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Electromechanical Department

Fourier series

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Advance Mathematics

Fourier series

3rd Class

Electromechanical Eng.

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Fourier series

Definition

The series

$$f(x) = \frac{1}{2}\, a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{l}) + b_n \sin\left(\frac{n\pi x}{l}\right)$$

Si the Fourier series of f on (-L , L) when the constant are chosen to be the Fourier coefficient of "f"on (-L , L)

<u>Where</u>

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad for\ \ n = 0\,, 1\,, 2$$

And

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \qquad for\ \ n = 1\,, 2\,, 3$$

And

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx$$

<u>Example</u>

Let f(x) = x for $-\pi \le$ x $\le \pi$. we will write the Fourier series of "f" on [$-\pi$, π] . the coefficient are

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x dx = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(nx) dx$$

$$= [\frac{1}{n^2 \pi} \cos(nx) + \frac{x}{n\pi} \sin(nx)]_{-\pi}^{\pi} = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) dx$$

$$= [\frac{1}{n^2\pi} \sin(nx) - \frac{x}{n\pi} \cos(nx)]_{-\pi}^{\pi} = 0$$

$$= -\frac{2}{n} \cos(n\pi) = \frac{2}{n} (-1)^{n+1}$$

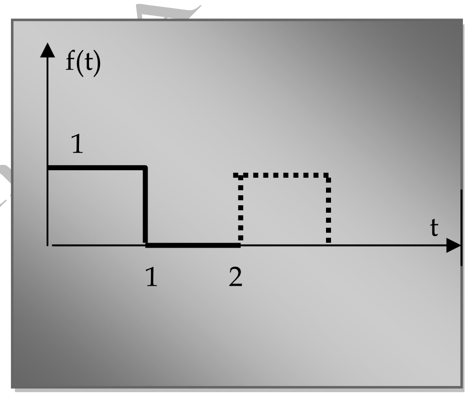
Since cos (n π) = $(-1)^n$ if n is an integer . the fourier series of "x" on [- π , π] is

$$\sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin(nx) = 2\sin(x) - \sin(2x) + \frac{2}{3}\sin(3x) - \frac{1}{2}\sin(4x) + \frac{2}{5}\sin(5x) \ \ldots$$

Example

Find the fourier series of the periodic function

$$f(x) \begin{cases} 1 & 0 < t < 1 \\ 0 & 1 < t < 2 \end{cases}$$



Solution

d = 0

2 p = 2 $\Rightarrow$ p = 1

$$f(t) = \frac{a_0}{2} + a_1 \cos\frac{\pi t}{p} + \ldots\ldots\ a_n \cos\frac{n\pi t}{p} + \ \ldots\ldots b \sin\frac{\pi t}{p} + \ \ldots\ldots b_n \sin\frac{n\pi t}{p}$$

$$a_0 = \frac{1}{p} \int_{d}^{d+2p} f(t)dt = \frac{1}{1} \int_0^2 f(t)\, dt = \int_0^1 (1)dt + \int_1^2 (0)dt$$

$$= t|_0^1 = 1 \qquad a_0 = 1$$

$$a_n = \frac{1}{p} \int_{d}^{d+2p} f(t) \cos\frac{n\pi t}{p} dt$$

$$= \frac{1}{1}\int_0^1 (1)\cos\frac{n\pi t}{p} dt + \int_1^2 (0)\cos\frac{\pi t}{p} dt$$

$$= \frac{1}{n\pi} \sin n\pi t \quad |_0^1 + 0 = \frac{1}{n\pi} \sin n\pi - 0 = 0$$

Always sin (n π) = 0

$a_n = 0$

$$b_n = \frac{1}{p} \int_{d}^{d+2p} f(t) \sin \frac{n\pi t}{p} \, dt$$

$$b_n = \frac{1}{p} \int_{0}^{1} (1) \sin n\pi t + \int_{1}^{2} 0 \sin n\pi t \; d\,t$$

$$b_n = -\frac{1}{n\pi} \cos n\pi t \,|_0^1 = -\frac{1}{n\pi} \cos n\pi + \frac{1}{n\pi} \cos 0$$

$$= \frac{1}{n\pi} (1 - \cos n\pi)$$

$$b_1 = \frac{1}{\pi} (1 - \cos n\pi) = 2/\pi$$

$$b_2 = \frac{1}{2\pi} (1 - \cos n\pi) = 0$$

$$b_3 = \frac{1}{3\pi} (1 - cos 2\pi) = \frac{2}{3\pi}$$

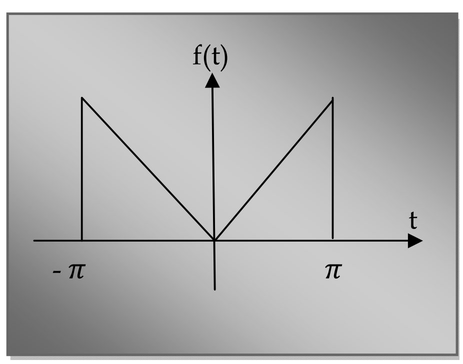
$$b_4 = \frac{1}{4\pi} (1 - cos 4\pi) = 0$$

