CHAPTER ONE

Definition

1-Partial Derivative

If f is a function of the variables x, and y in the region xy plane the **Partial Derivative** of f with respect to (w. r. to) x, at point (x, y) is

$$\partial f/\partial x = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

And (w. r. to) y at point (x, y) is

$$\partial f/\partial y = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

To find $\partial f/\partial x$ is simply regards y as constant in f (x, y) and Differential (w. r to) x is written in form

$\partial f/\partial x = \partial z/\partial x = \partial F/\partial x$ or

$D_X f = Z_x = F_x$

Using same way to find $\partial f/\partial y$ is simply regards x as constant in f (x, y) and Differential (w. r. to) y is written in form

$\partial f/\partial y = \partial z/\partial y = \partial F/\partial y$ or

$D_y f = Z_y = F_y$

Since a partial derivative of function twice variables to obtain second partial derivative as

1-$\partial f/\partial x = f_x$

2-$\partial f/\partial y = f_y$

3- $\partial/\partial x\,(\partial f/\partial x) = \partial^2 f/\partial x^2 = f_{x\,x}$

4- $\partial/\partial y\,(\partial f/\partial y) = \partial^2 f/\partial y^2 = f_{yy}$

5- $\partial/\partial x\,(\partial f/\partial y) = \partial^2 f/\partial x\partial y = f_{y\,x}$

6- $\partial/\partial y\,(\partial f/\partial x) = \partial^2 f/\partial y\partial x = f_{xy}$

Note I

It is easy to extend the partial derivative of function of three variables or more

$\partial/\partial x\,(\partial^2 f/\partial y\partial x) = \partial^3 f/\partial x\partial y = f_{x\,y\,x}$

Theorem

If f (x, y) and it's partial derivatives f_x , f_y , $f_{y\,x}$,and f_{xy} are define in region containing a point (a, b) and are all continuous at (a, b), then $f_{y\,x} = f_{xy}$.

Example1

Let $f(x, y) = x^2 - y^2 + xy + 7$.

Then find f_x , f_y , f_{xx} ,and f_{xy}

Solution

$f_x = 2x + y$

$f_y = -2y + x$

$f_{xy} = 1$.

Problem

1-Let $f(x, y) = e^{-x} \sin y + e^y \cos x + 8$

Then find f_x , f_y

2-Find f_x and f_y at point (1,3/2) if $f = \sqrt{4 - x^2 + y^2}$

3-If $f(x, y) = x e^y - \sin(x/y) + x^3y^2$.

Then find f_x , f_y , f_{xx} , F_{yy} and f_{xy}

4-If $U = x^2y + \arctan(xz)$, then find U_x , U_y and U_z .

5-If $V = x^2 + y^2 + z^2 + \text{Log}(xz)$, then find V_x , V_y , V_z ,V_{xy} and V_{zz} .

6-If $f = x^y$, then find f_x , f_y .

7-Prove that

$U_{xy} = U_{yx}$

If

a-$U = x \sin y + y \cos x$

b-$U = x \text{ Ln} y$

2-Chain Rule

1- Function of one variables

If y = f (x), and x = x (t), y = y (t) then

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial x}\,\frac{\partial x}{\partial t}$$

2- Function of two or three variables is

a- If Z = f(x, y), x = x (t), y = y (t) then

$$\frac{\partial Z}{\partial t} = \frac{\partial Z}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial Z}{\partial y}\,\frac{\partial y}{\partial t}$$

b- If Z = f(x, y, w), x = x (t), y = y (t) , w = w (t) then

$$\frac{\partial Z}{\partial t} = \frac{\partial Z}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial Z}{\partial y}\,\frac{\partial y}{\partial t} + \frac{\partial Z}{\partial w}\,\frac{\partial w}{\partial t}$$

Example2

Let $f(x, y) = e^{xy}$, $x = r\cos\theta$, $y = r\sin\theta$

Find f_r and f_θ, in term r and θ.

Solution

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial f}{\partial x} = ye^{xy}$$

$$\frac{\partial f}{\partial y} = xe^{xy}$$

$$\frac{\partial x}{\partial r} = \cos\theta$$

$$\frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial f}{\partial r} = ye^{xy}\cos\theta + xe^{xy}\sin\theta$$

$$= 2r\sin\theta\cos\theta\ e^{r^2\cos\theta\cos\theta}$$

$$= r\sin 2\theta\ e^{r^2\cos\theta\cos\theta}.$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial f}{\partial \theta} = ye^{xy}(-r\sin\theta) + xe^{xy}(r\cos\theta)$$

$$= -r^2\sin^2\theta\ e^{r^2\cos\theta\cos\theta} + r^2\cos^2\theta\ e^{r^2\cos\theta\cos\theta}$$

$$= r^2\ e^{r^2\cos\theta\cos\theta}(\cos^2\theta - \sin^2\theta)$$

$$= r^2\ \cos 2\theta\, e^{r^2\cos\theta\cos\theta}.$$

<u>Problem</u>

Find f_t, in the following function

1- $f(x, y) = x^2 - y^2$, $x = e^t$, $y = 2t - 6$

2- $f = x^2 - xy^2$, $x = \cos t$, $y = e^{-t}$

3- $f = y/x$, $x = \ln t$, $y = \cot t$

4- $f = e^{xy}\ln(x - y)$, $x = t^3$, $y = 2t - t^3$

5- $f = \sqrt{x + y}$, $x = \sin^{-1} t$, $y = \sin t$

6- $f = \frac{x}{x - y}$, $x = \cosh t$, $y = \sinh t$

7- $f = \sin(x + y - z)$, $x = \sin t$, $y = e^{t^2}$, $z = \ln t$

8- $f = \tan^{-1}(\frac{x}{y})$, $x = e^t\cos t$, $y = e^t\sin t$

9- $f = \frac{x}{y}$, x = e^t, y= 2 t-6

Find U_x, U_y and U_z, in the following function

10- $U = \tanh^{-1}(\frac{r}{s})$, r = x sin yz, s= x cos yz

11-U = ln(r+ s + t)if, r = xy, s= x z, t= y z

3- The Total Differential

The total differential of function

W = f(x, y, z),………………………………..(1)

Is defined to be

$$dw = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

Or

$$dw = f_x\,dx + f_y\,dy + f_z\,dz$$

In general the total differential of function

W = f(x, y, z, u,………..,v) is defined by

$$dw = f_x\,dx + f_y\,dy + f_z\,dz + f_u\,du + \ldots\ldots + f_v\,dv$$

where x , y, z, u…….and v are independent variables.

But if x, y and z are not independent variabl but are them can selves given by

x = x (t), y = y (t), z = z (t),

then we have

$$dx = \frac{\partial x}{\partial t}dt,\ dy = \frac{\partial y}{\partial t}dt,\quad dz = \frac{\partial z}{\partial t}dt.$$

Or in the form:-

x = x(r, s), y= y(r, s), z = z(r, s).

Then we ha

$$\left.\begin{aligned} dx &= \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial s}ds \\ dy &= \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial s}ds \\ dz &= \frac{\partial z}{\partial r}dr + \frac{\partial z}{\partial s}ds \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Then (1) become in case

W = f(x, y, z) = f(x(r, s), y(r, s), z(r, s))

= f(r, s).

Then from (2) and (3) we obtain:-

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

$$dw = \left[\frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial s}ds\right]\frac{\partial w}{\partial x} + \left[\frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial s}ds\right]\frac{\partial w}{\partial y} + \left[\frac{\partial z}{\partial r}dr + \frac{\partial z}{\partial s}ds\right]\frac{\partial w}{\partial z}$$

$=[\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial r}]\,dr+[\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial s}]ds$…….(4)

Example3

Find the total differential of function

$W = x^2 + y^2 + z^2$ if

x = r coss, y= r sins and z = r

Solution

$dw = w_x\, dx + w_y\, dy + w_z\, dz$

$= 2xdx + 2y\, dy + 2z\, dz$

$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial s}ds$, or

$dx = x_r\, dr + x_s\, ds$ =cossdr - r sins ds

$dy = y_r\, dr + y_s\, ds$ = sins dr + rcossds,

$dz = z_r\, dr + z_s\, ds$ = dr.

Now dw = 2x[cossdr - r sins ds] + 2y [sins dr + rcossds]
+ 2r [dr.]
= 2 rcoss [cossdr - r sins ds] + 2 r sins [sins dr + rcossds] + 2r [dr.]

dw = 2[$rcos^2s$]dr - r sins ds] + 2y [sins dr + rcossds]
+ 2r [dr.]
= 2 [$rcos^2s + r\, sin^2s$ +r] dr + 2 [$-r^2$ sins coss+ r^2 sins coss] ds + 2r dr.],

dw = 4r dr].

Problem

If U = f(x, y) Find dU, in the following:-

1- $U = 2Lnx + Lny^2$ if, $x = e^{-t}$, $y= e^t$.

2- $U = tan^{-1}x + = \sqrt{1-y^2}$ if, $x = t^2$, y= t-1.

3- U = sin(x+ y) +cos xy, , x = π +2t, y= π-4t.

4- $U = x^2 + y^2 + 6xy$, x = 3t-1, y= 4t-3.

5- $U = \frac{x}{y}$, x = sech t, y= coth t.

6- $U = \tanh^{-1}(\frac{r}{s})$, r = x sin yz, s= x cos yz

7-U = ln(r+ s + t)if, r = xy, s= x z, t= y z.

8- If f(x, y) = xcosy+y e^x. Prove that $f_{yx} = f_{xy}$.

9- If $f(x, y) = \tan^{-1}(\frac{x}{y})$. Prove that $f_{xx} + f_{yy} = 0$

10- If $f(x, y) = e^{-2y}$ xcos2x. Prove that $f_{xx} + f_{yy} = 0$

11- If $W = \sin(x + ct)$. Prove that $W_{tt} = c^2 W_{xx}$.
12- If $W = \cos(2x + 2ct)$. Prove that $W_{tt} = c^2 W_{xx}$.
13- If $W = \ln(2x + 2ct) + \cos(2x + 2ct)$. Prove that $W_{tt} = c^2 W_{xx}$.
14- If $W = \tan(x - ct)$. Prove that $W_{tt} = c^2 W_{xx}$.

CHAPTER TWO
Differential Equations (d.e)

Introduction:-

Definition 1.1

Differential Equations (d.e)

If y is a function of x, where y is called the dependent variable and x is called the independent variable. A differential equation is a relation between x and y which includes at least one derivative of y with respect to (w.r.to) x. Which has two types:-

1- Ordinary d.e.
   If (d.e) involves only a single independent variable this derivatives are called ordinary derivatives, and the equation is called ordinary (d.e).
2- Partial d.e.
   If there are two or more independent variables derivatives are called partial derivatives, and the equation is called partial (d.e).

For example

(a) $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 2e^x$

(b) $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$

(c) $\frac{df}{dx} + x = \sin x$

(d) $y''' - 3y'' + y = 0$

The Order of (d.e)

Is that the derivative of highest order in the equation for example (a) order 3 (b) order 2 (c) order 1 (d) order 3?

Solution of Differential Equations

Any relation between the variables that occur in (d.e) that satisfies the equation is called a solution or when y and it's derivatives are replace through out by f(x) and it's derivatives for example

Show that y= acos2x +bsin2x, of derivative a solution of (d.e)

$y'' + 4y = 0$ ………………………………………………….. (1),

Where a and b are arbitrary constant.

Solution

Since $y = a\cos 2x + b\sin 2x$,

$y' = -2a\sin 2x + 2b\cos 2x$

y'' $-4a\cos 2x - 4b\sin 2x$, put y and y'' in (1)

$-4a\cos 2x - 4b\sin 2x + 4(a\cos 2x + b\sin 2x) = 0$

0 =0, then this solution called the general solution.

Exercises

Show that each equation is a solution of the indicated (d.e)

(1) $y''' = y''$ where $y = c_1 + c_2x + c_3e^x$

(2) $x y'' + y' = 0$ where $y = c_1\ln x + c_2$

(3) $y'' + 9y = 4\cos x$ where $2y = \cos x$

(4) $y'' - y = e^{2x}$ where $y = e^{2x}$

(5) $y'' = 2y\sec^2 x$ where $y = \tan x$.

First Order Differential Equations

The first order differential equation take in the form:-

$M(x,y)\,dx + N(x,y)\,dy = 0$……………………………………(2)

Where M and N are functions of x and y or both.

To solve this type of (d.e), we consider the following methods:-

1-**Variable Separable**

Any (d.e) can be put in the form:-

$f(x)dx + g(x)dx = 0$, or x and derivative of x in term and y derivative of y in another term.

This equation called **Variable Separable**, this equation can be solve by take the integral of two sides of this equation

$\int f(x)dx + \int g(x)dy = c$, where c is arbitrary constant.

Example

Solve $xdy = ydx$

$ydx - xdy = 0$

$(ydx - xdy = 0)\ \frac{1}{xy}$,

$\frac{dx}{x} - \frac{dy}{y} = 0$, by integral of two sides

$\int \frac{dx}{x} - \int \frac{dy}{y} = c$,

$\ln x - \ln y = c$,

$\ln\frac{x}{y} = c$,

$\frac{x}{y} = e^c = c_1$,

$\therefore y = \frac{x}{c_1}$.

Problems

Solve the following differential equations:-

1- $x(2y-3)dx + (x^2+1)dy = 0$

2- $x^2(y^2+1)dx + y\sqrt{x^3+1}\,dy = 0$

3- $\frac{dy}{dx} = e^{x-y}$

4- $\sqrt{xy}\,\frac{dy}{dx} = 1$

5- $e^y \sec x dx + \cos x dy = 0$.

2-**Homogeneous Differential Equation** (H.d.e)

The differential equation as form

$M(x,y)\,dx + N(x,y)\,dy = 0$,

Where M and N are functions of x and y is called (H.d.e) if satisfy the condition

$$\left.\begin{array}{l} M(kx, ky) = k^n M(x, y) \\ N(kx, ky) = k^n N(x, y) \end{array}\right\} \text{Where k is constant.}$$

For example

1- $(x^2 - y^2)dx + 2xydy = 0$

$M = x^2 - y^2$, $N = 2xy$

$M(kx,ky) = (kx)^2 - (ky)^2 = k^2x^2 - k^2y^2 = k^2(x^2 - y^2)$

$k^2(M)$

$N(kx,ky) = 2(k^2xy) = k^2(2xy)$

$k^2(N)$.

The equation is (H.d.e).

3- Solve $(x-y)dx + xydy = 0$

$M = x - y$, $N = xy$

$M(kx,ky) = (kx) - (ky) = k(x - y)$

$k(M)$

$N(kx,ky) = k^2(xy)$

$k^2(N)$.

The equation is not (H.d.e).

If the equation is homogeneous we can solve by the following method:-

Put (H.d.e) in the form

$\frac{dy}{dx} = f(y/x)$………………………………………………… (3)

Let $v = y/x$…………………………………………………….. (4)

Put (4) in (3) $\Rightarrow \frac{dy}{dx} = f(v)$…………………………………..(5)

From (4) $y = xv$, and

$dy = xdv + vdx$, divided by dx

$\frac{dy}{dx} = x\frac{dv}{dx} + v$, since $\frac{dy}{dx}$ = f (v) from (5)

f(v) $= x\frac{dv}{dx} + v$ $\Rightarrow$ f (v)-v $= x\frac{dv}{dx}$

(f(v)-v)dx =xdv $\Rightarrow$ $\frac{dv}{f(v)-v} = \frac{dx}{x}$

$\frac{dx}{x} + \frac{dv}{v-f(v)} = 0$. Or

.... (6) $\frac{dx}{x} - \frac{dv}{f(v)-v} = 0$

After solving replace v by y/x.

Example

Solve $(x^2 + y^2)\,dx + 2xydy=0$

Solution

Since this equation (H.d.e). Now

$2xydy= -(x^2 + y^2)dx$,

$\frac{dy}{dx} = -\frac{x^2+y^2}{2xy}$, put y=xv

$\frac{dy}{dx} = -\frac{x^2+x^2v^2}{2x(xv)} = -\frac{1+v^2}{2v}$

$\therefore$ f(v) = $-\frac{1+v^2}{2v}$,

$\frac{dx}{x} + \frac{dv}{v-f(v)} = 0$,

$\frac{dx}{x} + \frac{dv}{v+\frac{1+v^2}{2v}} = 0$,

$\frac{dx}{x} + \frac{2vdv}{1+3v^2} = 0$, by integral both sides

$\ln x + 1/3\ln(1+3v^2) = c$,

$\ln x + 1/3\ln(1+3y^2/x^2) = c$

$\therefore \ \ln x\sqrt[3]{1+\frac{3y^2}{x^2}} = c$,

$x\frac{\sqrt[3]{x^2+3y^2}}{x} = c_1$, *where* $c_1 = e^c$

$\sqrt[3]{x^2+3y^2} = c_1$

Problems

Solve the following differential equations:-

(1) $2xydx - (xy+x^2)dy=0$

(2) $(x^2+2y^2+3xy)dx + x(x-2y)dy = 0$

(3) $(12x^2y - 4y^3)dx + x(3y^2 - 6x^2)dy = 0$
(4) $(xe^{y/x} - ye^{y/x})dx + xe^{y/x}dy = 0$
(5) $(3x + xe^{y/x} - ye^{y/x})dx + xe^{y/x}dy = 0$

3-**<u>Exact Differential Equation</u>**

The differential equation as form

$M(x,y)\,dx + N(x,y)\,dy = 0$ ……………………………………….……..(7)

There is function

$f(x,y) = c$……………………………………………….……...(8),

Which a solution of (7).

$df(x,y) = 0$…………………………………………….………..(9).

From total partial differential equation

$df(x,y) = \partial f/\partial x\, dx + \partial f/\partial y\, dy$……………………………….(10).

From 7,8,9 and 10,

$$\left.\begin{array}{l}\partial f/\partial x = M \\ \partial f/\partial y = N\end{array}\right\} \quad \ldots\ldots\ldots (11)$$

Now

$\partial^2 f/\partial y\partial x = \partial M/\partial y$, $\partial^2 f/\partial x\partial y = \partial N/\partial x$,

Since $\partial^2 f/\partial y\partial x = \partial^2 f/\partial x\partial y$

$\boldsymbol{\partial M/\partial y = \partial N/\partial x}$, …………………………………………………. (12).

Which condition of exact?

To solve equation (7) we must find **f** which the solution of equation (7).

$\partial f/\partial x = M$ from (11),

$\partial f = M\,\partial x$,

$f = \int M\,\partial x + A(y)$ ………………………………………………….(13),

Where A(y) is function of y.

We must find A(y)

$\partial f/\partial y = N = \partial/\partial y\,[\int M\,\partial x] + A'(y)$, since $\partial f/\partial y = N$, from (11),

$\therefore\ A'(y) = N - \partial/\partial y\,[\int M\,\partial x]$

$\therefore\ A(y) = \int\{N - \partial/\partial y\,[\int M\,\partial x]\} + c$………………………………..(14)

Now put (14) in (13) which complete solution.

<u>Example</u>

Solve the following differential equation

$$\frac{dy}{dx} = \frac{xy^2 - 1}{1 - x^2y}$$

<u>Solution</u>

$(1 - x^2y)dy - (xy^2 - 1)dx = 0$,

$N = 1 - x^2y$, $M = 1 - xy^2$,

$\partial M/\partial y = \partial N/\partial x = -2xy$,

$\partial f/\partial x = M$,

$f = \int M\,\partial x + A(y) = \int (1 - xy^2)\,\partial x + A(y)$

$f = (x - (x^2 y^2)/2 + A(y)$................(*)

(we must find A(y),

$\partial f/\partial y = - x^2 y + A'(y) = N = 1 - x^2 y$

$\therefore A'(y) = 1$,

$A(y) = y + c$ put in (*)

$F = (x - (x^2 y^2)/2 + y + c$.

Problems

Solve the following differential equations:-

(1) $(2x + y)dx + (x + y)dy = 0$

(2) $(3x - y)dx - (x-y)dy = 0$

(3) $(\cos x + y)dx + (2y + x)dy = 0$

(4) $(ye^x + y)dx + (x + e^x)dy = 0$

(5) $\tan y\, dx + x \sec^2 y\, dy = 0$.

Integrating Factor

If the equation

M dx+N dy =0.

Is not exact, then there is μ such that

$\mu M\, dx + \mu N\, dy = 0$..(*)

Is exact then

$\partial(\mu M)/\partial y = \partial(\mu N)/\partial x$.

To find μ (μ is called integrating factor).

Theorem I

(i) If $\dfrac{\dfrac{\delta M}{\delta y} - \dfrac{\delta N}{\delta x}}{N} = f(x)$ (function of x, or constant.

Then $\mu = e^{\int f(x)dx}$.

(ii) If $\dfrac{\dfrac{\delta N}{\delta x} - \dfrac{\delta M}{\delta y}}{M} = g(y)$ (function of y, or constant.

Then $\mu = e^{\int g(y)dx}$.

(iii) If μ is function of x and y, then there is no general method to find μ (integrating factor).

