" Integrals involving $ax^2 + bx +c$ "

First, We put the equation as $(ax^2 + bx) + c$.

Second, if $a\neq 1$ , we take a as a mutable by the sides of the equation which has x, $a[x^2 + (b/a)x] + c$.

Third, put and sub to the equation [(1/2) the number multiplied by $x]^2$ ,

$a[x^2 + (b/a)x + (1/4)(b/a)^2 - (1/4)(b/a)^2] + c$.

Fourth, rewrite the equation as $a[x^2 + (b/a)x + (1/4)(b/a)^2] + c - (1/4)(b^2/a)$

Last, the equation become $a[x+ (1/2) (b/a)]^2 + c - (1/4)(b^2/a)$ and suppose $u = x+ (1/2) (b/a)$ to become $a[u]^2 + c - (1/4)(b^2/a)$

Ex: Find $\int dx/(4x^2 + 4x + 2)$

Sol:

$4x^2 + 4x + 2 = (4x^2 + 4x) + 2 = 4(x^2 + x) + 2 =$

$4[x^2 + x +(1/4) -(1/4)] +2= 4[x^2 + x +(1/4)] +2 -1=$

$4[x+1/2]^2 +1$.

Let $u = x+1/2 \rightarrow 4[x+1/2]^2 +1 = 4u^2 +1$.

Since $u = x+1/2 \rightarrow x = u – (1/2) \rightarrow dx = du$

$\int dx/(4x^2 + 4x + 2) = \int du/(4u^2 +1) = 1/2 \int 2du/(4u^2 +1) =$

$1/2 \tan^{-1}2u = 1/2 \tan^{-1}2(x+1/2).$

6-"method of partial fractions"

If the integral of the form f(x)/g(x) s.t f(x) and g(x) are poly.

And degree of f(x)< degree of g(x) we can carry out two cases:

Case i

If all factor of g(x) are linear, by the following ex:

Ex: find $\int dx/ x^2 + x – 2$

Sol:

$1/ x^2 + x – 2 = 1/(x-1)(x+2) = A/(x-1) + B/(x + 2) =$

$[A(x + 2) + B/(x - 1)] /(x-1) (x + 2)$

$1= Ax +A2 +Bx –B = (A+B)x + (A2-B)$

$1=A2-B$

$\underline{0= (A+B)}$

$3A=1\rightarrow A=1/3$ put in eq.(2) $\rightarrow B =-1/3$

$\int dx/ x^2 + x – 2 = \int 1/(x-1)(x+2)dx = \int [A/(x-1) + B/(x + 2)]dx$

$= \int [(1/3)/(x-1) – (1/3)/(x + 2)]dx = 1/3\int /(x-1) – 1/3 \int /(x + 2)dx$

$= 1/3 \ln|x-1| -1/3 \ln|x+2|+c$

Case ii

If some of the factors of g(x) are quadratic, by the following ex:

Ex: find $\int (x^2 + x – 2)dx/ (3x^3- x^2 + 3x – 1)$

Sol:

$(x^2 + x – 2) / (3x^3- x^2 + 3x – 1) = (x^2 + x – 2)/ x^2(3x – 1) + (3x – 1)$

$= (x^2 + x – 2)/ (3x – 1) (x^2+1) = [A/(3x – 1)] +[(Bx + C)/ (x^2+1)]$

$= [A(x^2+1)+ (Bx + C) (3x – 1)]/ (3x – 1) (x^2+1)$

$x^2 + x – 2 = A(x^2+1)+ (Bx + C) (3x – 1)$

$x^2 + x – 2 = (A + 3B) x^2 + (B + 3C) x + (A-C)$

$A + 3B= 1$

$B + 3C = 1$

<u>$A - C = -2$</u>

$A = -7/5$, $B= 4/5$, $C = 3/5$

$(x^2 + x - 2)/ (3x - 1) (x^2+1) = (-7/5)/(3x - 1) +[(4/5)x + (3/5)]/ (x^2+1)$

And

$\int (x^2 + x - 2)dx/ (3x - 1) (x^2+1) = (-7/5) \int dx/(3x - 1) +$

$(4/5) \int x\, dx/(x^2+1) + (3/5) \int dx/ (x^2+1)$

$=-(7/15) \ln |3x-1| + (2/5) \ln | x^2+1| + 3/5 \tan^{-1} x$

<u>7-"further substitutions"</u>

Some integrals involving fractional powers of the variable x may be simplified by substitution $x = u^n$ where n is the least common multiple of the denominators of the exponents.