The fourier series become

$$f(t) = \frac{1}{2} + \frac{2}{\pi} \sin \pi t \; + \; 0 + \frac{2}{3\pi} \sin 3\pi t + 0 + \frac{3}{5\pi} \sin 5\pi t$$

Example

$$f(x) \begin{cases} -x & -\pi < x < 0 \\ +x & 0 < x < \pi \end{cases}$$



Solution

d = - π

d + 2p = π

p = π

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt = \frac{1}{\pi} \int_{-\pi}^{0} (-x) \, dx + \frac{1}{\pi} \int_{0}^{\pi} x \, dx = \pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi} \int_{-\pi}^{0} (-x)\cos(nx)\,dx + \int_{0}^{\pi} x \; \cos(nx)\,dx$$

Integral by partial we get $a_n = 0$ if n = even and $a_n = -\frac{4}{n^2\pi}$ if n = odd

$$b_n = \frac{1}{\pi} \int_{-\pi}^{0} (-x)\sin\,(nx)dx + \frac{1}{\pi} \int_{0}^{\pi} (x)\sin nx\,dx$$

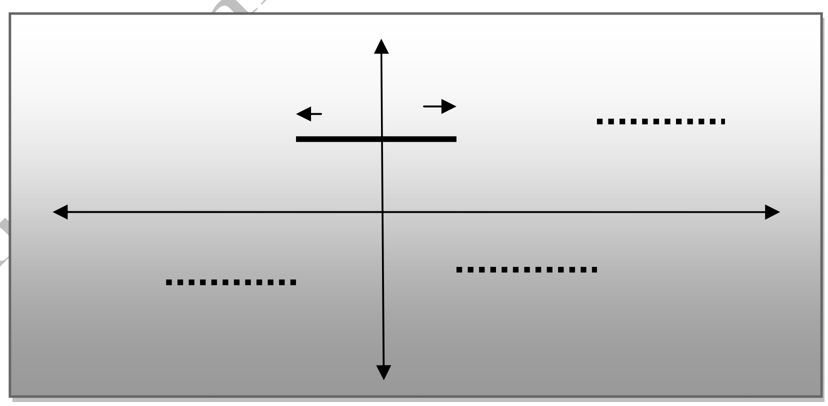
$$b_n = 0$$

Fourier series will be

$$f(x) = \frac{1}{2}\pi + \sum_{n=odd}^{\infty} \frac{\mu}{n^2\pi}\cos(nx) - \frac{x^2}{2}]_{-\pi}^{0} + \frac{\pi^2}{2} + \frac{\pi^2}{2}$$

Example

Let f (x) = x for $-\pi \le x \le \pi$



$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x\,dx = \frac{1}{\pi}\left(\frac{x^2}{2}\bigg|_{-\pi}^{\pi}\right) = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x\cos\left(\frac{n\pi x}{\pi}\right)dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x\cos(nx)dx$$

$$= [\frac{1}{n^2\pi}\cos(nx) + \frac{x}{n\pi}\sin(nx)]_{-\pi}^{\pi} = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x\cos\left(\frac{n\pi x}{\pi}\right)xd = \frac{1}{\pi} \int_{-\pi}^{\pi} x\cos(nx)dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin\left(\frac{n\pi x}{\pi}\right) xd = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin(nx)dx$$

$$= [\tfrac{1}{n^2\pi}\sin(nx) + \tfrac{x}{n\pi}\cos(nx)]_{-\pi}^{\pi}$$

$$= -\frac{2}{n}\cos(n\pi) = \frac{2}{n}(-1)^{n+1}$$

$\cos n\pi = (-1)^n$

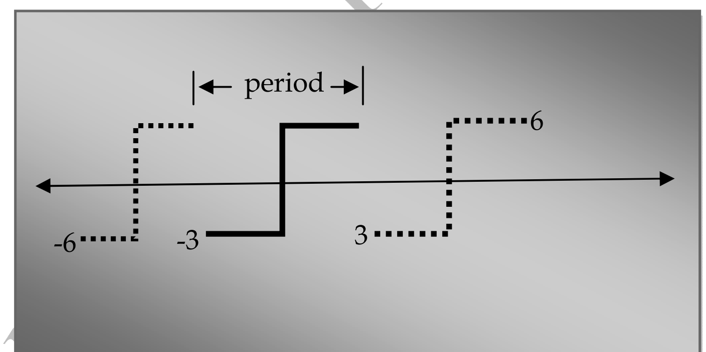
Fourier series will be

$$\sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin(nx) = 2\sin(x) - \sin(2x) + \frac{2}{3}\sin(3x) - \frac{1}{2}\sin(4x) + \dots\dots$$