Example

Solve the following differential equation

$y dx + (3+3x - y)dy = 0$

Solution

$M = y$, $N = 3 + 3x - y$.

$M_y = 1 \neq N_x = 3$, not exact

$$\frac{N_x - M_y}{M} = \frac{3-1}{y} = \frac{2}{y}, \textit{is function of } y$$

Then $\mu = e^{\int g(y)dy}$

$$\mu = e^{\int \frac{2}{y}dy} = e^{2\ln y} = y^2$$

$(ydx + (3+3x - y)dy = 0)\ y^2$

$y^3 dx + (3y^2 + 3xy^2 - y^3)dy = 0$

$M = y^3$, $N = 3y^2 + 3xy^2 - y^3$.

$M_y = 3y^2 = N_x = 3y^2$ exact

$\partial f/\partial x = M$,

$f = \int M\, \partial x + A(y) = \int (y^3)\, \partial x + A(y)$

$f = y^3 x + A(y)$……………. (*)

We must find A(y),

$\partial f/\partial y = 3xy^2 + A'(y) = N = 3y^2 + 3xy^2 - y^3$

$\therefore\ A'(y) = 3y^2 - y^3$,

$A(y) = y^3 - y^4/4 + c$ put in (*)

$f = y^3 x + y^3 - y^4/4 + c$.

Example

Solve the following differential equation

$$\frac{dy}{dx} = x - y$$

Solution

$dy = (x-y)dx$

$(x-y)dx - dy = 0$

$M = x - y$, $N = -1$.

$M_y = -1 \neq N_x = 0$, not exact

$$\frac{M_y - N_x}{N} = \frac{-1-0}{-1} = 1, \textit{is function of } x$$

Then $\mu = e^{\int f(x)dx} = e^{\int dx} = e^x$

$((x-y)\, dx - dy = 0)\ e^x$

$e^x (x-y)dx - e^x dy = 0$

$M = xe^x - y e^x$, $N = -e^x$.

$M_y = -e^x = N_x = -e^x$ exact

$\partial f/\partial x = M$,

$f = \int M\, \partial x + A(y) = \int (xe^x - y e^x)\, \partial x + A(y)$

$f = xe^x - e^x - y e^x + A(y)$……………. (*)

We must find A(y),

$\partial f/\partial y = -e^x + A'(y) = N = -e^x$

$\therefore\ A'(y) = 0$,

$A(y) = c$ put in (*)

$f = xe^x - e^x - y e^x + c$.

4- **First – Order Linear Differential Equation**

If the equation as form:-

$$\frac{dy}{dx} + P(x)\, y = Q(x) \ldots\ldots (15)$$

Where P and Q are functions of x.

To solve equation (15), we must find (I) where

$I = e^{\int p dx}$ { I is integrating factor}.

Now multiple both sides of (15) by I

$dy + Py\, dx = Q\, dx \ldots\ldots (15)$

$e^{\int p dx}$ { $dy + Py\, dx = Q\, dx$ }

$e^{\int p dx}\, dy + e^{\int p dx} Py\, dx = e^{\int p dx} Q\, dx$ }

$d\,[y e^{\int p dx}] = Q\, e^{\int p dx}\, dx \ldots\ldots (16)$

by integrate (16)

$y e^{\int p dx} = \int Q\, e^{\int p dx}\, dx + c,$

Which the solution of (15), or the solution is

$$\mathbf{Iy = \int IQ\ dx + c}$$

Example

Solve the following differential equation

$$\frac{dy}{dx} + \frac{y}{x} = 2$$

Sol

Since $P = 1/x$, $Q = 2$

$I = e^{\int p dx} = I = e^{\int dx/x} = e^{\ln x} = x.$

The solution

$Iy = \int IQ\ dx + c$

$xy = \int 2x\ dx + c,$

$xy = x^2 + c$

$y = x + c/x$

Problems

Solve the following differential equations:-

(1) $y' + 2y = e^x$

(2) $xy' + 3y = x^2$

(3) $y' + y\cot x = \cos x$

(4) $x y' + 2y = x^2 - x + 1$

(5) $y' - y \tan x = 1.$

4.1 **The Bernoulli Equation**

The equation

$$\frac{dy}{dx} + P(x)\, y = Q(x)\, y^n \ldots\ldots (*), \text{ if } n \neq 1.$$

Is similar to L-equation is called Bernoulli Equation.

We shall show how transform this equation to linear equation. In fact we must reduce this equation to linear, product (*) by (y^{-n}) or

$[\frac{dy}{dx} + P(x)\, y = Q(x)\, y^{n}]\, y^{-n}$

$\frac{dy}{dx} y^{-n} + P\, y^{1-n} = Q$ …………………. (**).

Let $w = y^{1-n} \Rightarrow dw = (1-n)\, y^{-n}\, dy$ or

$\frac{dw}{1-n} = y^{-n}\, dy$ put in (**)

$$\left.\begin{array}{l} \frac{dw}{(1-n)dx} + P\, y^{1-n} = Q, \\ \quad \text{or} \\ \frac{dw}{dx} + (1-n)P\, w = (1-n)\, Q \end{array}\right\} \text{Which is L-equation}$$

Example

Solve the following differential equation

$\frac{dy}{dx} + \frac{y}{x} = y^{2}$

Sol

$[\frac{dy}{dx} + \frac{y}{x} = y^{2}]\, y^{-2}$

$\frac{dy}{dx} y^{-2} + \frac{y^{-1}}{x} = 1$………… (#),

Let $w = y^{-1} \Rightarrow dw = - y^{-2}\, dy$

$-dw = y^{-2}\, dy$, put in (#)

$- \frac{dw}{dx} + \frac{w}{x} = 1$

$\frac{dw}{dx} - \frac{w}{x} = -1.$

$P = - \frac{1}{x}, Q = - 1,$

$I = e^{\int p dx} = I = e^{-\int dx/x} = e^{-\ln x} = \frac{1}{x}.$

The solution

$Iw = \int IQ\, dx + c,$

$\frac{w}{x} = \int - \frac{1}{x}\, dx + c,$

$$\frac{w}{x} = -\ln x + c,. \text{ Since } w = y^{-1} = \frac{1}{y}$$

$$\frac{1}{xy} = -\ln x + c,$$

$$y = \frac{1}{x(c - \ln x)}.$$

Problems

Solve the following differential equations:-

(1) $y' - 2y/x = 4x y^2$

(2) $y' + y/x = x y^2$

(3) $y' + y = y^3 e^{2x} \sin x$

(4) $y' + 2y/x = 5 y^2/x^2$

(5) $x y' + y = y^2$.

Second – Order Differential Equation

Special Types

Certain types of second order differential equation such that

$$F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}) \qquad (17)$$

Can be reduced to first order equations by a suitable of variables:-

Type I

Equation with dependent variable when equation as form

$$F(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}) \qquad (18)$$

It can be reduced to first order equation by suppose that:-

$$p = \frac{dy}{dx}, \qquad \frac{d^2y}{dx^2} = \frac{dp}{dx}$$

Then equation (18) takes the form

$$F(x, p, \frac{dp}{dx}) = 0,$$

Which is of the first order in p, if this can be solved for p as function of x says?

$p = q(x, c_1)$.

Then y can be found from one additional integration

$$y = \int \left(\frac{dy}{dx}\right) dx + c = \int p \, dx + c = \int q(x, c_1)\, dx + c.$$

Type II

Equation with independent variable when equation (17) does not contain x explicit but has the form

$$F(y, \frac{dy}{dx}, \frac{d^2y}{dx^2})=0 \ldots\ldots(19)$$

The substitution to use are :-

$$p = \frac{dy}{dx}, \qquad \frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = \frac{dp}{dy} p$$

The equation (19) become

$$F(y, p, p\frac{dp}{dx}) = 0,$$

Which is of the first order in p. Which solution gives p in terms of y, and then further integration gives the solution of equation (19).

Example 1

Solve the following differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0 \ldots\ldots (*)$$

Let $p = \frac{dy}{dx}$, $\frac{d^2y}{dx^2} = \frac{dp}{dy} p$, put in (*)

$$\frac{dp}{dy} p + p = 0, \quad \div p$$

$$\frac{dp}{dy} + 1 = 0,$$

$dp + dy = o.$ by integration

$p + y = c_1$

$$\frac{dy}{dx} + y = c_1$$

$$\frac{dy}{dx} = c_1 - y$$

$$\frac{dy}{c_1 - y} = dx \Rightarrow -\ln(c_1 - y) = x + c_2$$

$$\ln(c_1 - y) = -x + c_2 \Rightarrow c_1 - y = e^{-x+c_1} = ce^{-x}$$

$y = c_1 - ce^{-x}$

Example 2

Solve the following differential equation

$$X^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} = 1$$

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = 1/x^2 \ldots\ldots (**)$$

Let $p = \frac{dy}{dx}$ and $\frac{d^2y}{dx^2} = \frac{dp}{dx}$, put in (**)

$\frac{dp}{dx} + \frac{1}{x}p = 1/x^2$, which linear in p,

$I= e^{\int pdx} = I= e^{\int dx/x} =x.$

$Ip = \int IQ\ dx + c \Rightarrow Ip = \int x(1/x^2)\,dx + c = \ln x + c$

$\therefore\ xp = \ln x + c$

$P = (\ln x)/x + c/x$

Let $\frac{dy}{dx} = (\ln x)/x + c/x$

$dy= [(\ln x)/x]dx + (c/x)dx,$

$y = (\ln x)^2/2 + c \ln x + c_1$

Problems

Solve the following differential equations:-

(1) $y'' + y' = 0$

(2) $y'' + y\,y' = 0$

(3) $\mathbf{x}\,y'' + y' = 0$

(4) $y'' - y' = 0$

(5) $y'' + w^2 y = 0$, where w constant $\neq 0$.

Homogeneous-Second – Order (D. E) With Constant Coefficient

Consider linear equation with constant coefficient which in the form:-

$y'' + a\,y' + by = 0$…………………………………………… (20)

where a, b are constant.

How to solve this equation we shall now find how to determine m such that

$y= e^{mx}$ is a solution of (20) then

$y' = me^{mx}$ and $y'' = m^2e^{mx}$, put in (20)

$m^2e^{mx} + a\,me^{mx} + be^{mx} = 0$

$e^{mx}(m^2 + a\,m + b) = 0,$

since $e^{mx} \neq 0$, then

$m^2 + a\,m + b = 0$………………………………………… (21).

Which called **characteristic equation**.

Then we saw that e^{mx} is a solution of (20) $\Leftrightarrow$ m is root of (21).

Note

The general solution of (20), there is three cases:-

Case i

If $m_1 = m_2$ in equation (21), the solution of (20) (homogeneous equation) is

$y_h= (c_1 + xc_2)\,e^{mx}$

Case ii

If $m_1 \neq m_2$ in equation (21), the solution of (20) (homogeneous equation) is

$y_h= c_1e^{m_1x} + c_2e^{m_2x}$

<u>Case iii</u>

If m_1 and m_2 roots (m= $\alpha + i\beta$ where i= $\sqrt{-1}$) in equation (21), the solution of (20) (homogeneous equation) is

$y_h = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

<u>Ex i</u>

Solve $y'' + 4y' + 4y = 0$……………………………………………..(*)

<u>Sol</u>

let $y = e^{mx}$, $y' = me^{mx}$ and $y'' = m^2e^{mx}$, put in (*)

$m^2e^{mx} + 4\,me^{mx} + 4e^{mx} = 0$

$e^{mx}(m^2 + 4\,m + 4) = 0$,

since $e^{mx} \neq 0$, then

$m^2 + 4\,m + 4 = 0$

Which called **characteristic equation?**

$(m+2)^2 = 0 \Rightarrow m_1 = m_2 = -2$, the solution of (*) is

$y_h = (c_1 + xc_2)\,e^{-2x}$

<u>Ex ii</u>

Solve $y'' + y' - 6y = 0$……………………………………………..(**)

<u>Sol</u>

let $y = e^{mx}$, $y' = me^{mx}$ and $y'' = m^2e^{mx}$, put in (*)

$m^2e^{mx} + me^{mx} - 6e^{mx} = 0$

$e^{mx}(m^2 + m - 6) = 0$,

since $e^{mx} \neq 0$, then

$m^2 + m - 6 = 0$

This called characteristic equation

$(m+3)(m-2) = 0 \Rightarrow$ either $m_1 = -3$ or $m_2 = 2$, the solution of (**) is

$y_h = c_1\,e^{-3x} + c_2\,e^{2x}$

<u>Ex iii</u>

Solve $y'' - 4y' + 5y = 0$……………………………………………..(**)

<u>Sol</u>

let $y = e^{mx}$, $y' = me^{mx}$ and $y'' = m^2e^{mx}$, put in (*)

$m^2e^{mx} - 4me^{mx} + 5e^{mx} = 0$

$e^{mx}(m^2 - 4\,m + 5) = 0$,

since $e^{mx} \neq 0$, then

$m^2 - 4m + 5 = 0$

This called characteristic equation

$m_1 = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2}$,

$m_2 = 2 \pm i \Rightarrow \alpha + i\beta$, $\alpha = 2$, $\beta = 1$,

$y_h = e^{2x}(c_1 \cos x + c_2 \sin x)$.

Non-Homogeneous-Second – Order (D. E) With Constant Coefficient

Consider the equation which in the form:-

$y'' + a y' + by = f(x)$……………………………………………… (22)

where a, b are constant.

To find the general solution of (22). We find solution of homogeneous part

$y'' + a y' + by = 0$………………………………………………… (23),

let y_h be solution of (23).

Then the solution of (22) take by added the solution y_h to any another special solution y_p of (22)such that the general solution of (22) become

$y(x) = y_h + y_p$

Method of Undetermined Coefficient

The condition of this may that the form f(x), may be guessed for example f(x) may be a single power of x a polynomial an exponential function a sin, coin or sum function.

The general solution of (non- H. D.E) become.

$y(x) = y_h + y_p.$

We student to find y_h .We can select y_p from the following table.

Table 1

$f(x)$	y_p	mod
kx^n $(n = 1,2,.........)$	$k_n x^n + k_{n-1} x^{n-1} + ... + k_1 x + k_0$ $k_n, ..., k_1, k_0$ are constants	0
ke^{px}	ce^{px}, c constant	p
$k \sin \alpha x$ $k \cos \alpha x$	$m \cos \alpha x + n \sin \alpha x$ m and n constant	$i\alpha$

How to use the table to find y_p:-

(a) If f(x) function of first column of table, then we take y_p from second column which corresponding it.

(b) If f(x) is sum of two function of 1-st column then we selected y_p sum of function of 2-th column which corresponding.

(c) If the number of f(x) is root of y_h we must modify the solution of y_p, Modification Rule

If the number listed in the table 1 the last column root of y_h (H-part) of equation (23).

Then the function in second column of table must be multiplied by x^m where m is the multiplicity of the root in that equation [hence for a second –order equation m may be equal 1 or 2].

Example1

By use table 1 write y_p where

(a) $f(x)= 2x^3$
(b) $f(x)= 3e^{2x}$
(c) $f(x)= 4\sin 2x$
(d) $f(x)= \cos\alpha x + \sin\alpha x$
(e) $f(x)= e^{3x} + x$

Sol

(a) $y_p = k_3 x^3 + k_2 x^2 + k_1 x + k_0$
(b) $y_p= ce^{2x}$
(c) $y_p = m \cos\alpha x + n\sin\alpha x$
(d) $y_p = m \cos\alpha x + n\sin\alpha x$
(e) $y_p= ce^{3x} + k_1 x + k_0$

Example2

Find the general solution of the following (d.e)

$y'' \quad -4y= 8x^2$……………………………………………….(*)

Sol

$y'' \quad -4y= 0$………………………………………………….(**)

let $\quad y= e^{mx}$, $y' = me^{mx}$ and $y'' \quad =m^2e^{mx}$, put in (**)

$m^2e^{mx} - 4e^{mx} =0$

$e^{mx} (m^2 - 4) =0$,

since $e^{mx} \neq 0$, then

$m^2 - 4 =0$

This called characteristic equation

$m^2 =4 \Rightarrow m = \pm 2$, (or $m_1 = 2$, $m_2 = -2$;

$\mathbf{y_h= c_1 e^{-2x} + c_2 e^{-2x}}$,

$y_p = k_2 x^2 + k_1 x + k_0$, we must find k_2 , k_1 and k_0

$y_p = k_2 x^2 + k_1 x + k_0$, $y_p' = 2k_2 x + k_1$ and $y_p'' \quad = 2k_2$ put in (*)

$2k_2 -4(k_2 x^2 + k_1 x + k_0)= 8x^2$

$-4k_2 =8, \quad 2k_2 -4k_0=0$

$k_2 =-2, \; k_1 = 0, k_0=-1$

$\mathbf{y_p = -2 x^2 -1}$

$y (x) =y_h + y_p$

$\mathbf{y (x) = c_1 e^{-2x} + c_2 e^{-2x} -2 x^2 -1.}$

Variation of Parameter

Consider the equation which in the form:-

$y'' \;\; +a\,y' +by= f(x)$ …………………………………………… (24)

Where a, b are constant, f(x) be any function of x.

To solve (24)

(a) Find y_h (solution of (H-part),

$y_h = c_1\, u_1 \;+ c_2\, u_2$ ……………………………………………(25)

Where c_1 and c_2 are arbitrary constant, and u_1 and u_2 are two function as form:-

let e^{mx} or xe^{mx} $\;e^{\alpha x}\cos\beta x$ or $e^{\alpha x}\sin\beta x$), which solution of (H-part).

(b) We replace c_1 and c_2 by function of x say v_1 and v_2 then (#) become

$y_h = c_1\, v_1 \;+ c_2\, v_2$ ……………………………………………..(26),

Which solution of (24),

$y_h' = v_1 u'_1 \;\; v'_1 u_1 \;+ v_2\, u'_2 + v'_2\, u_2$

$y_h' = (v_1 u'_1 + v_2\, u'_2) + \;(v'_1\, u_1 \;+ + v'_2\, u_2)$ ………………….…(27),

from this

$y_h' = v_1 u'_1 + v_2\, u'_2,$

and

$v'_1\, u_1 \;+ + v'_2\, u_2=0$ ………………………………………………..(28)

Now

$y_h'' = v'_1\, u'_1 \;+ v_1 u''_1 \;+ v'_2\, u'_2 + v_2\, u_2''$

(c) Now put y_h, y_h' and y_h'' in (24)

$v'_1\, u'_1 \;+ v_1 u''_1 \;+ v'_2\, u'_2 + v_2\, u_2'' + a[v_1 u'_1 + v_2\, u'_2] + b[c_1\; v_1 \;+ c_2\, v_2]=f(x)$

$v_1\,[u_1'' \;\; +a\, u_1' +b\, u_1] \;+ v_2[u_2'' \;\; +a\, u_2' +b\, u_2] + v'_1\, u'_1 + v'_2\, u'_2=f(x)$

Since $y'' \;\; +a\, y' +by=0,$

$\therefore\; u_1'' \;\; +a\, u_1' +b\, u_1=o,\; u_2'' \;\; +a\, u_2' +b\, u_2 =o,$ and

$v'_1\, u'_1 + v'_2\, u'_2=f(x).$

{the value in brackets vanish because by hypothesis both u_1 and u_2 are solution of homogeneous equation corresponding to (24).

Then the equation (24) satisfy by equation

$\mathbf{v'_1\, u_1 \;+ v'_2\, u_2=0}$ ………………………………................... (29)

$\mathbf{v'_1\, u'_1 \;+ v'_2\, u'_2=f(x)}$ ………………………………………… (30).

(d) By solve (29) and (30) we find two unknown $\mathbf{v'_1, v'_2}$ and we find $\mathbf{v_1}$,$\mathbf{v_2}$ by integral.

(e) The general solution of (non- H. D.E) (24) is

$\mathbf{Y(x) = v_1\, u_1 \;+ v_2\, u_2}$

Example1

Find the general solution of the following (d.e)

$y'' \;\; -y'-2y= e^{-x}$ …………………………………………………. (*)

Sol

(1) Find the general solution of (H.d.e) $y'' - y' - 2y = 0$, or

$m^2 - m - 2 = 0$,

$(m-2)(m+1) = 0 \Rightarrow$

$y_h = c_1 e^{2x} + c_2 e^{-x}$,

$u_1 = e^{2x}$, $u_2 = e^{-x}$,

$u'_1 = 2e^{2x}$, $u'_2 = -e^{-x}$

$\mathbf{v'_1 u_1 + v'_2 u_2 = 0}$……………………#

$\mathbf{v'_1 u'_1 + v'_2 u'_2 = f(x)}$………………. ##

$v'_1 e^{2x} + v'_2 e^{2x} = 0$……………………#

$\underline{2v'_1 e^{2x} - v'_2 e^{-x} = e^{-x})}$………………. ##

$v'_1 e^{2x} = e^{-x} \Rightarrow v'_1 = 1/3e^{-3x} \Rightarrow v_1 = -1/9e^{-3x} + c_1$,

From (#)$v'_1 e^{2x} = - v'_2 e^{-x}$, or $v'_2 = - v'_1 e^{3x}$

$v'_2 = -1/3 \Rightarrow$

$v_2 = -1/3x + c_2$

Since $\mathbf{Y(x) = v_1 u_1 + v_2 u_2}$

$\mathbf{Y(x)} = (-1/9e^{-3x} + c_1)e^{2x} + (-1/3x + c_2)e^{2}$.