<u>Ex:</u>

$I=\int (x)^{1/2}dx/1+(x)^{1/3}$

<u>sol:</u>

let $x=u^6 \rightarrow dx = 6 u^5 du$

$I= \int (u^6)^{1/2}(6 u^5) du/(1+( u^6)^{1/3}) = 6 \int u^3 u^5du/1+u^2= 6 \int u^8du/1+u^2$

By long division

$u^8/1+u^2 = u^6 - u^4 + u^2 -1 +(1/1+ u^2)$

$I= 6\int u^8du/1+u^2 = 6 \int [u^6 - u^4 + u^2 -1 +(1/1+ u^2)] du$

$= (6/7) u^7 - (6/5) u^5 + 2 u^3 -6u + 6 \tan^{-1}u+c$

$= (6/7) x^{7/6} - (6/5) x^{5/6} + 2 x^{1/2} -6x^{1/6} + 6 \tan^{-1}(x^{1/6})+c$

<u>8-"rational functions of sin x and cos x"</u>

If the integral that is rational function of sin x or cos x or both, can be changed as following:

Let $z = \tan(x/2)$

$x/2 = \tan^{-1}z \rightarrow x = 2\tan^{-1}z \rightarrow dx = 2dz/(1+ z^2)$

$\cos(x/2) = 1/(1+ z^2)^{1/2}$, $\sin(x/2) = z/(1+ z^2)^{1/2}$

$\sin x = 2 \sin(x/2) \cos(x/2) = 2z/(1+ z^2)$

$\cos x = \cos^2 (x/2) – \sin^2(x/2) = (1- z^2)/(1+ z^2)$ from $[\cos(x/2 +x/2)]$

ex: find $I=\int dx/(1- \sin x + \cos x)$

sol:

$$I = \int \frac{2dz/(1+ z^2)}{(1- [2z/(1+ z^2)] + [(1- z^2)/(1+ z^2)] )}$$

$$= \int \frac{2dz/(1+ z^2)}{(1+ z^2 -2z+1- z^2)/(1+ z^2)}$$

$$=\int 2dz/(2-2z) = \int dz/(1-z) = -\ln|1-z|+c = -\ln |1-\tan(x/2)|+c$$

9-"evaluating integrals of the following types"

(A) $\sin(mx) \sin(nx) = (1/2) [\cos(m-n)x- \cos(m+n)x]$

(B) $\sin(mx) \cos(nx) = (1/2) [\sin(m-n)x+\sin(m+n)x]$

(C) $\cos(mx) \cos(nx) = (1/2) [\cos(m-n)x+ \cos(m+n)x]$

Ex:

$$\int 2\sin(4x) \sin(3x) dx = \int (2/2) [\cos(4-3)x- \cos(4+3)x]dx$$

$$=\int (\cos x- \cos 7x)dx = \sin x –(1/7)\sin 7x +c$$

{definite integral}

The definite integral like indefinite integral but there is a limit to the integral like $\int_a^b f(x)\, dx= F(a)- F(b)$.

Ex: evaluate $\int_0^3 x^3\, dx$

Sol:

$(3)^4/4 – 0 = 81/4$

Applications of definite integral

{area under the curve}

Ex: find the area under sin x bdd by x= 0 and x = 2π and x-axis

Sol:

A= $\int_0^{2\pi} \sin x\, dx = \int_0^{\pi} \sin x\, dx - \int_{\pi}^{2\pi} \sin x\, dx = (\cos \pi - \cos 0) + (\cos 2\pi - \cos \pi) = (-1-1) + (1-(-1)) = 4$

{area between two curves}

Ex: find the area bdd by $y = 2 - x^2$ and $y = -x$

Sol:

$Y = 2 - x^2 = y = -x$

$2 - x^2 = -x \rightarrow x^2 - x - 2 = 0 \rightarrow (x-2)(x+1) = 0 \rightarrow x=2,\ x=-1$

$A = \int_{-1}^{2} [(2-x^2)-(-x)]dx = [2x - (x^3/3) + (x^2/2)]^2_{-1} = 6.5$

<u>Double integrals</u>

When the integral have two signals of integral to two parameters x and y called double integral, like $\int\int f(x,y)\, dx\, dy$.

The benefit of like integrals is to find the volume of things.

Ex: find the volume of $f(x,y) = x^2y$ limited by x=(1,3) and y =(1.2)

$\int_1^2 \int_1^3 x^2y\, dx\, dy = \int_1^2 [(x^3/3)y\, dy]^3_1 = \int_1^2 [(3^3/3) - (1^3/3)]y\, dy$

$= \int_1^2 [(27/3) - (1/3)]y\, dy = \int_1^2 (26/3)y\, dy = (26/3)\int_1^2 y\, dy$

$= [(26/3)(y^2/2)]^2_1 = [(13/3)\, y^2]^2_1 = (13/3)\, 4 - (13/3) = 13$