Problems

Solve the following differential equations:-

(1) $y'' + 4y' = 3x$

(2) $y'' - 4y' = 8x^2$

(3) $y'' - y' - 2y = 10\cos x$

(4) $y'' - 4y' + 3y = e^x$

(5) $y'' + y = \sec x$.

Problems

Solve the following differential equations:-

1-$x(2y-3)dx + (x^2 + 1)dy = 0$

2- $x^2 (y^2+1)\, dx + y\sqrt{x^3 + 1}\, dy = 0$

3- $\sin x \frac{dx}{dy} + \cosh 2y = 0$

4 - $\sqrt{2xy}\frac{dy}{dx} = 1$

5- $\ln x \frac{dx}{dy} = \frac{x}{y}$

6- ($xe^y\, dy + \frac{x^2 + 1}{y}\, dx = 0$

7- $y\sqrt{1 + x^2}\, dy + \sqrt{y^2 - 1}\, dx = 0$

8- $x^2y \dfrac{dy}{dx} = (1+x) \csc y$

9- $\dfrac{dy}{dx} = e^{x-y}$

10- $e^y \sec x \, dx + \cos x \, dy = 0$

(H. d. e)

11- $(x^2 + y^2) dx + xy dy = 0$

12 - $x^2 dx + (y^2 - xy)dy = 0$

13 – $x e^{y/x} + y)dx - xdy = 0$

14 –$(x + y) dy + (x - y) dx = 0$

15 - $\dfrac{dy}{dx} = \dfrac{y}{x} + \cos \left(\dfrac{y-x}{x}\right)$

16- $xdy - 2ydx = 0$

17- $2xydy + (x^2 - y^2)dx = 0$

(Linear d. e)

18- $\dfrac{dy}{dx} + 2y = e^{-x}$

19- $x y' + 3y = \dfrac{\sin x}{x^2}$

20- $2 y' - y = e^{x/2}$

21- $xdy + ydx = \sin x \, dx$

22- $xdy + ydx = ydy$

23- $(x-1)^3 y' + 4 (x-1)^2 y = x+1$

24- $\cosh x \, dy + (y \sinh x + e^x) dx = 0$

25- $e^{2y} dx + 2(x e^{2y} - y)dy = 0$

26 – $(x-2y) dy + y dx = 0$

27 – $(y^2 + 1) dx + (2xy + 1) dy = 0$

(Exact d. e)

Use the given integrating factor to make (d. e) exact then solve the equation

28 - $(x+2y) dx - x dy = 0$, $\left(I = \dfrac{1}{x^3}\right)$

29 – $y dx + x dy = 0$, $\left(I = \dfrac{1}{xy}\right)$ or $\left(I = \dfrac{1}{(xy)^2}\right)$

Solve (exact d. e)

30 - $(x + y) dx + (x+y^2) dy = 0$

31 – $(2xe^y + e^x) dx + (x^2 + 1) e^y dy = 0$

32 – $(2xy + y^2)dx + (x^2 + 2xy - y)dy = 0$

33- $(x+\sqrt{y^2+1})dx-(y-\dfrac{xy}{\sqrt{y^2+1}})dy=0$

34- $x\,dy+y\,dx+x^3dx=0$

35- $x\,dy-y\,dx=x^2\,dx$

36- $(x^2+x-y)\,dx+xdy=0$

37- $(e^x+\ln y+\dfrac{y}{x})\,dx+(\dfrac{x}{y}+\ln x+\sin y)\,dy=0$

38- $(\dfrac{y^2}{1+x^2}-2y)\,dx+(2y\tan^{-1}x-2x+\sinh y)dy=0$

39- $dy+\dfrac{y-\sin x}{x}dx=0$

(Second- Order)

40- $y''+2y=0$

41- $y''+5y'+6y=0$

42- $y''+6y'+5y=0$

43- $y''-6y'+10y=0$

44- $y''+y=0$

45- $y''+y'\quad x$

46- $y''+y=\sin x$

47- $y''-2y'+y=e^{-x}$

48- $y''+2y'+y=e^x$

49- $y''-y=\sin x$

50- $y''+4y'+5y=x+2.$

51- $y''-y=e^x$

52- $y''+y=\sec x$

53- $y''+y=\tan x$

54- $y''+y=\cot x$

CHAPTER THREE

Laplace Transformation (L. T)

Definition 2.1

Let f (t) be function of variable t which define on all value of t such that (t >0).

The Laplace transformation of f (t) which written as L {f (t)} is

$$F(s) = L\ \{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \ldots\ldots\ldots (1)$$

Note 1

The Laplace transformation is define in (1) is converge to value of s, and no define if the integral in (1) has no value of s.

Laplace Transformation of Some Function:-

Using the definition (1) to obtain the following transforms:-

1- If f(t) =1

Solution

Since $L\ \{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = -\frac{1}{s} e^{-st} \Big|_0^\infty = 0 + \frac{1}{s} e^0 = \frac{1}{s}$

$\therefore\ L\ (1) = \frac{1}{s}$.

2-If f (t) = e^{at}

Solution

Since $L\ \{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$L\ \{f(t)\} = \int_0^\infty e^{-st} e^{at}\, dt = L\ \{f(t)\} = \int_0^\infty e^{(a-s)t}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt = -\frac{1}{s-a} e^{-(s-a)t} \Big|_0^\infty$$

$$= \frac{1}{s-a}$$

$$\therefore\ l\{e^{at}\} = \frac{1}{s-a}\ .$$

Note 2

Let f(t) be function and c constant then

(i) L {cf (t)} = cL {f (t)}

(ii) L $\{f_1(t) \pm f_2(t)\}$ = L $\{f_1(t)\} \pm$ L $\{f_2(t)\}$

3-If f(t) = cos(wt)

4-If f(t) = sin(wt).

Solution

From Euler formula

e^{-iwt} = cos (wt) + isin (wt).

L{ e^{-iwt} } = L{ cos (wt)} + i L{sin (wt)}……………………(*).

But

$l\{e^{iwt}\} = \dfrac{1}{s-iw}$, from (2)

$$\frac{1}{s-iw} = \frac{1}{s-iw} \times \frac{s+iw}{s+iw} = \frac{s+iw}{s^2+w^2}$$

$$= \frac{s}{s^2+w^2} + i\,\frac{w}{s^2+w^2}$$

$\therefore\ l\{e^{iwt}\} = \dfrac{s}{s^2+w^2} + i\,\dfrac{w}{s^2+w^2}$, from (*)

L{ cos (wt)} + i L{sin (wt)}= $l\{e^{iwt}\} = \dfrac{s}{s^2+w^2} + i\,\dfrac{w}{s^2+w^2}$

From this

3- L {cos (wt)} = $\dfrac{s}{s^2+w^2}$

4- L {sin (wt)} = $\dfrac{w}{s^2+w^2}$

5-If f(t) = sinh (wt).

6-If f(t) = cosh (wt).

Solution

Since sinhx = $\frac{1}{2}$[$e^x - e^{-x}$], coshx = $\frac{1}{2}$[$e^x + e^{-x}$].

Now sinh (wt) = $\frac{1}{2}$[$e^{wt} - e^{-wt}$],

L {sinh (wt)} = $\frac{1}{2}$ {L (e^{wt}) – L(e^{-wt})},

$$= \frac{1}{2}\left\{ \frac{1}{s-w} - \frac{1}{s+w} \right\}.$$

$$= \frac{1}{2}\,\frac{2w}{s^2-w^2} = \frac{w}{s^2-w^2}$$

$\therefore$ L {sinh (wt)} = $\dfrac{w}{s^2-w^2}$, and

$L\{\cosh(wt)\} = \dfrac{s}{s^2 - w^2}$.

Example 1

Find $L\{8-6e^{3t} + e^{-4t} + 5\sin 3t + 7\cosh 3t\}$

Solution

$L(8) = 8L(1) = 8\,\dfrac{1}{s} = \dfrac{8}{s}$.

$L(6e^{3t}) = 6L(e^{3t}) = \dfrac{6}{s-3}$

$L(e^{-4t}) = \dfrac{1}{s+4}$

$L\{5\sin(3t)\} = 5L\{\sin(3t)\} = 5\,\dfrac{3}{s^2+9} = \dfrac{15}{s^2+9}$

$L\{7\cosh(3t)\} = 7L\{\cosh(3t)\} = \dfrac{7s}{s^2-9}$

Laplace Transformation of Differential

Theorem :-

If f(t)is continuous function of exponential on [o, ∞) whose derivative is also exponential then the (L.T) of $f'(t)$ is given by formula

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

Proof

$\int u\,dv = uv - \int v\,du$.

Let $u = e^{-st} \Rightarrow du = -s\,e^{-st}\,dt$,

$dv = f'(t)\,dt \Rightarrow v = f(t)$,

$$\Rightarrow \int_0^\infty e^{-st} f'(t)\,dt = e^{-st} f(t)\Big|_0^\infty + \int_0^\infty s\,e^{-st} f(t)\,dt$$

$$= o - e^0 f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$= -f(0) + s\,L\{f(t)\}$. Where $s\int_0^\infty e^{-st} f(t)\,dt = s\,L\{f(t)\}$.

$\therefore\ L\{f'(t)\} = s\,L\{f(t)\} - f(0)$.

Corollary

If both f(t) and $f'(t)$ are continuous functions of exponential order on [o, ∞) , and if $f''(t)$ is also exponential then :-

$L\{f''(t)\} = s^2 L\{f(t)\} - s f(0) - f'(0)$

Proof

$$L\{f''(t)\} = L\{f'(t)\}' = sL\{f'(t)\} - f'(0)$$
$$= s[s L\{f(t)\} - f(0)] - f'(0)$$
$$= s^2 L\{f(t)\} - s f(0) - f'(0).$$

Now in general

$L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) \text{-----------} f^{n-1}(0).$

Problem

Prove that

$L\{t^n\} = \frac{n!}{s^{n+1}}$, where n=1, 2, 3, ……….., and n!= n(n-1)(n-2)…..(n-n).

And 0! =1.

Properties of L. T

(1) **Shifting**

If $L\{f(t)\} = f(s) = L\{e^{at} f(t)\}\ f(s-a)$

Example

Find $L\{e^{-4t}\cos 3t\}$

Solution

$f(t) = \cos 3t$, $a=-4$, then $f(s) = L\{f(t)\} = L\{\cos 3t\} = \frac{s}{s^2+9}$

$$L\{e^{-4t}\cos 3t\} = \frac{s-(-4)}{(s-(-4))^2+9} = \frac{s+4}{(s+4)^2+9}$$

(2) **L. T of Integrals**:

$$L\{\int_0^t f(u)\,du\} = \frac{f(s)}{s}$$

Example

Find $L\{\int_0^t \sinh 2t\,dt\}$

Solution

F (u) = L{sinh2t} = $\frac{2}{s^2-4}$, L $\{\int_0^t$ sinsh2tdt$\}$= $\frac{\frac{2}{s^2-4}}{s}$

$= \frac{2}{s(s^2-4)}$.

(3) **Multiplication by t^n**

If L {f (t)} = f(s), then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n f(s)}{ds^n}$$

Example

Evaluate L $\{t^2 e^{3t}\}$

Solution

$f(t)= e^{3t}$

$L(f(t)) = L(e^{3t}) = \frac{1}{s-3}$ =f(s)

$f'(s)= \partial f/\partial s = \frac{-1}{(s-3)^2}$

$f''(s) = \partial^2 f/\partial s^2 = \frac{2}{(s-3)^3}$

$L\{t^2 e^{3t}\} = (-1)^2 \frac{2}{(s-3)^3} = \frac{2}{(s-3)^3}$

(4) **Division by t**

If L {f (t)} = f(s) , and $\lim_{t\to 0} \frac{f(t)}{t}$ exist

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} f(u)du$$

Example

Evaluate

$L\left\{\frac{\sin t}{t}\right\}$

Solution

$\lim_{t\to 0} \frac{\sin t}{t} = 1$, exists.

f(t) = sint $\longrightarrow$ L{sint} = $\frac{1}{s^2+1}$ =f(s)

$\therefore$ f(u)= $\frac{1}{u^2+1}$

$$L\left\{\frac{f(t)}{t}\right\} = L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{u^2+1}\,du = \tan^{-1}u \Big|_s^\infty = \tan^{-1}\infty - \tan^{-1}s$$

$= \Pi/2 - \tan^{-1}s.$

Example

Let f(t) =2cos3t.

Find

L {f ″ (t) }

Solution

$L\{f''(t)\} = s^2 L\{f(t)\} - s\,f(0) - f'(0)$

f(t) =2cos3t, f(0)=2cos(0)=2, $f'(t) = -6\sin 3t$, $f'(0) = -6\sin(0)=0$,

$L\{f(t)\} = L\{\cos 3t\} = \frac{s}{s^2+9}$,

$L\{f(t)\} = L\{2\cos 3t\} = 2L\{\cos 3t\} = \frac{2s}{s^2+9}$ =f(s).

$L\{f''(t)\} = s^2\left[\frac{2s}{s^2+9}\right] - s[2] - 0$

$= \frac{3s}{s^2+9} - 2s$

$L\{f''(t)\} = \frac{-18s}{s^2+9}$.

Unit Step Function $u_a(t)$

Definition 2.2

The unit step function is defined by:-

$$u_a(t) = \begin{cases} 0 & \text{when } t < a \\ 1 & \text{when } t > a \end{cases}$$

If a=0, then

$$u_0(t) = \begin{cases} 0 & \text{when } t < 0 \\ 1 & \text{When } t > 0. \end{cases}$$

If a=2, then

$$U_2(t) = \begin{cases} 0 & \text{when } t < 2 \\ 1 & \text{when } t > 2 \end{cases}$$

Definition 2.3.1

To find Laplace Transformation of unit step function (L { u_a (t)}) is defined as :-

Since

$$u_a(t) = \begin{cases} 0 & \text{when } t < a \\ 1 & \text{when } t > a \end{cases}$$

$$\therefore L\{u_a(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt = \int_0^{\infty} e^{-st} u_a(t)\, dt$$

$$= \int_0^{a} e^{-st}(0)\, dt + \int_a^{\infty} e^{-st}(1)\, dt = -\frac{1}{s} e^{-st} \Big|_a^{\infty} = -\frac{1}{s}\,[0 - e^{-sa}] =$$

$$\therefore \mathbf{L\{u_a(t)\}} = \frac{e^{-sa}}{s}.$$

Definition 2.3.2

To find the terms of the unit step function is defined by:-

$$f(t) = \begin{cases} 1 & \text{when } 0<t<1 \\ 2 & \text{when } 1<t<2 \\ -1 & \text{when } 2<t<3. \end{cases}$$

$$\therefore\ f(t) = 1[u_0 - u_1] + 2[u_1 - u_2] - 1[u_2 - u_3]$$
$$= u_0 - u_1 + 2u_1 - 2u_2 - u_2 + u_3$$

$$\therefore\ \mathbf{f(t) = u_0 + u_1 - 3u_2 + u_3}$$

Problem

Find Laplace Transformation of f (t) which defines in (Definition 2.3.2).

Solution

Since $f(t) = u_0 + u_1 - 3u_2 + u_3$

$$\therefore L\{f(t)\} = L\{u_0(t)\} + L\{u_1(t)\} - 3\,L\{u_2(t)\} + L\{u_3(t)\}$$

$$= \frac{e^{-s(0)}}{s} + \frac{e^{-s}}{s} - 3\frac{e^{-2s}}{s} + \frac{e^{-3s}}{s}$$

$$= \frac{1}{s} + \frac{e^{-s}}{s} - 3\frac{e^{-2s}}{s} + \frac{e^{-3s}}{s}.$$

<u>L. T of Periodic Functions</u>

If f (t) is Periodic function of period T>0 satisfy such that f(x+T) = f(x), then

$$L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)dt}{1-e^{-sT}}$$

<u>Gamma Function</u>

<u>Definition</u> 2.4

If (n >0), then the gamma (n) becomes:-

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\, dt \quad \text{...( )}$$

<u>Important Properties of gamma function</u>

(i) $\Gamma(n+1) = n\,\Gamma(n)$

ii) $\Gamma(n+1) = n!$

(iii) $\Gamma(\frac{1}{2}) = \Gamma(\Pi)$

<u>Table I</u>

Some elementary function f (t) their Laplace Transforms L {f (t)} = f(s).

	$f(t)$	$L\{f(t)\} = f(s)$
1	1	$\frac{1}{s}$
2	t	$\frac{1}{s^2}$
3	t^2	$\frac{2!}{s^3}$
4	$t^n, n = 1,2,3,...$	$\frac{n!}{s^{n+1}}$
5	e^{at}	$\frac{1}{s-a}$
6	$\cos wt$	$\frac{s}{s^2 + w^2}$
7	$\sin wt$	$\frac{w}{s^2 + w^2}$
8	$\cosh at$	$\frac{s}{s^2 - a^2}$
9	$\sinh at$	$\frac{a}{s^2 - a^2}$
10	$y'(t)$	$sL\{y(t)\} - y(0) = sf(s) - y(0)$
11	$y''(t)$	$s^2 L\{y(t)\} - sy(0) - y'(0) = s^2 f(s) - sy(0) - y'(0)$
12	$\int_0^t f(u)du$	$\frac{f(s)}{s}$
13	$t^n f(t)$	$\frac{(-1)^n \partial^n f(s)}{\partial s^n}$
14	$t^n (npositive)$	$\frac{\Gamma(n+1)}{s^{n+1}}$

Problems

Find Laplace transform (f(s)) of the following functions :-

1- $f(t) = \sin^2 t$
2- $f(t) = t^4 e^{3t}$
3- $f(t) = e^{-t} \cosh 3t$
4- $f(t) = \dfrac{\sinh t}{t}$
5- $f(t) = t^2 e^{3t}$
6- $f(t) = 3t+4$
7- $f(t) = t^2 + at + b$
8- $f(t) = (a+bt)^2$
9- $f(t) = t e^{-t}$
10- $f(t) = (e^{2t} - 4)^2$
11- $f(t) = t e^{at} \sin at$
12- $f(t) = \cosh at \cos at$

13- $f(t) = \begin{cases} 0 & \text{when } 0<t<2 \\ 4 & \text{when } 2<t. \end{cases}$

14 -Prove that $\int_0^\infty t\, e^{-3t} \sin t\, dt = \dfrac{3}{50}$

15- Prove that

(a) $= L\{a+bt\} = \dfrac{as+b}{s^2}$

(b) $= L\{t \cos at\} = \dfrac{s^2 - a^2}{(s^2 + a^2)^2}$,

Inverse Laplace Transformation

If $L\{f(t)\} = f(s)$. Then we call f(t) is the inverse of (L. T) of function f(s) and which written as:

$f(t) = L^{-1}\{f(s)\}$, for example

$L(e^{3t}) = \dfrac{1}{s-3} = f(s).$

$L^{-1}\{f(s)\} = L^{-1}\{\dfrac{1}{s-3}\} = e^{3t}.$

Some Properties of Inverse L. T

We see the L. T of first (9) in table I, we can inverse there Laplace to find inverse of this for example

L(1) = f(s)= $\frac{1}{s}$,

L^{-1} {f(s)} = L^{-1} { $\frac{1}{s}$ } =1,

Example1

Find f (t), if f(s) = $\frac{5}{s+3}$

Solution

$$f(t) = L^{-1}\{ \frac{5}{s-3} \} = 5L^{-1}\{ \frac{1}{s+3} \} = 5\ e^{-3t}$$

Example1

Find f (t), if f(s) = $\frac{s+1}{s^2+1}$

Solution

$$f(s) = \frac{s}{s^2+1} + \frac{1}{s^2+1} ,$$

$$f\ (t) = L^{-1}\{ \frac{s}{s^2+1} \} + L^{-1}\{ \frac{1}{s^2+1} \},$$

= cost + sint.

Partial Fraction

If we want to find the inverse transform of a rational function as $\frac{f(x)}{g(x)}$, where f and g are polynomials which the degree of f less than degree of g then.

We can take advantage of partial transform is easily found as see in examples:-

Example1

Find the inverse Laplace transform (f (t)), if f(s) = $\frac{1}{s^2(s^2+1)}$

Solution

$$f(s) = \frac{1}{s^2(s^2+1)} ,$$

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1} ,$$

$1= As^3 + Bs^2 + As + B + Cs^3 + Ds^2$,

B =1,

B+D =0,

A+ C =0,

D = -1 $\longrightarrow$ A=0 $\longrightarrow$ C=0,

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1} ,$$

$$f(t) = L^{-1}\{ \frac{1}{s^2(s^2+1)} \} = L^{-1}\{ \frac{1}{s^2} \} - L^{-1}\{ \frac{1}{s^2+1} \},$$

$$= t^2 - \sin t.$$

Example2

Find the inverse Laplace transform (f (t)), if $f(s) = \frac{s+1}{s^2+s-6}$

Solution

$$f(s) = \frac{s+1}{s^2+s-6} = \frac{s+1}{(s+3)(s--2)} = \frac{A}{s+3} + \frac{B}{s-2}.$$

S+1 =As - 2A + Bs + 3B,

-2A +3B = 1,

A+B =1,

-2A +3B = 1

2A + 2B = 2 +

5B=3 $\longrightarrow$ B =3/5, A = 2/5,

$$f(t) = L^{-1}\{ F(s)\} = 2/5L^{-1}\{ \frac{1}{s+3} \} - 3/5L^{-1}\{ \frac{1}{s-2} \},$$

$$= 2/5\ e^{-3t} + 3/5\ e^{2t}$$

Problems

Find f(t) { the inverse Laplace transform} of the following:-

(1) $f(s) = \frac{1}{s^2-1}$,

(2) $f(s) = \frac{1}{s^2(s^2+1)}$,

(3) $f(s) = \frac{s^2-6}{s^3+4s^2+3s}$,

(4) $f(s) = \frac{1}{s\,(s^2+4)}$,

(5) $f(s) = \frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+4}$,

(6) $f(s) = \frac{1}{s+3}$,

(7) f(s) = $\frac{1}{s^2+9}$,

(8) f(s) = $\frac{2s+3}{s^2+9}$,

(9) f(s) = $\frac{s+3}{s^2+s-6}$.

Application of Laplace Transformation

Linear (D. E) With Constant Coefficient

To solve L- non homogeneous (d. e) of order n with constant coefficient.

We use same way as second- order (d. e) as form :-

$a_0 y'' + a_1 y' + a_2 y = f(x)$……………………………………… … (*)

Where a_0, a_1 and a_2 are constant, which satisfy initial condition:

y(0)= A and y′ (0) =B……………………………………………(**)

Where A and B are choice constant.

Example1

Find the solution of the following (d.e) by (L. T)

$y'' + 3y' + 2y = 0$……………………………………………… (*)

Which satisfies initial condition?

y (0)= 0 and y′ (0) =1.

Solution

$L\{y''(t)\} = s^2\{y(s)\} - s\,y(0) - y'(0)$

$L\{y'(t)\} = s\{y(s)\} - y(0).$

$L\{y(t)\} = y(s)$

Since $L\{y\} = y(s)$

} Put in (*)

$s^2\{y(s)\} - s\,y(0) - y'(0) + 3[s\{y(s)\} - y(0)] + 2y(s) = 0$

By use y (0) = y′ (0) =1,

$(s^2 + 3s + 2)\,y(s) = (s+3)\,y(0) + y'(0),$

$(s^2 + 3s + 2)\,y(s) = s+3+1 = s+4,$

$$y(s) = \frac{s+4}{s^2+3s+2} = \frac{s+4}{(s+1)(s+2)} = \frac{A}{s+2} + \frac{B}{s+1}.$$

S+4 = As+ 2B + As + A,

A + B = 1

A +2B = 4 -

-B =-3 ⟶ B =3 ⟶ A =-2,

$$y(s) = \frac{-2}{s+2} + \frac{3}{s+1},$$

$$y(t) = L^{-1}\{ y(s)\} = 3L^{-1}\{ \frac{1}{s+1}\} - 2L^{-1}\{\frac{1}{s+2}\},$$

$\therefore$ $y(t) = 3e^{-t} - 2e^{-2t}$

Example2

Find the solution of the following (d.e) by (L. T)

$y'' + 4y' + 4y = 2$…………………………………………………… (i)

Which satisfies initial condition?

$y(0) = 1$ and $y'(0) = 1$.

Solution

$L\{y''(t)\} = s^2\{y(s)\} - s\,y(0) - y'(0)$

$L\{y'(t)\} = s\{y(s)\} - y(0)$.

$L\{y(t)\} = y(s)$

Since $L\{y\} = y(s)$

} Put in (i)

$$s^2\{y(s)\} - 1 + 4[s\,y(s)] + 4y(s) = \frac{2}{s}$$

$$[s^2 + 4s + 4]y(s) = \frac{2}{s} + 1 = \frac{2+s}{s},$$

$$y(s) = \frac{s+2}{s(s^2+4s+4)} = \frac{s+2}{s(s+2)^2} = \frac{s+2}{s(s+2)},$$

$$y(s) = \frac{s+2}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2},$$

$1 = A(s+2) + Bs,$

$\longrightarrow B = -½$ and $A = ½$,

$$y(s) = \frac{\frac{1}{2}}{s} + \frac{-\frac{1}{2}}{s+2},$$

$y(t) = L^{-1}\{ y(s)\} = 1/2L^{-1}\{\frac{1}{s}\} - 1/2L^{-1}\{\frac{1}{s+2}\}$,

$\therefore\ y(t) = ½ - ½e^{-2t}$.

Problems

Find the solution of the following (d.e) by (L. T), which satisfies the given initial conditions:-

(1) $y'' + 4y' + 3y = 0$, at $y(0) = 3$ and $y'(0) = 1$,
(2) $y'' + 4y' + 4y = 2$, at $y(0) = 0$ and $y'(0) = 1$,
(3) $y'' - y = 0$, at $y(0) = 0$ and $y'(0) = 1$,
(4) $y'' - 5y' + 6y = 0$, at $y(0) = 0$ and $y'(0) = 1$,
(5) $y'' - 9y = \sin t$, at $y(0) = 1$ and $y'(0) = 0$,
(6) $y'' - 9y = e^{t}$, at $y(0) = 1$ and $y'(0) = 0$,
(7) $y'' + 4y = \sin t$, at $y(0) = 0$ and $y'(0) = 1$,
(8) $y'' + 4y' + 4y = 4\cos 2t$, at $y(0) = 2$ and $y'(0) = 5$,

CHAPTER FOUR

Fourier series

Periodic Function

Definition 3.1

The function f(x) satisfy the condition

$f(x+T) = f(x)$

For all value of x where T is real number then f(x) is called Periodic function, and if T least positive number satisfies (1), then T is called periodic number of function. We can find that:-

$F(x) = f(x+T) = f(x+2T) = (x+3T) = \ldots\ldots = (x+nT).$

And

$F(x) = f(x-T) = f(x-2T) = (x-3T) = \ldots\ldots . = (x-nT).$

This means that

$F(x) = (x \pm nT)$, where n integer.

Some Properties of Series

1-f(x+T) = f(x) Periodic function

2- n=No of terms positive integer.

3- $\text{Cos} n\Pi = \begin{cases} 1 & \text{if n even } (2, 4, 6\ldots \\ -1 & \text{if n odd } (1, 3, 5\ldots\ldots \end{cases}$

4- - Cos $2n\Pi = 1$,

5- Sin $n\Pi$ = sin $2n\Pi = 0$,

6- Cos nx = Cos (-nx).

Some Important Integrals:-

1- $\int_0^{2\Pi} \sin nx \, dx = \int_0^{2\Pi} \cos nx \, dx = 0$, where n integer.

2 - $\int_0^{2\Pi} \sin mx \sin nx \, dx = \frac{1}{2} \int_0^{2\Pi} [\cos(m-n)x - \cos(m+n)x] dx = 0.$

3- $\int_0^{2\Pi} \sin^2 nx\, dx = \frac{1}{2}\int_0^{2\Pi} [1 - \cos 2nx]dx = \Pi$, where n integers.

4- $\int_0^{2\Pi} \text{Cos}\, nx \sin nx\, dx == \frac{1}{2}\int_0^{2\Pi} \sin 2nxdx = 0.$

5- $= \int_0^{2\Pi} \cos^2 nx\, dx = \int_0^{2\Pi} [\cos nx \cos nx\, dx = 0.$

Fourier series

Suppose that f(x) is periodic function to x, and 2Π is periodic number of it.

And the function f(x) is defined on the interval ($0 < x < 2\Pi$).

Then we can write f(x) in the form:-

$f(x) = a_0 + a_1\cos x + a_2\cos 2x + \ldots\ldots\ldots + a_n \cos nx + b_1 \sin x + b_2 \sin 2x + \ldots\ldots\ldots\ldots\ldots\ldots + b_n \sin nx$……………………………………….(1)

This means that

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$= \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

Such that

$$a_0 = \frac{1}{2\Pi}\int_0^{2\Pi} f(x)\, dx$$

$$a_n = \frac{1}{\Pi}\int_0^{2\Pi} f(x) \cos nx\, dx, \quad n = 1, 2, 3, \ldots.$$

$$b_n = \frac{1}{\Pi}\int_0^{2\Pi} f(x) \sin nx\, dx.$$

The series (1) is called Fourier series of the function f(x).

If the function f(x) defined on interval $-\Pi < x < \Pi$, then

$$a_0 = \frac{1}{2\Pi}\int_{-\Pi}^{\Pi} f(x)\, dx$$

$$a_n = \frac{1}{\Pi}\int_{-\Pi}^{\Pi} f(x) \cos nx\, dx, \quad n = 1, 2, 3, \ldots.$$

$$b_n = \frac{1}{\Pi}\int_{-\Pi}^{\Pi} f(x) \sin nx\, dx.$$

Example 1

Find Fourier series of the function $f(x) = x$, from $x = 0$ to $x = 2\Pi$ or $(0 < x < 2\Pi)$.

Solution

Use the rule to find a_0, a_n and b_n ,

$$a_0 = \frac{1}{2\Pi}\int_0^{2\Pi} f(x)\,dx = \frac{1}{2\Pi}\int_0^{2\Pi} x\,dx = \frac{1}{2\Pi}\int_0^{2\Pi} f(x)\,dx$$

$$a_0 = \frac{1}{4\Pi}x^2\Big]_0^{2\Pi} = \Pi.$$

$$a_n = \frac{1}{\Pi}\int_0^{2\Pi} f(x)\cos nx\,dx = \frac{1}{\Pi}\int_0^{2\Pi} x\cos nx\,dx,$$

$$= \frac{1}{\Pi}\Big[x\frac{\sin nx}{n} - \frac{-\cos nx}{n^2}\Big]\Big]_0^{2\Pi}$$

$$= \frac{1}{\Pi}\Big[2\Pi\frac{\sin 2n\Pi}{n} + \frac{\cos 2n\Pi}{n^2}\Big] - \Big[\frac{\cos 0}{n^2}\Big]$$

$$= \frac{1}{\Pi}\Big[\frac{2\Pi\sin 2n\Pi}{n} + \frac{\cos 2n\Pi - 1}{n^2}\Big] = 0.$$

$$b_n = \frac{1}{\Pi}\int_0^{2\Pi} f(x)\sin nx\,dx = b_n = \frac{1}{\Pi}\int_0^{2\Pi} x\sin nx\,dx$$

$$= \frac{1}{\Pi}\Big[\frac{-x\cos nx}{n} - \frac{\sin nx}{n^2}\Big]\Big]_0^{2\Pi}$$

$$= -\frac{2}{\Pi}.$$

The equation (1) becomes:

$$f(x) = \Pi - 2\sum_{n=1}^{\infty}\Big(\frac{\sin nx}{n}\Big)$$

$$f(x) = \Pi - 2\Big(\sin x + \frac{\sin 2x}{} + \frac{\sin 3x}{3} + \ldots\ldots\Big)$$

Even and Odd Function

If $f(x) = f(-x)$, is called even function.

If $f(-x) = -f(x)$, is called odd function.

Fourier series of Even and Odd Function

If $f(x)$ is even when $\{ x^2, x^4, x^6 \ldots \cos x, \sin^2 x, | f(x)|$.

If f(x) is odd when { x, x^3, x^5,……, sinx.

(i) If f(x) is even then

$b_n = 0$

(ii) If f(x) is odd then

$a_0 = a_n = 0$.

Example 1

Find Fourier series of the function

f(x) = x, for ($-\Pi < x < \Pi$).

Solution

Since f(-x) = -x = -f(x), $\therefore$ the function is odd.

$\therefore$ $a_0 = a_n = 0$.

$$b_n = \frac{1}{\Pi}\int_{-\Pi}^{\Pi} f(x)\sin nx\,dx = \quad b_n = \frac{1}{\Pi}\int_{-\Pi}^{\Pi} x\sin nx\,dx = \frac{2}{\Pi}\int_{0}^{\Pi} x\sin nx\,dx$$

$$= \frac{2}{\Pi}\left[\frac{-x\cos nx}{n} - \frac{-\sin nx}{n^2}\right]_0^{\Pi}$$

$$= \frac{2}{\Pi}\left[.\frac{-\Pi\cos n\Pi}{n}\right]$$

$$= -\frac{2}{n}\cos n\Pi$$

$$= -\frac{2}{n}(-1)^n$$

Then the series becomes:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = -2\sum_{n=1}^{\infty}(-1)^n\left(\frac{\sin nx}{n}\right)$$

$$f(x) = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\ldots\right)$$

Example 2

Find Fourier series of the function

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \Pi. \\ 2 & \text{if } \Pi < x < 2\Pi. \end{cases}$$

Solution

$$a_0 = \frac{1}{2\Pi}\int_0^{2\Pi} f(x)\,dx = \frac{1}{2\Pi}\int_0^{\Pi} f(x)dx + \frac{1}{2\Pi}\int_{\Pi}^{2\Pi} f(x)\,dx$$

$$= \frac{1}{2\Pi}\int_0^{\Pi} dx + \frac{1}{2\Pi}\int_{\Pi}^{2\Pi} 2\,dx$$

$$= \frac{1}{2\Pi} x\Big]_0^{\Pi} + \frac{1}{\Pi} x\Big]_{\Pi}^{2\Pi} dx$$

$a_0 = 3/2.$

$$a_n = \frac{1}{\Pi}\int_0^{2\Pi} f(x)\cos nx\,dx = \frac{1}{\Pi}\int_0^{\Pi} f(x)\cos nx\,dx + \frac{1}{\Pi}\int_{\Pi}^{2\Pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\Pi}\int_0^{\Pi} \cos nx\,dx + \frac{1}{\Pi}\int_{\Pi}^{2\Pi} 2\cos nx\,dx\ ,$$

$$= \frac{1}{\Pi}\frac{\sin nx}{n}\Big]_0^{\Pi} + \frac{1}{\Pi} 2\frac{\sin nx}{n}\Big]_{\Pi}^{2\Pi}$$

$$\frac{1}{\Pi}\int_0^{2\Pi} x\cos nx\,dx,$$

$$= \frac{1}{\Pi}\Big[x\frac{\sin nx}{n} - \frac{-\cos nx}{n^2}\Big]\Big]_0^{2\Pi} = 0.$$

$a_n = 0.$

$$b_n = \frac{1}{\Pi}\int_0^{\Pi} \sin nx\,dx + \frac{1}{\Pi}\int_{\Pi}^{2\Pi} 2\sin nx\,dx$$

$$= \frac{1}{\Pi}\frac{-\cos nx}{n}\Big]_0^{\Pi} + \frac{1}{\Pi} 2\frac{-\cos nx}{n}\Big]_{\Pi}^{2\Pi},$$

$$= \frac{1}{\Pi}\Big[\frac{\cos n\Pi - 1}{n}\Big] = \frac{(-1)^n - 1}{n\Pi},$$

$$\therefore\ b_n = \frac{(-1)^n - 1}{n\Pi},$$

$$a_0 = 3/2\ ,\ a_n = 0.\ b_0 = \frac{-2}{\Pi},\ b_1 = 0\ b_3 = \frac{-2}{3\Pi},$$

$$\therefore\ f(x) = 3/2 - \frac{2}{\Pi}\Big[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\ \Big].$$

<u>Half-Range Series</u>

If we want find Fourier series on interval ($0 < x < \Pi$), does not on all interval ($-\Pi < x < \Pi$), then we can find the Fourier series by :-

1- Fourier Cosine series or f(x) an even function as:-

$f(x)= a_0 + a_1\cos x + a_2\cos 2x + \dots\dots\dots + a_n \cos nx.$
2- Fourier Sine series or f(x) an odd function as:-

$f(x)= b_1 \sin x + b_2 \sin 2x+\dots\dots\dots\dots\dots + b_n \sin nx$

Such that

$$a_0 = \frac{1}{\Pi}\int_0^{\Pi} f(x)\,dx$$

$$a_n = \frac{2}{\Pi}\int_0^{\Pi} f(x)\cos nx\,dx\ ,\ \ n = 1, 2, 3,\dots.$$

$$b_n = \frac{2}{\Pi}\int_0^{\Pi} f(x)\sin nx\,dx.$$

Example 3

Find cosine Half-range series for the function defined as

$f(x) = x,$ for $0< x< \Pi$.

Solution

Use the rule to find a_0 and a_n

$$a_0 = \frac{1}{\Pi}\int_0^{\Pi} f(x)\,dx = \frac{1}{\Pi}\int_0^{\Pi} x\,dx = \frac{1}{\Pi}\int_0^{\Pi} f(x)\,dx$$

$$a_0 = \frac{1}{2\Pi}x^2\,\Big]_0^{\Pi} = \frac{\Pi}{2}.$$

$$a_n = \frac{2}{\Pi}\int_0^{\Pi} f(x)\cos nx\,dx = \frac{2}{\Pi}\int_0^{\Pi} x\cos nx\,dx,$$

$$= \frac{2}{\Pi}\Big[x\frac{\sin nx}{n} - \frac{-\cos nx}{n^2}\Big]\Big]_0^{\Pi}$$

$$= \frac{2(\cos n\Pi - 1)}{\Pi n^2}$$

$$a_n = \begin{cases} 0 & \text{if n even.} \\ \dfrac{-4}{\Pi n^2} & \text{if n odd} \end{cases}$$

$\therefore$ f(x)= $\frac{\Pi}{2}$ - $\frac{4}{\Pi}$[cosx $+ \frac{\cos 3x}{3} + \frac{\cos 5x}{5}$ +……..].

Example 4

Find sine Half-range series for the function defined as

f(x) = x, for 0< x< Π.

Solution

Use the rule to find b_n

$$b_n = \frac{2}{\Pi}\int_0^{\Pi} f(x) \sin nx \, dx = \quad b_n = \frac{2}{\Pi}\int_0^{\Pi} x \sin nx \, dx = \frac{2}{\Pi}\int_0^{\Pi} x \sin nx \, dx$$

$$= \frac{2}{\Pi}\left[\frac{-x\cos nx}{n} - \frac{-\sin nx}{n^2} \right]_0^{\Pi}$$

$$= \frac{2}{\Pi}[.\frac{-\Pi\cos n\Pi}{n}]$$

$$= -\frac{2}{n} \cos n\Pi$$

$$= -\frac{2}{n} (-1)^n$$

Then the series becomes:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = -2 \sum_{n=1}^{\infty} (-1)^n \left(\frac{\sin nx}{n}\right)$$

$$f(x) = 2 \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} \; ..... \right).$$

CHAPTER FOUR

Partial Differential Equations

Partial Differential Equations (P. D.E)

Partial Differential Equations are Differential Equations in which the unknown function of more than one independent variable.

Types of (P. D.E)

The following some type of (P. D.E):-

1-Order of (P. D.E)

The order of **(P. D.E)** is the highest derivative of equation for example:-

$U_x = U_y$ First-order (p. d. e).

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$$ Second -order (p. d. e).

2-The Number of Variables

For example:-

$U_x = U_{tt}$ (two variables x and t).

$$U_x = U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta}$$ (Three variables t, r and □).

3-Linearity

The **(P. D.E)** is linear or non-linear, is linear **(P. D.E)** if u and whose derivative appear in linear form (non- linear if product two dependent variable or power of this variable greater than one).

For example {the general second L. P. D.E in two variable}

$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$…………………(*)

Where A, B, C, D, E, F and G are constant or function of x and y for example

$u_{tt} + e^{-t} u_{xx} = \sin t$ (Linear)

$u_{xx} = y u_{yy}$ (Linear)

$u\, u_x + u_y = 0$ (Non-Linear)

$x\, u_x + y\, u_y + u^2 = 0$ (Non-Linear).

4-Homogeneity

If each term of (P. D.E) contain the unknown function and which derivative is called (H. P. D.E) otherwise is called (non-H. P. D.E), in special case in (*) is homogeneous if [G =0]. Otherwise non-homogeneous.

$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$ (H. P. D.E)

Where A, B, C, D, E and F are constant or function of x and y.

Example1

Determine which (L. P. D.E) is, order and dependent or independent variable in following:-

$$1-\frac{\partial u}{\partial t}=4\frac{\partial^2 u}{\partial x^2}$$

Linear second degree u, dependent variable, x and t are independent variable.

$$2-x^2\frac{\partial^3 r}{\partial y^3}=y^3\frac{\partial^2 r}{\partial x^2}$$

Linear 3- degree(r, dependent variable, x and y are independent variable.

$$3-w\frac{\partial^3 w}{\partial y^3}=rst$$

Non-Linear 3- degree(w, dependent variable, r, s and t are independent variable.

$$4-\frac{\partial^2 Q}{\partial x^2}+\frac{\partial^2 Q}{\partial y^2}+\frac{\partial^2 Q}{\partial z^2}=0$$

Linear 2- degree(Q, dependent variable, x, y and z are independent variables, homogeneous.

$$5-(\frac{\partial u}{\partial t})^2+(\frac{\partial u}{\partial x})^2=0$$

Non-Linear 1- degree(u, dependent variable, t and x are independent variables, homogeneous.

Solution of (P. D.E)

A solution of (P. D.E) mean that the value of dependent variable which satisfied the (P. D.E) at all points in given region R.

For Physical Problem, we must be given other conditions at boundary, these are called boundary if these condition are given at t=0 we called them as initial conditions its order.

For a linear homogeneous equation if

$u_1, u_2 \ldots u_n$ are n solution then the general solution can be written as (n-th order p. d. e)

$u= c_1 u_1+ c_2 u_2+ \ldots + c_n u_n$.

Note i

We can find the solution of (P. D.E) by sequence of integrals as see in the following examples:-

Example2

Find the solution of the following (P. D.E)

$$\frac{\partial^2 z}{\partial x \partial y}=0$$

Solution

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial y}) = 0$$ By integrate (w. r. to) x gives

$$\frac{\partial z}{\partial y} = c(y)$$

Where c(y) is arbitrary parametric of y. Also by integrate (w. r. to) y gives

$$z = \int c(y)\partial y + c(x)$$

Where c(x) is arbitrary parametric of x.

Example3

Find the solution of the following (P. D.E)

$$\frac{\partial^2 z}{\partial x \partial y} = x^2 y$$

Solution

By integrate (w. r. to) x gives

$$\frac{\partial z}{\partial y} = \frac{x^3 y}{3} + c(y)$$

By integrate (w. r. to) y gives

$$z = \frac{x^3 y^2}{6} + \int c(y)\partial y + c(x)$$

$$z = \frac{x^3 y^2}{6} + F(y) + c(x)$$

Example4

Find the solution of the following (P. D.E)

$$\frac{\partial^2 u}{\partial x \partial y} = 6x + 12y^2$$

With boundary condition, $u(1,y) = y^2 - 2y$, $u(x,2) = 5x - 5$

Solution

By integrate (w. r. to) x gives

$$\frac{\partial u}{\partial y} = 3x^2 + 12y^2 x + c(y)$$

By integrate (w. r. to) y gives

$$u = 3x^2 y + 4y^3 x + \int c(y)\partial y + g(x)$$

$$\therefore u(x,y) = 3x^2 y + 4y^3 x + h(y) + g(x)$$

$$u(1,y) = 3y + 4y^3 + h(y) + g(1) = y^2 - 2y$$

$$h(y) = y^2 - 4y^3 - 5y - g(1)$$

$$\therefore u(x,y) = 3x^2 y + 4y^3 x + y^2 - 4y^3 - 5y - g(1) + g(x)$$

$$\therefore u(x,2) = 6x^2 + 32x + 4 - 32 - 10 - g(1) + g(x) = 5x - 5$$

$$g(x) = 33 - 27x - 6x^2 + g(1)$$

$$\therefore u(x,y) = 3x^2y + 4y^3x + y^2 - 4y^3 - 5y + 33 - 27x - 6x^2$$

Formation of (P. D.E)

A (P. D.E) may formed by a eliminating arbitrary constants or arbitrary function from a given relation and other relation obtained by differentiating partially the given relation.

Note ii

Suppose the following relation:-

$1 - \frac{\partial z}{\partial x} = z_x = p$

$2 - \frac{\partial z}{\partial y} = z_y = q$

$3 - \frac{\partial^2 z}{\partial x^2} = z_{xx} = r$

$4 - \frac{\partial^2 z}{\partial y^2} = z_{yy} = t$

$5 - \frac{\partial^2 z}{\partial x \partial y} = z_{yx} = s$

Example 5

Form a **Partial Differential Equations** from the following equation:-

Z= $(x - a)^2 + (y-b)^2$……………………………(1)

Solution

$$\frac{\partial z}{\partial x} = z_x = 2(x - a)$$

$$\frac{\partial z}{\partial y} = z_y = 2(y - b)$$

□Eq(1) become

$$Z = (\frac{1}{2} z_x)^2 + (\frac{1}{2} z_y)^2$$

$$4Z = (z_x)^2 + (z_y)^2$$

$$4Z = (p)^2 + (q)^2$$

Example 6

Form a **Partial Differential Equations** from the following equation:-

Z= $f(x^2 + y^2)$……………………………(2)

Solution

$Z_x = 2xf'(x^2 + y^2)$

$Z_y = 2yf'(x^2 + y^2)$

Eq(2) become

$$\frac{z_x}{z_y} = \frac{x}{y},$$

$-x Z_y + y Z_x = 0$

$yp - xq = 0$

Example 7

Form a **Partial Differential Equations** from the following equation:-

$Z= ax+by+a^2 +b^2$……………………………. (3).

Solution

$Z_x=a$

$Z_y=b$

Eq(3) become

$Z= x Z_x +y Z_y +(Z_x)^2 +(Z_y)^2$

$Z= x p +y q +(p)^2 +(q)^2$

Example 8

Form a **Partial Differential Equations** from the following equation:-

$v= f(x -ct) +g(x+ct)$

Solution

$v_x= f'(x -ct)+g'(x+ct)$

$v_t= -cf'(x -ct)+cg'(x+ct)$

$v_{xx}= f''(x -ct)+g''(x+ct)$

$v_{tt}= c^2f''(x -ct)+c^2g''(x+ct)$

$v_{tt}= c^2 [f''(x -ct)+g''(x+ct)]$

□ $v_{tt}= c^2 v_{xx}$, or

$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2}$ One dimensional Wave equation

Solution of First Order Linear (P. D. E)

Let the Partial Differential Equation as form:-

$Pp+ Qq =R$…………………………………..……. (4)

Where P, Q and R are function of x, y and z.

So the solution of this equation is the same as the solution of simultaneous

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{..(5)}$$

Eq (5) are calle LaGrange Auxiliary Equations or (characteristic equation).

A soluation of Eq(5), can be written as

$U(x, y, z) = c_1$,

$V(x, y, z) = c_2$

The general solution written as

$F (U, V) =0$, or $F (c_1, c_2) =0$.

Note iii

To solve Eq(5), we note that:-

(i) If P or Q or R equal to zero then dx or dy or dz equal to zero respectivly, For example

If R=0→ dz =0→ Qdx =Pdy from Eq(5), which can easily to solve it.

(ii)In case sparable the variable in problem, then we can write characteristic Eq(5), in the following form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \rightarrow \frac{\lambda dx + \mu dy + \beta dz}{\lambda P + \mu Q + \beta R}$$

We slected the value of λ, μ and β such that gives

λP +μQ + βR =0, → λdx +μdy + βdz =0.

Which helpe to find of **<u>Solution of (P. D.E).</u>**

<u>Example 9</u>

Solve the following Partial Differential Equation

xzp+yzq =xy

<u>Solution</u>

Suppose the following relation:-

Where

$$\frac{\partial z}{\partial x} = z_x = p, \text{ and } \quad \frac{\partial z}{\partial y} = z_y = q$$

P= xz, Q= yz, and R= xy

$$\frac{dx}{xz} = \frac{dy}{yz} \rightarrow \frac{dx}{x} = \frac{dy}{y}$$

Lnx=lny =ln c_1

$$\frac{x}{y} = c_1 = V\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

$$\frac{dy}{yz} = \frac{dz}{xy} \rightarrow \frac{dy}{z} = \frac{dz}{x} \rightarrow xdy= zdz$$

zdz = $c_1$ydy

$$\frac{z^2}{2} = c_1 \frac{y^2}{2} + c$$

$$\frac{z^2}{2} - \frac{xy}{2} = c$$

z^2-xy =2c = c_2=V

The general solution

F (c_1, c_2) =0, or

F ($\frac{x}{y}$, z^2-xy) =0.

<u>Example 10</u>

Solve the following Partial Differential Equation
(x+z)p –(x+z)q =x-y………………………………………. (7)

Solution

P= x+z, Q= -(x+z), and R= x-y

$$\frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y} \rightarrow \frac{\lambda dx + \mu dy + \beta dz}{\lambda(y+z) - \mu(x+z) + \beta(x-y)}$$

$$\therefore \quad \frac{dx + dy + dz}{0}$$

Where $\lambda = 1$, $\mu = 1$, $\beta = 1$.

$\therefore dx + dy + dz = 0.$

$x + y + z = c_1 = \mathbf{U}$.

For $\lambda = x$, $\mu = y$, $\beta = -z$

$$\frac{xdx + ydy - zdz}{0}$$

xdx +ydy - zdz = 0.

$x^2 + y^2 - z^2 = 2c = c_2 = V$

The general solution

$F(c_1, c_2) = 0$, or

$F(x + y + z, x^2 + y^2 - z^2) = 0.$

Example 11

Solve the following :-

$xz\, Z_x + yz\, Z_y + (x^2 + y^2) = 0$

Solution

$xz\, Z_x + yz\, Z_y = -(x^2 + y^2)$

1- $\frac{dx}{xz} = \frac{dy}{yz}$

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\frac{dx}{x} - \frac{dy}{y} = 0$$

$\ln x - \ln y = \ln c_1$

$$Ln\frac{y}{x} = \ln c_1$$

$$\frac{y}{x} = c_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots 1$$

2- $\frac{dx}{xz} = \frac{dz}{-(x^2 + y^2)}$

From (1) $y = x\, c_1$

$$\frac{dx}{xz} = \frac{dz}{-(x^2 + c_1^2 x^2)}$$

$$\frac{dx}{xz} = \frac{dz}{-x^2(1 + c_1^2)}$$

$-x(1 + c_1^2)\, dx = zdz$

$x(1 + c_1^2)\, dx + zdz = 0$

$$\frac{x^2}{2}(1 + c_1^2) + \frac{z^2}{2} = c_2$$

$x^2 + x^2 c_1^2 + z^2 = 2c_2$,

$x^2 + y^2 + z^2 = c_3$, where $c_3 = 2c_2$.

The general solution

$F(c_1, c_3) = 0$, or

$F(\frac{y}{x}, x^2 + y^2 + z^2) = 0$

Problems

Find the solution of the following Partial Differential Equation:-

1- $2p + 3q = 1$
2- $p - xq = z$
3- $y^2 zp - x^2 zq = x^2 y$
4- $(y+z)p + (x+z)q = x + y$
5- $ap + bq + cz = 0$
6- $(y^2 + z^2 - x^2)p - 2xyq + 2xz = 0$

Theorem1

If u_1 u_2........are solution of equation

$F(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \ldots\ldots\ldots\ldots)u = 0$, Then

$U = c_1 u_1 + c_2 u_2 + \ldots$ is solution also, where $u = c_1, c_2 \ldots$ .are constants.

Method of Variable Sparable

Let the Partial Differential Equation as

$F(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \ldots\ldots\ldots\ldots)u = 0.$

Let the general solution of above equation is

Let u (x, t) = XT, or u (x, t) = X(x)T(t) Be solution **of (P. D.E)** where X ifs function of x only, and Y function of y only. As see in the following problems:-

Examp 12

Solve the following Partial Differential Equation with boundary codition

$\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = 0$ With boundary condition.

$u(0, y) = 4e^{-2y} - 3e^{-6y}$ ………………………………… (8)

Solution

To solve Eq(8) suppose

u (x, t) = XT. Be solution of (8) where X ifs function of x only, and Y function of y only.

$$\frac{\partial u}{\partial x} = YX', \quad \frac{\partial u}{\partial y} = XY'$$

$$X' = \frac{dX}{dx} \quad Y' = \frac{dY}{dy}$$

Put in eq (8)

$YX' + 3XY' = 0$

$$\frac{X'}{3X} = -\frac{Y'}{Y}$$

Now let

$$\frac{X'}{3X} = -\frac{Y'}{Y} = c$$

$$\frac{X'}{3X} = c$$

$$-\frac{Y'}{Y} = c$$

$X' - 3CX = 0$, $Y' - CY = 0$,

$X = a_1e^{3cx}$, $Y = a_2e^{-cy}$

$u(x, t) = XT = a_1a_2e^{3cx-cy} = Be^{c(3x-y)}$, where $B = a_1a_2$, are constant.

Now let

$u_1 = b_1e^{c_1(3x-y)}$, and $u_2 = b_2e^{c_2(3x-y)}$ solution of (8) (theorem 1)

$u = u_1 + u_2 = b_1e^{c_1(3x-y)} + b_2e^{c_2(3x-y)}$,from boundary condion

$u(0, y) = b_2e^{-c_2y} + b_1e^{-c_1y} = 4e^{-2y} - 3e^{-6y}$

$b_1=4$, $b_2=-3$, $c_1=2$, $c_2=6$

$u(x, y) = 4e^{2(3x-y)} - 3e^{6(3x-y)}$

Example 13

Find the solution of following [Heat equation] by using partial differential equation:-

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \quad \text{……………………..……………..…………. (9)}$$

With boundary condition.

(1) u (0, t) = 0, (2) u (10, t) = 0, for all t,

(3) $u(x, 0) = 50\sin\frac{3\Pi}{2}x + 20\sin 2\Pi x - 10\sin 4\Pi x$

Solution

Let u (x, t) = XT. Be solution of (9)

$$\frac{\partial u}{\partial t} = XT'$$

$$\frac{\partial^2 u}{\partial x^2} = TX''$$

Put in(1)

XT′ =2 TX′′ ………………….…………..………… (10)

We can write (10)in the form:-

$$\frac{T'}{2T} = \frac{X''}{X}$$

Let

$$\frac{T'}{2T} = \frac{X''}{X} \quad = c$$

Where c be constant

T′ -2cT=0, X′′ - cX=0 there three cases OF C (C=0,C>0 and c<0)

CaseI. If c=0

T′=0, →

T= c_1

and

X′′ =0,X= $c_2x + c_3$

U= TX=$c_1(c_2x+c_3)$

U=Ax+B

Where A=c_1c_2, B= c_1c_3

U(0,t)= B=0

U(x, t)=Ax

U(10,t)= 10A=0 →

A=0

$\therefore \quad U = 0$

Which trivial solution c≠ 0

CaseII. If C>0

$Te^{-2cx} = c_1 \rightarrow T = c_1e^{2ct}$

X=$c_2\, e^{\sqrt{cx}} + c_3 e^{-\sqrt{cx}}$

u (x, t) = XT,

$= c_1e^{2ct}\, (c_2\, e^{\sqrt{cx}} + c_3 e^{-\sqrt{cx}})$

u = $e^{2ct}\, (A e^{\sqrt{cx}} + B_3 e^{-\sqrt{cx}})$

A= c_1, c_2, and B= c_1, c_3

U(0,t)= e^{2ct} (A + B)=0

$e^{2ct} \neq 0 \rightarrow$ A + B=0→ A= -B

$U(x,t)=B\ e^{-2ct}\ )\ (e^{\sqrt{cx}} - e^{-\sqrt{cx}})$

$U(10,t)=B\ e^{2Ct}\ (e^{10\sqrt{c}} - e^{-10\sqrt{c}}) = 0$

If $B=0 \rightarrow A=0 \rightarrow U=0$ Which trivial solution $B \neq 0$

$e^{10\sqrt{c}} - e^{-10\sqrt{c}} = 0 \rightarrow e^{10\sqrt{c}} - e^{-10\sqrt{c}} = 0 \rightarrow e^{10\sqrt{c}} = e^{-10\sqrt{c}}$,

$\rightarrow e^{10\sqrt{c}} - e^{-10\sqrt{c}} = 0 \rightarrow e^{20\sqrt{c}} = 1$ which impossible since $e^{20\sqrt{c}} \square 1$

There is no solution if $C > 0$.

CaseIII. If $c<0$, let $c=-k^2 \quad -k^2 \rightarrow$

$k^2 \square 0 \rightarrow$

$T' + 2k^2T = 0, \quad X'' + k^2X = 0 \rightarrow T = c_1 e^{-2k^2t}, \quad X = c_2 \cos kx + c_3 \sin kx.$

$U(x,t) = c_1 e^{-2k^2t} (c_2 \cos kx + c_3 \sin kx)$

$U(x,t) = e^{-2k^2t} (A \cos kx + B \sin kx).$

Where $A=c_1c_2$, $B=c_1c_3$

$U(0,t) = e^{-2k^2t} (A) = 0$

$\rightarrow A = 0$, because $e^{-2k^2t} \neq \mathbf{0}$

$U(x,t) = B\ e^{-2k^2t} (\sin kx).$

$U(10,t) = B\ e^{-2k^2t} (\sin 10k.) = 0$

Since $B \neq 0$, $e^{-2k^2t} \neq 0$

$\rightarrow \sin 10k = 0$

$\leftrightarrow 10k = n\Pi$, where $n = 0 \pm 1 \pm 2 \pm \ldots$

$\leftrightarrow k = \frac{n\Pi}{10}$

$U(x,t) = B\ e^{-2\frac{n^2\Pi^2}{100}t} (\sin \frac{n\Pi}{10} x) = B\ e^{-\frac{n^2\Pi^2}{50}t} (\sin \frac{n\Pi}{10} x) =$

$U(x,t) = b_1\ e^{-\frac{n_1^2\Pi^2}{50}t} (\sin \frac{n_1\Pi}{10} x)$

$U(x,t) = b_2\ e^{-\frac{n_2^2\Pi^2}{50}t} (\sin \frac{n_2\Pi}{10} x)$

$U(x,t) = b_3\ e^{-\frac{n_3^2\Pi^2}{50}t} (\sin \frac{n_3\Pi}{10} x)$

$$U(x,t) = b_1\ e^{-\frac{n_1^2\Pi^2}{50}t} (\sin \frac{n_1\Pi}{10} x) + b_2\ e^{-\frac{n_2^2\Pi^2}{50}t} (\sin \frac{n_2\Pi}{10} x) + b_3\ e^{-\frac{n_3^2\Pi^2}{50}t} (\sin 2 \frac{n_3\Pi}{10} x)$$

$U(x,0)= b_1 \sin\frac{n_1\Pi}{10}x + b_2 \sin\frac{n_2\Pi}{10}x + b_3 \sin\frac{n_3\Pi}{10}x =$

$50\sin\frac{3\Pi}{2}x$ +20sin2Πx - 10sin4Πx

$b_1 =50, b_2 =20, + b_3 = -10$,

$\frac{n_1\Pi}{10}=\frac{3\Pi}{2} \rightarrow n_1=15, n_2=20, n_3=40$

$U(x,t)= 50 e^{-\frac{9\Pi^2}{2}t} \sin\frac{n3\Pi}{2}x + 20 e^{-8\Pi^2 t} \sin 2\Pi x -10 e^{-32\Pi^2 t} \sin 4\Pi x$

Examp 14

Find the solution of following [Wave equation] by using partial differential equation:-

(1) $\frac{\partial^2 u}{\partial t^2}=4\frac{\partial^2 u}{\partial x^2}$ With boundary condition.

(2) u (0, t) = 0, (3) u (L, t) = 0, for all t, $L,\succ 0$0 (4) u(x, 0) = f(x).

(5) $\frac{\partial u}{\partial t}$= g(x), at t=0.

Solution

Let u (x, t) = XT. Be solution of (1) where X ifs function of x only, and Y function of y only.

$XT'' \quad \frac{\partial^2 u}{\partial t^2} =$

$TX'' \quad \frac{\partial^2 u}{\partial x^2} =$

Put in(1)

XT′′ =4 TX′′

$\frac{T''}{4T}=\frac{X''}{X}$

Let

$\frac{T''}{4T}=\frac{X''}{X}=k^2$

Where k be constant

$T'' -4k^2T=0$, $X'' - k^2X=0$(there three cases)

CaseI. If

$k^2 = 0$

T′′ =0,

T=at+b

∴

X′′ =0,X= cx +d

$U = TX = (at+b)(cx+d)$

$U(0,t) = (at+b)(d) = 0$

$at + b \neq 0 \rightarrow b = 0$

$U(x,t) = (at+b)\,cx$

$U(L,t) = (at+b)\,cL = 0$

$cL = 0$

$L \neq 0 \rightarrow c = 0$

$cx + d = 0$

$U(x,t) = 0$

CaseII. If

$k^2 \succ 0$

$T'' - 4k^2T = 0, \quad X'' - k^2X = 0$

$T = a e^{2kt} + b e^{-2kt}$, $X = c e^{kx} + d e^{-kx}$

$U(x,t) = (a e^{2kt} + b e^{-2kt})(c e^{kx} + d e^{-kx})$

$U(0,t) = (a e^{2kt} + b e^{-2kt})(c + d) = 0$

$c + d = 0 \rightarrow d = -c$

$U(x,t) = c(a e^{2kt} + b e^{-2kt})(e^{kx} - e^{-kx})$

$U(L,t) = c(a e^{2kt} + b e^{-2kt})(e^{kL} - e^{-kL}) = 0$

If $c = 0 \rightarrow X = 0 \rightarrow U = 0$

$e^{kL} - e^{-kL} = 0 \rightarrow e^{kL} = e^{-kL} \rightarrow e^{2kL} = 1$, which impossible since

L, k□0

There is no solution if k^2 □0

CaseIII. If $-k^2 \rightarrow k^2$ □0 $\rightarrow$

$T'' + 4k^2T = 0, \quad X'' + k^2X = 0 \rightarrow$

$T = A\cos 2kt + B\sin 2kt$, $X = C\cos kx + D\sin kx$.

$U(x,t) = (A\cos 2kt + B\sin 2kt)(C\cos kx + D\sin kx)$

$U(0,t) = (A\cos 2kt + B\sin 2kt)(C) = 0$

$\rightarrow c = 0$, because $A\cos 2kt + B\sin 2kt \neq 0$

$U(x,t) = (A\cos 2kt + B\sin 2kt)\,D\sin kx$

$U(L,t) = D\sin kL\,(A\cos 2kt + B\sin 2kt) = 0$

Since $A\cos 2kt + B\sin 2kt \neq 0 \rightarrow D\sin kL = 0$

If $D = 0 \rightarrow U = 0$

$\rightarrow D\sin kL = 0 \leftrightarrow kL = n\Pi$, where $n = 0 \pm 1 \pm 2 \pm \ldots$

$\leftrightarrow k = \dfrac{n\Pi}{L}$

$U(x,t) = D\sin n\Pi x\ (A\cos 2\dfrac{n\Pi}{L} t + B\sin 2\dfrac{n\Pi}{L} t)$

$U(x,t) = (A_n\cos 2\dfrac{n\Pi}{L} t + B_n\sin 2\dfrac{n\Pi}{L} t)\sin n\Pi x$.

Where $A_n = AD$, $B_n = BD$

$$U(x,t)=\sum_{n=1}^{\infty}U_n(x,t)$$

$$\sum_{n=1}^{\infty}U_n(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos 2\frac{n\Pi}{L}\ t+B_n\sin 2\frac{n\Pi}{L}\ t\right)\sin n\Pi x.$$

U(x, 0) = f(x).

$$f(x)=\sum_{n=1}^{\infty}A_n\sin n\Pi x.$$

$U_t(x,0)=$ g(x),

$$g(x)=2\frac{\Pi}{L}\sum_{n=1}^{\infty}B_n(n\ \sin n\Pi x).$$

Problems

Find the solution of the following Partial Differential Equation:-

(1) $\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}=0$

With boundary condition.

u (0, t) = 0, u (10, t) = 0, for all t,

u(x, 0) = $50\sin\frac{3\Pi}{2}x$ +20sin2Πx - 10sin4Πx.

(2) $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$ With boundary condition.

u (0, y) = e^{2y},

(3) $2\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$

With boundary condition.

u (0, t) = 0, u (π, t) = 0, for all t,

u(x, 0) = 2sin3x - 5sin4x.

(4) $\frac{\partial^2 u}{\partial t^2}=4\frac{\partial^2 u}{\partial x^2}$ With boundary condition.

(i) u (0, t) = 0, (ii) u (L, t) = 0, for all t, L > 0

(iii) u(x, 0) = f(x). (iv) $\frac{\partial u}{\partial t}$ = g(x), at t=0.

(5) $\frac{\partial^2 u}{\partial y^2}=4\frac{\partial^2 u}{\partial x^2}$ With boundary condition.

(i) u (0, y) = 0, (ii) u (10, y) = 0, for all t,

(iii) $\frac{\partial u}{\partial y}(x,0)=0$, at t=0.

(iv) $u(x, 0) = 3\sin 2\Pi x - 4\sin\frac{\Pi}{2}x$.

//الرياضيات المرحلة الثانية الحاسبات

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CHAPTER FIFE

Numerical Analysis
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Numerical Analysis
Solution of Non-Linear Equation
1-Newton-Raphson Method for Approximating

We use tangent to approximate the graph of y = f(x), near the point P (x_n, y_n), where
y_n = f(x_n), is small. Let x_{n+1} be the value of x where that tangent line crosses the x-axis.
Let tangent = The slope between (x , y) and (x_n, y_n), is

$$f`(x_n) = \frac{y - y_n}{x - x_n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Since the tangent line crosses the x-axis, y = 0, and y_n = f(x_n), put in Eq (1) which becomes

$$f`(x_n) = \frac{-f(x_n)}{x - x_n},$$

$$x - x_n = \frac{-f(x_n)}{f'(x_n)},$$

$$x = x_n - \frac{f(x_n)}{f'(x_n)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Put x= x_{n+1} in Eq (2) gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Eq (3) called **Newton-Raphson Method,** can using this method by the following

1-Give first approximating to root of equation f(x) = 0. A graph of
y = f(x).

2-Use first approximating to get a second. The second to get a third, and so on. To go from nth approximation x_n to the next approximation x_{n+1}, by using Eq (3), where f `(x) the derivative of f at x_n.

Example 1

Solve the following using Newton-Raphson Method

$\frac{1}{x}+1 = 0$, start with $x_0 = -0.5$, error % = 0.5 %

Where e % = $\left|\frac{x_{n+1}-x_n}{x_{n+1}}\right| \otimes \%$

Sol

$f(x) = \frac{1}{x}+1$, $x_0 = -0.5$,

$f`(x_n) = -\frac{1}{x^2}$

$f(x_0) = \frac{1}{-0.5}+1 = -1$,

$f`(x_0) = -\frac{1}{(-0.5\ )^2} = -4$, from Eq (3)

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = -0.5 - \frac{-1}{-4} = -0.75.$$

By use e % = $\left|\frac{x_{n+1}-x_n}{x_{n+1}}\right| \otimes \%$ as

$$e\ \% = \left|\frac{-0.75-(-0.5)}{-0.75}\right| \otimes \%$$

e % = 33%

By use same of new of x_1 in Eq (3) as

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, $\therefore$ $x_2 = -0.937$, in same we can find x_3 and x_4

which use in the following table

n	x_n	f(x)	$f`(x_n)$	x_{n+1}	e %
0	- 0.5	- 1	- 4	-0.75	33%
1	- 0.75	- 0.333	- 1.77	-0.937	19 %
2	- 0.937	-0.067	- 1.137	-0.997	6 %
3	-0.997	-0.003	- 1.006	- 1.000	0.3 %

To check the answer as:-

$\frac{1}{-1}+1 = -1+1 = 0.$

Interpolation

2-Lagrange Approximation

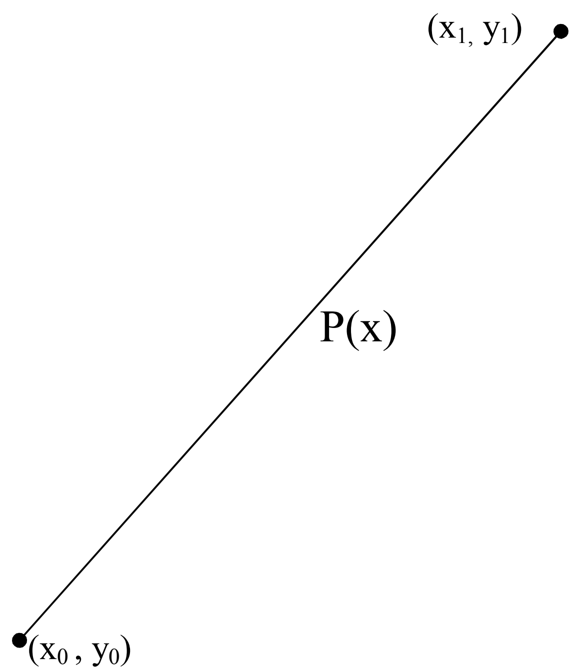
Interpolation means to estimate amassing function value by taking a weighted average of known function value of neighboring point.

Linear Interpolation

Linear Interpolation uses a line segment passes through two distinct pointes (x_0 , y_0) and (x_1, y_1) is the same as approximating a function f for which $f(x_0) = y_0$,and $f(x_1) = y_1$ by means of first-degree polynomial interpolation.

The slope between (x_0, y_0) and (x_1, y_1) is

Slope =m= $\frac{y_1 - y_0}{x_1 - x_0}$



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The point- slope formula for the line

$y = m(x-x_0) + y_0$

$y = P(x) = m(x - x_0) + y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0) + y_0$

$= y_0 + (y_1 - y_0)\frac{x - x_0}{x_1 - x_0}$

$P_1(x) = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}$ .... (4)

Each term of the right side of (4) involve a linear factor hence the sum is a polynomial of degree ≤ 1.

$L_{1,0}(x) = \frac{x - x_1}{x_0 - x_1}$, and $L_{1,1}(x) = \frac{x - x_0}{x_1 - x_0}$ .... (5)

When $x = x_0$, $L_{1,0}(x_0)=1$ and $L_{1,1}(x_0)=0$. When $x = x_1$, $L_{1,0}(x_1)=0$ and $L_{1,1}(x_1)=1$.

In terms $L_{1,0}(x)$ and $L_{1,1}(x)$ in Eq (5) called **<u>Lagrange</u>** coefficient of polynomial hazed on the nodes x_0 and x_1,

$P_1(x_0) = y_0 = f(x_0)$,and $P_1(x_1) = y_1 = f(x_1)$.

Using this notation in Eq (4), can be write in summation

$P_1(x) = y_0 L_{1,0}(x) + y_1 L_{1,1}(x)$

$P_1(x) = \sum_{k=0}^{1} y_k L_{1k}(x)$.

Suppose that the ordinates

$y_k = f(x_k)$.

If $P_1(x)$ is uses to approximante f(x) over intervalle $[x_0\ x_1]$.

<u>Example 2</u>

Consider the graph $y = f(x) = \cos(x)$ on ($x_0 = 0.0$, and $x_1=1.2$), to find the linear interpolation polynomial.

<u>Sol</u>

Now $y_0 = f(x_0) = f(0.0) = \cos(0.0) = 1.0000$, and

$y_1 = f(x_1) = f(1.2) = \cos(1.2) = 0.3624$,

$L_{1,0}(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - 1.2}{0.0 - 1.2} = -\frac{x - 1.2}{1.2}$, and

$L_{1,1}(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 0.0}{1.2 - 0.0} = \frac{x}{1.2}$.

$P_1(x) = \sum_{k=0}^{1} y_k L_{1k}(x)$.

$P_1(x) = y_0 L_{1,0}(x) + y_1 L_{1,1}(x)$

$P_1(x) = -(1.0000)\frac{x - 1.2}{1.2} + (0.3624)\frac{x}{1.2}$

$P_1(x) = -0.8333(x - 1.2) + 0.3020x$.

<u>Quadratic Lagrange Interpolation</u>

Interpolation of given pointes (x_0, y_0), (x_1, y_1) and (x_2, y_2) by a second degree polynomial $P_2(x)$, which by Lagrange summation as
$P_2(x) = y_0\, L_{1,0}(x)+ y_1\, L_{1,1}(x) + y_2\, L_{1,2}(x)$.

$$P_2(x) = \sum_{k=0}^{2} y_k L_{1k}(x) = \sum_{k=0}^{2} f(x_k) L_{1k}(x).$$

$$L_{1,0}(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)},$$

$$L_{1,1}(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$L_{1,2}(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

approximating a function f for which $f(x_0) = y_0$,and $f(x_2) = y_2$ by means of second -degree polynomial interpolation.

Example 3

Using the nodes (x_0 =2, x_1=2.5 and x_2 =4), to find the second interpolation polynomial for $f(x) = \frac{1}{x}$.

Sol

We must find

$$L_{1,0}(x) = \frac{(x-2.5)(x-4)}{(2-2.5)(2-4)} = (x-6.5)x+10,$$

$$L_{1,1}(x) = \frac{(x-2)(x-4)}{(2.5-2)(2.5-4)} = \frac{(-4x+24)x-32}{3}$$

$$L_{1,2}(x) = \frac{(x-2)(x-2.5)}{(4-2)(4-2.5)} = \frac{(x-4.5)x+5}{3}.$$

Now $f(x_0) = f(2) = 0.5$, $f(x_1) = f(2.5) = 0.4$, and $f(x_2) = f(4) = 0.25$, and

$$P_2(x) = \sum_{k=0}^{2} y_k L_{1k}(x) = \sum_{k=0}^{2} f(x_k) L_{1k}(x).$$

$P_2(x) = y_0\, L_{1,0}(x)+ y_1\, L_{1,1}(x) + y_2\, L_{1,2}(x)$

$$= 0.5[x-6.5)x+10]+ 0.4\left[\frac{(-4x+24)x-32}{3}\right] +0.25 \left[\frac{(x-4.5)x+5}{3}\right];$$

$P_2(x) =[0.05 x-0.425]x +1.15$

$f(3)= \frac{1}{3}$

$P_2(3) = 0.325.$

$f(3)= P_2(3) = 0.325.$

Cubic Lagrange Interpolation

Interpolation of given pointes (x_0, y_0), (x_1, y_1), (x_2, y_2) and (x_3, y_3) by a third degree polynomial $P_3(x)$, which by Lagrange summation as

$P_3(x) = y_0\, L_{1,0}(x) + y_1\, L_{1,1}(x) + y_2\, L_{1,2}(x) + y_3\, L_{1,3}(x)$,

$$P_3(x) = \sum_{k=0}^{3} y_k L_{1k}(x) = \sum_{k=0}^{3} f(x_k) L_{1k}(x).$$

$$L_{1,0}(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)},$$

$$L_{1,1}(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$L_{1,2}(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)},$$

$$L_{1,3}(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

Approximating a function f for which $f(x_0) = y_0$,and $f(x_3) = y_3$ by means of third -degree polynomial interpolation.

Example 4

Consider the graph $y = f(x) = \cos(x)$ on ($x_0 = 0.0$, $x_1 = 0.4$, $x_2 = 0.8$ and $x_3 = 1.2$), to find the cubic interpolation polynomial.

Sol

Now $y_0 = f(x_0) = f(0.0) = \cos(0.0) = 1.0000$,

$y_1 = f(x_1) = f(0.4) = \cos(0.4) = 0.9210$,

$y_2 = f(x_2) = f(0.8) = \cos(0.8) = 0.6967$, and

$y_3 = f(x_3) = f(1.2) = \cos(1.2) = 0.3624$,

$$L_{1,0}(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-0.4)(x-0.8)(x-1.2)}{(0.0-0.4)(0.0-0.8)(0.0-1.2)},$$

$y_0\, L_{1,0}(x) = -2.6042(x- 0.4)(x- 0.8)(x- 1.2)$,

$y_1\, L_{1,1}(x) = 7.1958(x- 0.0)(x- 0.8)(x- 1.2)$,

$y_2\, L_{1,2}(x) = -5.4430(x- 0.0)(x- 0.4)(x- 1.2)$

$y_3 L_{1,3}(x) = 0.9436(x- 0.0)(x- 0.4)(x- 0.8)$.

$P_3(x) = y_0\, L_{1,0}(x) + y_1\, L_{1,1}(x) + y_2\, L_{1,2}(x) + y_3\, L_{1,3}(x)$,

$$P_3(x) = \sum_{k=0}^{3} y_k L_{1k}(x) = \sum_{k=0}^{3} f(x_k) L_{1k}(x).$$

$P_3(x) = y_0\, L_{1,0}(x) + y_1\, L_{1,1}(x) + y_2\, L_{1,2}(x) + y_3\, L_{1,3}(x)$,

$P_3(x) = -2.6042(x- 0.4)(x- 0.8)(x- 1.2) + 7.1958(x- 0.0)(x- 0.8)(x- 1.2) + -5.4430(x- 0.0)(x- 0.4)(x- 1.2) + 0.9436(x- 0.0)(x- 0.4)(x- 0.8)$.

, In general case we construct, for each k= 0, 1…n, we can write

$$L_{n,k}(x_i)\begin{cases} = 1 \text{ if } k = i \\ = 0 \text{ if } k \neq i \end{cases}$$

Where

$$L_{n,k}(x)=\frac{(x-x_0)(x-x_1)...(x-x_{k-1})(x-x_{k+1})...(x-x_n)}{(x_k-x_0)(x_k-x_1)...(x_k-x_{k-1})(x_k-x_{k+1})...(x_k-x_n)}$$

or

$$L_{n,k}(x)=\prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)}.$$

Problems

1-If y(1) =12, y(2) =15, y(5) =25, and y(6) =30. Find the four points Lagrange interpolation polynomial that takes some value of function (y) at the given points and estimate the value of y (4) at given points.

2-Fit a cubic through the first four points y(3.2) =22.0, y(2.7) =17.8, y(1.0) =14.2, y(3.2) =22.0and y(5.6) =51.7, to find the interpolated value for x= 3.0 function (y) at the given points and estimate the value of y (4) at given points.

3-If f(1.0) =0.7651977, f(1.3) =0.6200860, f(1.6) = 0.4554022, f(1.9) = 0.2818186 and f(2.2) = 0.1103623. Use Lagrange polynomial to approximation to f(1.5).

Numerical Differentiation and Integration

Approximate Integration

Integration Equal Space

We begin our development of numerical integration by giving well-known numerical methods. If the function f(x) such a nature that $\int_a^b f(x)$dx cannot be evaluated by method of integration. In such cases, we use method to approximation to value. A geometric interpolation of $\int_a^b f(x)$dx is the area of the region bounded by the graph of y = f(x), x = a x = b, and y = 0. We can obtain an estimate of the value of integral by sketching the boundaries of the region and estimating the area of the enclosed region.

3-The Trapezoidal Rule

We shall obtain an approximation to $\int_a^b f(x)dx$ by finding the sum of areas of trapezoids. We begin by dividing [a, b] into n equal subintervals and constructed a trapezoid.
Let the lengths of the ordinates drawn at the points of subdivision by $f_0, f_1, \ldots, f_{n-1}$, and f_n and the width of each trapezoid by $\Delta x = \frac{b-a}{n}$, we find the sum of the area of the trapezoid is:-

$$A = \frac{1}{2}[f_0 + f_1]\Delta x + \frac{1}{2}[f_1 + f_2]\Delta x + \ldots + \frac{1}{2}[f_{n-1} + f_n]\Delta x$$

Or

$$\int_a^b f(x)dx = \frac{\Delta x}{2}[f_0 + 2f_1 + 2f_2 + \ldots + 2f_{n-1} + f_n] \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Eq (6) called **The Trapezoidal Rule**.

Example 5

Find $\int_0^1 \frac{1}{x^2+1}dx$, for n = 6 by **Trapezoidal** rule

Sol

$f(x) = \frac{1}{x^2+1}$, $x_0 = 0$, $x_6 = 1$

$h = \frac{x_6 - x_0}{h} = \frac{1-0}{6} = \frac{1}{6}$

$x_0 = 0$, $f_0 = \frac{1}{0^2+1} = 1$

$x_1 = x_0 + h$

$x_1 = \frac{1}{6}$, $f_1 = \frac{1}{(\frac{1}{6})^2+1} = 0.9729$

$x_2 = \frac{2}{6}$, $f_2 = \frac{1}{(\frac{2}{6})^2+1} = 0.90$

$x_3 = \frac{3}{6}$, $f_3 = \frac{1}{(\frac{3}{6})^2+1} = 0.8$

$x_4 = \frac{4}{6}$, $f_4 = \frac{1}{(\frac{4}{6})^2+1} = 0.6923$

$x_5 = \frac{5}{6}$, $f_5 = \frac{1}{(\frac{5}{6})^2 + 1} = 0.5901$

$x_6 = 1$, $f_6 = \frac{1}{(1)^2 + 1} = \frac{1}{2} = 0.5$

$A = \frac{h}{2}[\ f_0 + 2(\ f_1 + f_2 + f_3 + f_4 + f_5) + f_6]$

$A = \frac{1}{12}[\ 1 + 2(0.9729 + 0.90 + 0.8 + 0.6923 + 0.5901) + 0.5\ ]$

$A = \frac{1}{12}[\ 1 + 2(0.9729 + 0.90 + 0.8 + 0.6923 + 0.5901) + 0.5\ ]$

A = 0.7842.

4-Simpson's Rule

We obtain another approximation to $\int_a^b f(x)\,dx$. We dividing the interval from x = a to x = b into an even number of equal subintervals. We can drive the formula of Simpson by connected any three non-collinear points in the plane can be fitting with parabola and Simpson's Rule is based on approximating curves with parabola as shown in the following:-

Let the equation of parabola as

$f = Ax^2 + Bx + C$.

The area under it from x = -h to x = h as

$$\int_a^b f(x)\,dx = \int_{-h}^{h}(Ax^2 + Bx + C)\,dx = [A\frac{x^3}{3} + B\frac{x^2}{2} Cx\ ]_{-h}^{h}$$

$$= 2A\frac{h^3}{3} + 2Ch = \frac{h}{3}[2Ah^2 + 6C].$$

Since the curve passes through the three points $(-h, f_0)$, $(0, f_1)$ and (h, f_2)

$f_0 = Ah^2 - Bh + C$

$f_1 = \qquad C$

$f_2 = Ah^2 + Bh + C$.

From above equation can see that

$C = \qquad f_1$

$Ah^2 - Bh = f_0 - f_1$

$Ah^2 + Bh = f_2 - f_1$

$Ah^2 \qquad = f_0 + f_2 - 2f_1$.

Now the area $\int_a^b f(x)\,dx$ in terms of ordinates f_0, f_1 and f_2, we have

$$\int_a^b f(x)\,dx = \frac{h}{3}[2Ah^2 + 6C]. = \frac{h}{3}[\ f_0 + f_2 - 2f_1 + 6\ f_1],$$ or

$$\int_a^b f(x)\,dx = \frac{h}{3}[f_0 + 4f_1 + f_2] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Eq (7) called **Simpson's** Rule of two intervals [the with 2h]. Now in general to even number of equal subintervals by pass a parabola through [f_0, $f_1$and f_2], another through [f_2, $f_3$and f_4]…and through [f_{n-2}, $f_{n-1}$and f_n]. We then find the sum of the areas under the parabolas.

$$\int_a^b f(x)\,dx = \frac{h}{3}[f_a + 4f_1 + f_2] + \frac{h}{3}[f_2 + 4f_3 + f_4] + \ldots + \frac{h}{3}[f_{n-2} + 4f_{n-1} + f_b]$$

$$\int_a^b f(x)\,dx = \frac{h}{3}[f_a + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \ldots + 2f_{n-2} + 4f_{n-1} + f_b].$$

Where $h = \frac{b-a}{n}$, and n = even.

And the truncation error for Simpson's rule is:-

$$e_s = \frac{(b-a)^5}{180n^4} f^{(4)}(c) \quad = \frac{(b-a)}{180} h^4 f^{(4)}(c)$$

<u>Example6</u>

Use **Simpson's** rule to evaluate $\int_0^1 \frac{1}{x^2+1}\,dx$, for n = 6.

<u>Sol</u>

$f(x) = \frac{1}{x^2+1}$, $x_0 = 0$, $x_6 = 1$

$h = \frac{x_6 - x_0}{h} = \frac{1-0}{6} = \frac{1}{6}$

$x_0 = 0$, $f_0 = \frac{1}{0^2+1} = 1$

$x_1 = x_0 + h$

$x_1 = \frac{1}{6}$, $f_1 = \frac{1}{(\frac{1}{6})^2+1} = 0.9729$

$x_2 = x_1 + h$

$x_2 = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$,

$f_2 = \frac{1}{(\frac{2}{6})^2+1} = 0.90$

$x_3 = \frac{3}{6}$, $f_3 = \frac{1}{(\frac{3}{6})^2+1} = 0.8$

$x_4 = \frac{4}{6}$, $f_4 = \frac{1}{(\frac{4}{6})^2 + 1} = 0.6923$

$x_5 = \frac{5}{6}$, $f_5 = \frac{1}{(\frac{5}{6})^2 + 1} = 0.5901$

$x_6 = 1$, $f_6 = \frac{1}{(1)^2 + 1} = \frac{1}{2} = 0.5$

$A = \frac{h}{3}[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + 4f_5 + f_6]$

$A = \frac{1}{12}[\,1 + 4(0.9729) + 2(\,0.90\,) + 4(\,0.8) + 2(0.6923) + 4(0.5901\,) + 0.5\,]$

$A = 0.78593$.

5-Simpson's (3/8) Rule

If f(x) approximated by polynomial of higher degree then an accurate approximation in computing the area so if the interval divided into n subinterval that (n is odd number divided by 3) and by calculating the area of three strips by approximating f(x) by a cubic polynomial as in Simpson's Rule. And for the n formulas we obtain the three eight rule

$$\int_a^b f(x)\,dx = \frac{3h}{8}[f_a + 3f_1 + 3f_2 + 2f_3 + 3f_4 + 3f_5 + 2f_6 + \ldots + 3f_{n-2} + 3f_{n-1} + f_b].$$

Where $h = \frac{b-a}{n}$, and n = odd

And the truncation error is:-

$e_r = \frac{(b-a)^5}{6480} f^{(4)}(c)$.

Example7

Use **Simpson's** $\frac{3}{8}$ rule to evaluate $\int_0^1 x^4\,dx$, for n = 6.

Sol

$f(x) = x^4$, $x_0 = 0$, $x_6 = 1$

$h = \frac{b-a}{n} = \frac{x_6 - x_0}{h} = \frac{1-0}{6} = \frac{1}{6}$

$x_0 = 0$, $f_0 = (\,x)^4 = = (\,0)^4 = 0$

$x_1 = x_0 + h$

$x_1 = \frac{1}{6}$, $f_1 = (\frac{1}{6})^4 = 0.00077$

$x_2 = x_1 + h$

$x_2 = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$,

$f_2 = \quad (\frac{2}{6})^4 = 0.01234$

$x_3 = \frac{3}{6}$, $f_3 = (\frac{3}{6})^4 = 0.06251$

$x_4 = \frac{4}{6}$, $f_4 = (\frac{4}{6})^4 = 0.1975$

$x_5 = \frac{5}{6}$, $f_5 = (\frac{5}{6})^4 = 0.482253$

$x_6 = 1$, $f_6 = (\frac{6}{6})^4 = 1.0$

$$\int_a^b f(x)\,dx = \frac{3h}{8}[f_a + 3f_1 + 3f_2 + 2f_3 + 3f_4 + 3f_5 + f_6].$$

$$A = \frac{3h}{8}[f_a + 3(f_1 + f_2 + f_4 + f_5) + 2f_3 + f_6].$$

A = 0.2002243.

Problems

1- Approximate $\int_0^1 4x^3\,dx$, by the trapezoidal rule and by the Simpson's rule, with n = 6.

2- Approximate each of the integrals in the following problems with n = 4, by

(i) The trapezoidal rule and (ii) The Simpson's rule.

Compare your answers with

(a) The exact value in each case.

(b) Use the error in terms in Trapezoidal rule.

(c) Use the error in terms in Simpson's rule.

(1) $\int_0^2 x\,dx$

(2) $\int_0^2 x^2\,dx$

(3) $\int_0^2 x^4\,dx$

(4) $\int_1^2 \frac{1}{x^2}\,dx$

(5) $\int_1^4 \sqrt{x}\,dx$

(6) $\int_{0}^{\pi} \sin x \, dx$.

Solutions of Ordinary Differential Equation

Numerical Differentiation

Let f(x, y) be a real valued function of two variable defined for ($a \le x \le b$), and all real value of y.

6-Euler Method

The Step by Step Methods

This starts from

$y_1 = y(y_0)$, and compute an approximate value y_1 of the solution at y for $y`(x) = f(x, y(x))$ at

$x_1 = x_0 + h$, in second step computes the value y_2 of solutions at

$x_2 = x_1 + h$

$x_2 = x_0 + 2h$,

where h is fixed increment, in each step the computation are done by the same formula such formula suggested by Taylor series

$$y(x+h) = y(x) + hy`(x) + \frac{h^2}{2} y``(x) + \frac{h^3}{3} y```(x) + \ldots..$$

$$y`(y) = f(x, y(x)), \quad y``(x) = f`(x, y(x)) + \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \; y`$$

$$\therefore \; y(x+h) = y(x) + hy`(x) + \frac{h^2}{2} y``(x) + \frac{h^3}{3} y```(x) + \ldots..$$

For small h and neglected terms of h^2, h^3 …..

$y(x+h) = y(x) + h\, f(x, y)$

$y_1 = y_0 + h\, f(x_0, y_0)$,

$y_2 = y_1 + h\, f(x_1, y_1)$,

….

….

$y_{n+1} = y_n + h\, f(x_n, y_n)$.

Which called Euler's method for first order.

Example 8

Use Euler's method to solve the D. E

$\frac{dy}{dx} = x^2 + 4x - \frac{y}{2}$, with, $x_0 = 0$ $y_0 = 4$, for x = 0 to $x_0 = 0.2$, h = 0.05 work to (4D).

Sol

$f(x, y) = \frac{dy}{dx} = x^2 + 4x - \frac{y}{2}$

$y_{n+1} = y_n + h\, f(x_n, y_n)$.

$n = 0, x_0 = 0, y_0 = 4$

$y_1 = y_0 + h\, f(x_0, y_0)$.

$y_1 = 4 + .o5\, f(0, 4)$.

$y_1 = 4 + .o5[\, 0^2 + 4 \otimes 0 - \frac{4}{2}\,]$.

$y_1 = 4 - 0.1$

$y_1 = 3.9$

$x_1 = x_0 + h$

$x_1 = 0 + 0.05$

$x_1 = 0.05$

$y_2 = y_1 + h\, f(x_1, y_1)$.

$y_2 = 3.9 + .o5\, [(0.05)^2 + 4 \otimes (0.05) - \frac{3.9}{2}\,]$.

$y_2 = 3.81$

$x_2 = 0.05 + 0.05 = 0.10$

$x_3 = 0.15$, $y_3 = 3.73$

$x_4 = 0.20$, $y_4 = 3.67$

$x_5 = 0.25$, $y_5 = 3.37$.

7-Modified Euler Method (Euler Trapezoidal Method)

The Modified Euler Method gives from modified the value of (y_{n+1}) at point (x_{n+1}) by gives the new value (y_{n+1}) by the following method

$x_1 = x_0 + h$

$y^{(0)}{}_1 = y_0 + h\, f(x_0, y_0)$.

$y^{(1)}{}_1 = y_0 + \frac{h}{2}\, [\, f(x_0, y_0). + f(x_1, y^{(0)}{}_1)]$,

$y^{(2)}{}_1 = y_0 + \frac{h}{2}\, [\, f(x_0, y_0). + f(x_1, y^{(1)}{}_1)]$

....

....

$y^{(r+1)}{}_1 = y_0 + \frac{h}{2}\, [\, f(x_0, y_0). + f(x_1, y^{(r)}{}_1)]$, we can go to five iteration.

Example 9

Use Euler's **Modified** method to solve the D. E

$\frac{dy}{dx} + \frac{y}{2} = x^2 + 4x$, with, y = 4, for x = 0(0.05) 0.20, work to (3D).

Sol

Step 1

$f(x, y) = x^2 + 4x - \frac{y}{2}$

$y^{(0)}{}_1 = y_0 + h\, f(x_0, y_0).$

$n = 0, x_0 = 0, y_0 = 4$

$y_1 = y_0 + h\, f(x_0, y_0).$

$y_1 = 4 + .o5\, f(0, 4).$

$y^{(0)}{}_1 = 4 + 0.05[\, 0^2 + 4 \otimes 0 - \frac{4}{2}\,].$

$y^{(0)}{}_1 = 3.9$

$y^{(1)}{}_1 = y_0 + \frac{h}{2}\, [\, f(x_0, y_0). + f(x_1, y^{(0)}{}_1)],$

$= 4 + \frac{0.05}{2}\, [-\frac{4}{2} + (-0.05)^2 + 4(0.05) - \frac{1}{2} \otimes 3.9] = 3.906$

$y^{(1)}{}_1 = 3.906.$

$y^{(2)}{}_1 = y_0 + \frac{h}{2}\, [\, f(x_0, y_0). + f(x_1, y^{(1)}{}_1)]$

$= 4 + \frac{0.05}{2}\, [-\frac{4}{2} + (-0.05)^2 + 4(0.05) - \frac{1}{2} \otimes 3.906] = 3.906$

$y^{(2)}{}_1 = 3.906$

Step 2

$x_2 = x_1 + h = 0.05 + 0.05 = 0.05 + 0.1$

$y^{(0)}{}_2 = y_1 + h\, f(x_1, y_1).$

$n = 1, x_1 = 0.05, y_1 = 3.906$

$y^{(0)}{}_2 = y_1 + h\, f(x_1, y_1).$

$= 3.906 + 0.05[(0.05)^2 + 4(0.05) - \frac{1}{2} \otimes 3.906] = 3.912$

$y^{(0)}{}_2 = 3.912$

$y^{(1)}{}_2 = y_1 + \frac{h}{2}\, [\, f(x_1, y_1). + f(x_2, y^{(0)}{}_2)],$

$= 3.906 + \frac{0.05}{2}\, [(0.05)^2 + 4(0.05) - \frac{1}{2} \otimes 3.906 + (0.1)^2 + 4(0.1) - \frac{1}{2} \otimes 3.91] =$
3.868

$y^{(1)}{}_2 = 3.868.$

$y^{(2)}{}_2 = y_1 + \frac{h}{2}\, [\, f(x_1, y_1). + f(x_2, y^{(1)}{}_2)]$

$= 3.906 + \frac{0.05}{2}\, [(0.05)^2 + 4(0.05) - \frac{1}{2} \otimes 3.906 + (0.1)^2 + 4(0.1) - \frac{1}{2} \otimes 3.868] =$
3.824

$y^{(2)}{}_2 = 3.824.$

$y^{(3)}_2 = y_1 + \frac{h}{2}$ [$f(x_1, y_1). + f(x_2, y^{(2)}_2)$]

$= 3.906+\frac{0.05}{2}$ [$(0.05)^2 +4(0.05) - \frac{1}{2} \otimes 3.906+(0.1)^2 +4(0.1) - \frac{1}{2} \otimes 3.824$] = 3.825

$y^{(3)}_2 = 3.825.$

Step 3

$x_3 = x_2 + h = 0.1+0.05 = 0.15$

$n = 2, x_2 = 0.1, y_2 = 3.825$

$y^{(0)}_3 = y_2 + h f(x_2, y_2).$

$= 3.825+0.05[(0.1)^2 +4(0.1) - \frac{1}{2} \otimes 3.825] = 3.750$

$y^{(0)}_3 = 3.750$

$y^{(1)}_3 = y_2 + \frac{h}{2}$ [$f(x_2, y_2). + f(x_3, y^{(0)}_3)$],

$= 3.825+\frac{0.05}{2}$ [$(0.1)^2 +4(0.1) - \frac{1}{2} \otimes 3.825+(0.15)^2 +4(0.15) - \frac{1}{2} \otimes 3.750$] = 3.756

$y^{(1)}_3 = 3.756.$

In same way we find

$y^{(2)}_3 = 3.756.$

Step 4

$x_4 = x_3 + h = 0.15+0.05 = 0.2$

$n = 3, x_3 = 0.15, y_3 = 3.756$

$y^{(0)}_4 = y_3 + h f(x_3, y_3).$

$= 3.756+0.05[(0.15)^2 +4(0.15) - \frac{1}{2} \otimes 3.756] = 3.693$

$y^{(0)}_4 = 3.693$

$y^{(1)}_4 = y_3 + \frac{h}{2}$ [$f(x_3, y_3). + f(x_4, y^{(0)}_4)$],

$= 3.756+\frac{0.05}{2}$ [$(0.15)^2 +4(0.15) - \frac{1}{2} \otimes 3.756+(0.2)^2 +4(0.2) - \frac{1}{2} \otimes 3.693$] = 3.699

$y^{(1)}_4 = 3.699.$

$y^{(2)}_4 = y_3 + \frac{h}{2}$ [$f(x_3, y_3). + f(x_4, y^{(1)}_4)$],

$= 3.756+\frac{0.05}{2}$ $[(0.15)^2 +4(0.15) -\frac{1}{2}\otimes 3.756+(0.2)^2 +4(0.2) -\frac{1}{2}\otimes 3.699] =$ 3.699

$y^{(2)}{}_4 = 3.699.$

The following table gives the above resulted of x and y.

x	y
0	4
0.05	3.906
0.1	3.825
0.15	3.756
0.2	3.699

Problems

Apply Euler's methods to the following initials value problems. Do 5 steps. Solve the problem exactly. Compute the errors to see that the method is too inaccurate for Practical purposes

(1) $y' +$ o.1 $y= 0$ with $y(0) = 2$, $h = 0.1$.

(2) $y' = \frac{\pi}{2}\sqrt{1-y^2}$, with $y(0) = 0$, $h = 0.1$.

(3) $y' + 5x^4 y^2= 0$ with $y(0) = 1$, $h = 0.2$.

(4) $y' = (y+ x)^2$ with $y(0) = 1$, $h = 0.1$.

Find the exacted solution and the error

(5) $y' + 2x y^2= 0$ with $y(0) = 1$, $h = 0.2$.

(6) $y' = 2(1+ y^2)$, with $y(0) = 0$, $h = 0.5$.

(7) Use Euler's methods to find numerical solution of the following d. e.

(8) $y'= 4x +x^2 -\frac{1}{2}y$, with $y(0) = 4$, $h = 0.05$, find to 3-decimal.

8-Runge Kutta Method

When

$$\frac{dy}{dx} = f(x, y)$$

$$\therefore y_{n+1} = y_n +\frac{1}{6}[k_1+ 2k_2 +2k_3 + k_4]$$

Where

$k_1= h\, f(x_n, y_n)$.

$k_2 = h\, f(x_n +\frac{h}{2}, y_n+\frac{k_1}{2})$.

$K_3 = h\, f(x_n +\frac{h}{2}, y_n+\frac{k_2}{2})$.

$K_4 = h\, f(x_n +h, y_n+ k_3)$.

Where h and (x_n, y_n) are given.

Example 10

Use Runge Kutta Method to solve the D. E

$\frac{dy}{dx} = x + y$, with $x_0 = 0$, $y_0 = 1$, with h = 0.1 work to (4D).

Sol

$f(x, y) = \frac{dy}{dx} = x + y$

$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$k_1 = h\, f(x_0, y_0)$.

$n = 0,\ x_0 = 0,\ y_0 = 1$

$k_1 = 0.1\ f(0, 1) = 0.1[0 + 1] = 0.1$

$k_1 = 0.1$

$k_2 = h\, f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$.

$= 0.1\ f(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}) = 0.1[0.05 + 1.05]$

$K_2 = 0.11$.

$K_3 = h\, f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.1\ f(0.05, 1 + \frac{0.11}{2})$

$= 0.1[0.05 + 1.055]$

$K_3 = 0.1105$.

$K_4 = h\, f(x_0 + h, y_0 + k_3) = 0.1[0.1, 1 + 0.1105]$

$= 0.1[0.1, 1.1105]$

$K_4 = 0.12105$.

$y_1 = 1 + \frac{1}{6}[0.1 + 2 \otimes 0.11 + 2 \otimes 0.1105 + 0.12105]$,

$y_1 = 1.11034$

$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$k_1 = h\, f(x_1, y_1)$.

$n = 1,\ x_1 = x_0 + h = 0 + 0.1 = 0.1,\ y_1 = 1.11034$

$k_1 = 0.1\ f(0.1, 1.11034) = 0.1[0.1 + 1.11034] = 0.12103$

$k_1 = 0.12103$

$k_2 = h\, f(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$.

$= 0.1\ f(0.1 + \frac{0.1}{2}, 0.12103 + \frac{0.1}{2}) = 0.13208$

$K_2 = 0.13208$.

$K_3 = 0.132638$.

$K_4 = 0.1442978$.

$y_2 = 1.24306.$

$\therefore (x_2, y_2) = (0.2, 1.24306).$

9-Runge- Kutta-Merson Method

The problem of Runge Kutta Method is not compute an approximate decimal error[Rounding Error or Truncation Error], we think Runge-Kutta-Merson Method give the an approximate the error of this problem at any step as see in the following:-

$y_{n+1} = y_n + \frac{1}{6}[k_1 + 4k_4 + k_5],$

$k_1 = h\, f(x_n, y_n),$

$k_2 = h\, f(x_n + \frac{h}{3}, y_n + \frac{k_1}{3}),$

$K_3 = h\, f(x_n + \frac{h}{3}, y_n + \frac{k_1}{6} + \frac{k_2}{6}),$

$K_4 = h\, f(x_n + \frac{h}{2}, y_n + \frac{k_1}{8} + \frac{3k_3}{8}),$

$K_5 = h\, f(x_n + h, y_n + \frac{k_1}{2} - \frac{3k_3}{2} + 2k_4).$

We compute the error as

$\text{Error} = \frac{1}{30}[2k_1 - 9k_3 + 8k_4 - k_5].$

Example 11

Use Runge- Kutta-Merson Method to solve the D. E

$\frac{dy}{dx} = x + y$, with $x_0 = 0$, $y_0 = 1$, for $x = 0$ to $x_0 = 1.0$, with $h = 0.1$ work to (4D).

Sol

$f(x, y) = \frac{dy}{dx} = x + y$

$k_1 = h\, f(x_n, y_n).$

$n = 0, x_0 = 0 , y_0 = 1$

$k_1 = h\, f(0, 1) = 0.1[0 + 1] = 0.1$

$k_1 = 0.1$

$k_2 = h\, f(x_n + \frac{h}{3}, y_n + \frac{k_1}{3}),$

$= h\, f(0 + \frac{0.1}{3}, 1 + \frac{0.1}{3}).$

$= h\, f(0.113, 1.0333) = 0.1[0.113 + 1.0333]$

$K_2 = 0.1067$

$K_3 = h\, f(0 + \frac{0.1}{3}, 1 + \frac{0.1}{6} + \frac{0.1067}{6}),$

$= h\ f(0.0333+ 1.0344),$

$= 0.1[0.0333+ 1.0344],= 0.1068.$

$K_4 = h\ f(x_n + \frac{h}{2}, y_n + \frac{k_1}{8} + \frac{3k_3}{8}).$

$= h\ f(0 + \frac{0.1}{2}, 1 + \frac{0.1}{8} + \frac{3(0.1068)}{8}).$

$= h\ f(0.05, 1.0525) = 0.1[0.05+ 1.0525]. = 0.1103.$

$K_5 = h\ f(x_n + h, y_n + \frac{k_1}{2} - \frac{3k_3}{2} + 2k_4).$

$= 0.1\ f(0 + 0.1, 1 + \frac{0.1}{2} - \frac{3(0.1068)}{2} + 2(0.1103)).$

$= 0.1\ f(0.1, 1.1103),$

$= 0.1[0.1, 1+1.1103]. = 0.1210.$

$y_{n+1} = y_n + \frac{1}{6}[k_1 + 4k_4 + k_5],$

$y_1 = y_0 + \frac{1}{6}[k_1 + 4k_4 + k_5],$

$y_1 = 1 + \frac{1}{6}[0.1 + 4(0.1103) + 0.1210],$

$y_1 = 1.1104.$

$x_1 = x_0 + h$

$x_1 = 0 + 0.1 = 0.1$

$\therefore (x_1, y_1) = (0.1, 1.1104).$

$\text{Error} = \frac{1}{30}[2k_1 - 9k_3 + 8k_4 - k_5].$

$= \frac{1}{30}[2(0.1) - 9(0.1068) + 8(0.1103) - 0.1210].$

$\therefore \text{Error} = 6.667 \otimes 10^{-6}.$

Problems

1-Apply Range –Kutta methods to the initials value problem, choosing $h = 0.2$, and computing$(y_1 + y_2 + y_3 + y_4 + y_5)$ of $y' = x + y$ with $y(0) = 0$.

2- Use Range –Kutta methods to find numerical solution of the following d. e.

(a) $y' = 3x + \frac{y}{2}$, with $y(0) = 1$, $h = 0.1$. On interval $(0 \leq x \leq 1)$

(b) $y' = x + y$ with $y(0) = 1$, in the range $(0 \leq x \leq 1)$ 1, with $h = 0.1$.

3- Comparison of Euler and Range –Kutta methods to solve $y' = 2x^{-1}\sqrt{y - \ln x} + x^{-1}$, with $y(1) = 0$, $h = 0.1$. On interval $(1 \leq x \leq 1.8)$. And compute the error.

4- Solve problem (3) by classical Range –Kutta methods, with h = 0.4, determine the error, and compute with (3).

System of Linear Equation

Definition 1

Let the system of linear equation as

$$\left.\begin{array}{l} a_{11}x_1, a_{12}x_2 ... ,a_{1n}x_n = b_1 \\ a_{21}x_1, a_{22}x_2 ... ,a_{2n}x_n = b_2 \\ \ldots\ldots\ldots \\ \ldots\ldots\ldots \\ A_{m1}x_1, a_{m2}x_2 ... ,a_{mn}x_n = b_m \end{array}\right\} \ldots\ldots\ldots\ldots \quad (8)$$

Can put the apove system in matrix form as:-

$$\begin{pmatrix} a_{11} & a_{12} & - & - & - & a_{1n} \\ a_{21} & a_{22} & - & - & - & a_{2n} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ a_{m1} & a_{m1} & - & - & - & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ - \\ - \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ - \\ - \\ b_m \end{pmatrix} \ldots\ldots\ldots\ldots \quad (8)$$

Or

AX=B, ……………………………………………………….. (8)

$$A=\begin{pmatrix} a_{11} & a_{12} & - & - & - & a_{1n} \\ a_{21} & a_{22} & - & - & - & a_{2n} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ a_{m1} & a_{m1} & - & - & - & a_{mn} \end{pmatrix}, B=\begin{pmatrix} b_1 \\ b_2 \\ - \\ - \\ b_m \end{pmatrix}, \text{ and } X=\begin{pmatrix} x_1 \\ x_2 \\ - \\ - \\ x_n \end{pmatrix}$$

Where A=mxn, matrix, $a_{11}, a_{12}..., a_{mn}$ are constant, X= nx1, B = mx1and, $b_1, b_2..., b_m$, are constant $x_1, x_2 ... x_n$, variable.

Now we study the following methods {Cramer's Rule, Inverse Matrices, and Elimination Method}

10-Cramer's Rule

To solve the system (8) by Cramer's Rule. Find determinate of A (| A|) such that $|A| \neq 0$.
Let

$$|A| = D = \begin{vmatrix} a_{11} & a_{12} & - & - & - & a_{1n} \\ a_{21} & a_{22} & - & - & - & a_{2n} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ a_{m1} & a_{m1} & - & - & - & a_{mn} \end{vmatrix}, \quad D_1 = \begin{vmatrix} b_1 & a_{12} & - & - & - & a_{1n} \\ b_2 & a_{22} & - & - & - & a_{2n} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ b_m & a_{m1} & - & - & - & a_{mn} \end{vmatrix},$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 & - & - & - & a_{1n} \\ a_{21} & b_2 & - & - & - & a_{2n} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ a_{m1} & b_m & - & - & - & a_{mn} \end{vmatrix}, \ldots, D_n = \begin{vmatrix} a_{11} & a_{12} & - & - & - & b_1 \\ a_{21} & a_{22} & - & - & - & b_2 \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ a_{m1} & a_{m1} & - & - & - & b_m \end{vmatrix},$$

To solve system (8), we must find unknown $x_1, x_2 \ldots x_n$ as

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \ldots x_n = \frac{D_n}{D}.$$

11-Solution of Linear Equations by using Inverse Matrices

To solve the system (8) by **using Inverse Matrices** Find determinate of A (| A|) such that $|A| \neq 0$.
Or
AX=B,
Turing to the relation between the solution of linear equation and matrix inversion multiplying both sides by A^{-1} thus
A^{-1} [AX=B]
$A^{-1}AX = A^{-1}B$.
$X = A^{-1}B$.
This equation gives the values of the entire unknown X by a simple multiplication of matrix A by inverse of it matrix. As see in the following example

Example12

Use the matrix inversion method; find the values of (x_1, x_2, x_3) for the following set of linear algebraic equations:-

$$\left.\begin{aligned} 3x_1 - 6x_2 + 7x_3 &= 3 \\ 4x_1 \qquad - 5x_3 &= 3 \\ 5x_1 - 8x_2 + 6x_3 &= -4 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

Solution

Put the system (9) in the following matrix form as
AX=B,

$$\begin{pmatrix} 3 & -6 & 7 \\ 4 & 0 & -5 \\ 5 & -8 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}$$

Where| A|

$$|A| = \begin{vmatrix} 3 & -6 & 7 \\ 4 & 0 & -5 \\ 5 & -8 & 6 \end{vmatrix} = 462 \neq 0.$$

We can find the inverse matrix of A (A^{-1}), by any method.

$\therefore A^{-1} = \begin{pmatrix} 0.26 & 0.14 & -0.2 \\ 0.52 & 0.12 & -0.52 \\ 0.48 & 0.04 & -0.36 \end{pmatrix}$, now we can see the following

A^{-1} [AX=B]

A^{-1} AX= A^{-1} B.

X= A^{-1} B.

$$\therefore X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.26 & 0.14 & -0.2 \\ 0.52 & 0.12 & -0.52 \\ 0.48 & 0.04 & -0.36 \end{pmatrix}\begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}$$

$\therefore X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$, which gives the solution of system as $x_1 = 2$, $x_2 = 4$, $x_3 = -4$.

12-Gauss Elimination Method

We can use Gauss Elimination Method to solve the system of linear equation in (8), as see in the following example

Example 13

$$\left.\begin{aligned} 3x_1 - x_2 + 2x_3 &= 12 \\ 3x_1 + 2x_2 + 3x_3 &= 11 \\ 2x_1 - 2x_2 - x_3 &= 2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Solution

Put the system (10) in the following matrix form

$$\left[\begin{array}{ccc:c} 3 & -1 & 2 & 12 \\ 3 & 2 & 3 & 11 \\ 2 & -2 & -1 & 12 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

Where R_i (i= 1, 2, 3) row of system.

Step 1

By using

$R_2 - R_1$, and $3R_3 - 2R_1$

System (11) become

$$\begin{bmatrix} 3 & -1 & 2 & : & 12 \\ 0 & 7 & 7 & : & 21 \\ 0 & -4 & -7 & : & -8 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \qquad (12)$$

Step 2

By using

$7R_3 + 4R_2$

System (11) become

$$\begin{bmatrix} 3 & -1 & 2 & : & 12 \\ 0 & 7 & 7 & : & 21 \\ 0 & 0 & -21 & : & -42 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \qquad (13)$$

Step 3

From last system (13) we the following equation

$$3x_1 - x_2 - 2x_3 = 12$$
$$7x_2 + 7x_3 = 21$$
$$-21x_3 = -42$$

Which can easily to solve this system to find:-

$x_3 = 2$, $x_2 = 1$, $x_1 = 3$.

13- Iterative Methods (Gauss Siedle Methods)

We can use Gauss Siedle Method to solve the system of linear equation in (8), as see in the following example

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \qquad (14)$$

To solve the system (13) by using Gauss Siedle Method can see the following steps:-

Step 1

Re write system (13) as form

$$\left.\begin{aligned} x_1 &= [\, b_1 - a_{12}x_2 - a_{13}x_3] / (a_{11}) \\ x_2 &= [\, b_2 - a_{21}x_1 - a_{23}x_3] / (a_{22}) \\ x_3 &= [\, b_3 - a_{31}x_1 - a_{32}x_2] / (a_{33}). \end{aligned}\right\} \qquad (15)$$

Step 2

Selected initial values of x_1, x_2 and x_3 put in system (16). For example (Let $x_1 = x_2 = x_3 = 0$ initial values)

Step 3

By using the new value of x_1, x_2 and x_3 as Step 2.Repeated Step2 until no change of values of x_1, x_2 and x_3.As see in following example

Example14

$$\left.\begin{aligned} 5x_1 - 2x_2 + x_3 &= 4 \\ x_1 + 4x_2 - 2x_3 &= 3 \\ x_1 + 4x_2 + 4x_3 &= 17 \end{aligned}\right\} \quad \ldots\ldots (16)$$

Solution

Step 1

Re write system (16) as form

$x_1 = [4 + 2x_2 - x_3]/(5)$ .... (17)

$x_2 = [3 - x_1 + 2x_3]/(4)$ .... (18)

$x_3 = [17 - x_1 - 4x_2]/(4)$. .... (19)

Step 2

Selected initial values of x_1, x_2 and x_3 put in system (15). For example (Let $x_1 = x_2 = x_3 = 0$ initial values).

Then get x_1 from Eq (17){by using $x_2 = x_3 = 0$ } $\rightarrow x_1 = 4/5 = 0.8$, x_2 from Eq (18) {by use new of $x_1 = 0.8$, $x_3 = 0$ } gives$\rightarrow x_2 = 0.55$.Find x_3 from Eq (19) {by use new of $x_1 = 0.8$, $x_2 = 0.55$} gives $\rightarrow x_3 = 0.55$.

Step 3

By using the new value of x_1, x_2 and x_3 as Step 2.Repeated Step2 until no change of values of x_1, x_2 and x_3.As see in following values

n	X_1	X_2	X_3
0	0	0	0
1	0.8	0.55	3.775
2	0.265	2.572	2.898
3	1.247	1.889	3.007
4	0.956	2.008	2.998
5	1.002	2.003	3.000
6	1.001	1.999	3.000
7	0.999	2.000	3.000

In general let k (where k integer number) denoted repeated to number of iteration. Then we can rewrite the system (15) as form:-

$$\left.\begin{aligned} x_1^k &= [b_1 - a_{12}\, x_2^{k-1} - a_{13}\, x_3^{k-1}]/(a_{11}) \\ x_2^k &= [b_2 - a_{21}\, x_1^{k} - a_{23}\, x_3^{k-1}]/(a_{22}) \\ x_3^k &= [b_3 - a_{31}\, x_1^{k} - a_{32}\, x_2^{k}]/(a_{33}). \end{aligned}\right\} \quad \ldots\ldots (20)$$

Suppose that $a_{11} \neq 0$, $a_{22} \neq 0$, $a_{33} \neq 0$.

Problems

(a) Use Gauss Elimination Method to solve the following system of linear equation

(1)

$3x_1 - x_2 + 3x_3 = 12$

$x_1 + x_2 + 3x_3 = 11$

$2x_1 - 2x_2 - x_3 = 2$

(2)

$2x_1 - x_2 + x_3 = 1$

$3x_1 - 2x_2 + x_3 = 0$

$5x_1 + x_2 \; 2x_3 = 9$

(3)

$x_1 + 2x_3 = 3$

$2x_2 + 3x_3 = 5$

$2x_3 + x_4 = 7$

$x_1 + 4x_4 = 5$

(4)

$x_1 + 2x_2 - 4x_3 = 4$

$5x_1 - 3x_2 - 7x_3 = 6$

$3x_1 - 4x_2 + 3x_3 = 1$

(5)

$2x_1 + x_2 - 3x_3 = 1$

$5x_1 + 2x_2 - 6x_3 = 5$

$3x_1 - x_2 - 4x_3 = 7$

(6)

$2x_1 - 4x_2 + 6x_3 = 5$

$x_1 + 3x_2 - 7x_3 = 2$

$7x_1 + 5x_2 + 9x_3 = 4$

(7)

$-x_1 + x_2 + 2x_3 = 2$

$3x_1 - x_2 + x_3 = 6$

$-x_1 + 3x_2 + 4x_3 = 4$

(b) Use Gauss Siedle Method to solve the following system of linear equation

(1)

$3x_1 - x_2 + 3x_3 = 12$

$x_1 + x_2 + 3x_3 = 11$

$2x_1 - 2x_2 - x_3 = 2$

(2)

$2x_1 - x_2 + x_3 = 1$

$3x_1 - 2x_2 + x_3 = 0$

$5x_1 + x_2 \; 2x_3 = 9$

(5)

$2x_1 + x_2 - 3x_3 = 1$

$5x_1 + 2x_2 - 6x_3 = 5$

$3x_1 - x_2 - 4x_3 = 7$

(6)

$2x_1 - 4x_2 + 6x_3 = 5$

$x_1 + 3x_2 - 7x_3 = 2$

$7x_1 + 5x_2 + 9x_3 = 4$

(3)

$x_1 \quad +2x_3 \quad = 3$
$2x_2 + 3x_3 \quad = 5$
$2x_3 + x_4 = 7$
$x_1 \quad +4x_4 = 5$

(7)

$-x_1 + x_2 + 2x_3 = 2$
$3x_1 - x_2 + x_3 = 6$
$-x_1 + 3x_2 + 4x_3 = 4$

(4)

$x_1 + 2x_2 - 4x_3 = 4$
$5x_1 - 3x_2 - 7x_3 = 6$
$3x_1 - 4x_2 + 3x_3 = 1.$

14-Least Squares Approximations

Let y denoted to real value, $\bar{y}$ denoted to approximation value, and d denoted to deferent between the real value (y) from tables, and approximation value ($\bar{y}$), denoted to it in general as:-

$d_i = y_i - \bar{y}_i$, where i= 1, 2…m.

Let there are m value y as (y_1 … y_m) corresponding to m value of x as(x_1 … x_m) gives m of different d as (d_1 … d_m), where

$d_1 = y_1 - \bar{y}_1$,

$d_2 = y_2 - \bar{y}_2$,

…

…

$d_m = y_m - \bar{y}_m$.

The method of Least Squares Approximations using, the summation of difference ($\sum_{i=1}^{m} d_i$) at minimum. We square the difference because the negative sign.

$$\sum_{i=1}^{m}(d_i)^2 = \sum_{i=1}^{m}(y_i - \bar{y}_i)^2 .$$

Let the relation between x and y at linear form as:-

$\bar{y}_1 = a + bx_1$,

The difference become as

$d_i = y_i - a - bx_i$, let

$$q = \sum_{i=1}^{m}(d_i)^2 \text{ , or}$$

$$q = \sum_{i=1}^{m}(d_i)^2 = \sum_{i=1}^{m}(y_i - a - bx_i)^2 \text{ or}$$

$$q = \sum_{i=1}^{m}(y_i - a - bx_i)^2 \quad \text{……..…………………..…} (21)$$

There are only two unknown (a and b) in Eq (21).

Now if q at minimum, then first partial derivative of q (w .r .to) a and b must equal to zero as:-

$$\frac{\partial q}{\partial a} = \sum_{i=1}^{m} -2(y_i - a - bx_i) \quad = 0$$

$$\frac{\partial q}{\partial b} = \sum_{i=1}^{m} -2x_i(y_i - a - bx_i) \quad = 0.$$

Re-write above equations as

$$ma + (\sum_{i=1}^{m} x_i)b = \sum_{i=1}^{m} y_i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

$$(\sum_{i=1}^{m} x_i)a \quad + (\sum_{i=1}^{m} x^2{}_i)b = \sum_{i=1}^{m} x_i y_i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23).$$

Put Eq (22and 23) in the following matrix form

$$\begin{pmatrix} m & \sum_{i=1}^{m} x_i \\ \sum_{i=1}^{m} x_i & \sum_{i=1}^{m} x^2{}_i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{m} y_i \\ \sum_{i=1}^{m} x_i y_i \end{pmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$$

We can find two unknown (a and b) in Eq (21). By using crammers rule as:-

$$\text{Let } D = \begin{vmatrix} m & \sum_{i=1}^{m} x_i \\ \sum_{i=1}^{m} x_i & \sum_{i=1}^{m} x^2{}_i \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25)$$

Where D the determinant such that $D \neq 0$, and

$$D_1 = \begin{vmatrix} \sum_{i=1}^{m} y_i & \sum_{i=1}^{m} x_i \\ \sum_{i=1}^{m} x_i y_i & \sum_{i=1}^{m} x^2{}_i \end{vmatrix}, \quad D_2 = \begin{vmatrix} m & \sum_{i=1}^{m} y_i \\ \sum_{i=1}^{m} x_i & \sum_{i=1}^{m} x_i y_i \end{vmatrix},$$

$$a = \frac{D_1}{D}, \; b = \frac{D_2}{D}.$$

Example 15

Find the following points to linear form $y = a + b\,x$, where

x	y
1	3
2	5
3	8
4	13
5	16

Solution

	x	y	x^2	xy
	1	3	1	3
	2	5	4	10
	3	8	9	24
	4	13	16	52
	5	16	25	80
Sum	15	45	55	169

From Eqs. (21 and 23)

5a+15 b = 45,

15a+55 b = 169,

$$D = \begin{vmatrix} 5 & 15 \\ 15 & 55 \end{vmatrix} = 50$$

$$a = \frac{D_1}{D},\ a = \frac{\begin{vmatrix} 45 & 15 \\ 169 & 55 \end{vmatrix}}{50} = \frac{-6}{5} \quad b = \frac{D_2}{D} = \frac{\begin{vmatrix} 5 & 45 \\ 15 & 196 \end{vmatrix}}{50} = \frac{17}{5}$$

$$y = \frac{-6}{5} + \frac{17}{5}x,$$

5y = -6+ 17 x,

Example 16

Find the following points to linear form $y = a e^{bx}$. Where

X	Y
0	1.5
1	2.5
2	3.5
3	5
4	7.5

Sol

$\text{Ln } y = \text{Ln}(a\, e^{bx}) \rightarrow \text{Ln } y = \text{Ln}(a) + \text{Ln}(e^{bx})$

$\rightarrow \text{Ln } y = \text{Ln}(a) + bx$, compare with standard equation $Y = A + b\,X$

$Y = \text{Ln } y$ $\text{Ln}(a) = A$, $b = b$, $X = x$.

X	Y	X=x	Y=Lny	$X_i^{\ 2}$	$X_i\,Y_i$
0	1.5	0	0.40547	0	0
1	2.5	1	0.91629	1	0.91629
2	3.5	2	1.25276	4	2.50553
3	5	3	1.60944	9	4.82831
4	7.5	4	2.01490	16	8.05961
Sum=10		10	6.19866	30	16.30974

$Y = A + b\,X \longrightarrow ma + (\sum_{i=1}^{m} x_i)b = \sum_{i=1}^{m} y_i$

$$(\sum_{i=1}^{m} x_i)a + (\sum_{i=1}^{m} x^2{}_i)b = \sum_{i=1}^{m} x_i y_i$$

$5a + 10\,b = 6.19866$,

$10a + 30\,b = 16.30974$,

$D = \begin{vmatrix} 5 & 10 \\ 10 & 30 \end{vmatrix} = 50$

$a = \frac{D_1}{D}$, $a = \frac{\begin{vmatrix} 6.19866 & 10 \\ 16.30974 & 30 \end{vmatrix}}{50} = 0.45736$, $b = \frac{D_2}{D} == \frac{\begin{vmatrix} 5 & 6.19866 \\ 10 & 16.30974 \end{vmatrix}}{50} = 0.39120$.

$A = \text{Ln}(a) \longrightarrow e^{A} = e^{Lna} \longrightarrow e^{A} = a \longrightarrow e^{A} = e^{0.45736}$,

$\longrightarrow a = 1.5799$, $b = 0.39120$.

$Y = 1.5799\, e^{0.39120X}$

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