Example

Let

$$f(x)\begin{cases} 0 & for \quad -3 < x < 0 \\ x & for \quad 0 < x < 3 \end{cases}$$



Solution

L = 3

$$a_0 = \frac{1}{3}\int_{-3}^{3} f(x)xd = \frac{1}{3}\int_{0}^{3} x\,dx = \frac{3}{2}$$

$$a_n = \frac{1}{3}\int_{-3}^{3} f(x)\cos\left(\frac{n\pi x}{3}\right)dx = \frac{1}{3}\int_{0}^{3} x\,cos\,\left(\frac{n\pi x}{3}\right)dx$$

$$= \frac{3}{n^2\pi^2}\cos\left(\frac{n\pi x}{3}\right) + \frac{x}{n\pi}\sin\left(\frac{n\pi x}{3}\right)]_0^3$$

$$= \frac{3}{n^2\pi^2}(-1)^n = 1$$

$$b_n = \frac{1}{3}\int_{-3}^{3} f(x)\sin\left(\frac{n\pi x}{3}\right)dx = \frac{1}{3}\int_{0}^{3} x\sin\left(\frac{n\pi x}{3}\right)dx$$

$$= \frac{3}{n^2\pi^2}\sin\left(\frac{n\pi x}{3}\right) - \frac{x}{n\pi}\cos\left(\frac{n\pi x}{3}\right)]_0^3$$

$$= \frac{3}{n\pi}(-1)^{n+1}$$

The Fourier series

$$\frac{3}{4}+\sum_{n=1}^{\infty}\frac{3}{n^2\pi^2}[\,(-1)^n-1]\cos\left(\frac{n\pi x}{3}\right)+\frac{3}{n\pi}(-1)^{n+1}\sin\left(\frac{n\pi x}{3}\right)$$

Example

Let

$f(x) = e^{-4x}\ for\ -2 \le x \le 2$ find fourier series

$$a_0=\frac{1}{2}\int_{-2}^{+2} e^{-4x}\mathrm{dx}=\frac{1}{8}(e^8-e^{-8})$$

$$a_n=\frac{1}{2}\int_{-2}^{+2} e^{-4x}\cos\left(\frac{\mathrm{n\pi x}}{2}\right)\mathrm{dx}$$

$$=(e^8-e^{-8})\frac{8(-1)^n}{64+\pi^2 n^2}$$

$$b_n=\frac{1}{2}\int_{-2}^{+2} e^{-4x}\sin\left(\frac{n\pi x}{2}\right)dx$$

$$=\frac{3}{n^2\pi^2}\sin\left(\frac{n\pi x}{3}\right)-\frac{x}{n\pi}\cos\left(\frac{n\pi x}{3}\right)\Big]_0^3$$

$$=\frac{3}{n\pi}\,(-1)^{n+1}$$

∴ The Fourier series will be

$$\frac{3}{4}+\sum_{n=1}^{\infty}\left(\frac{3}{n^2\pi^2}\,[\,(-1)^n-1]\cos\left(\frac{n\pi x}{3}\right)+\frac{3}{n\pi}(-1)^{n+1}\sin\left(\frac{n\pi x}{3}\right)\right.$$

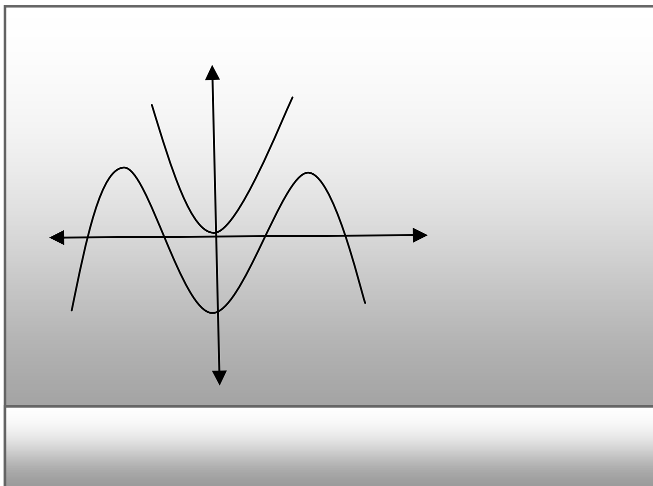
Even and odd function

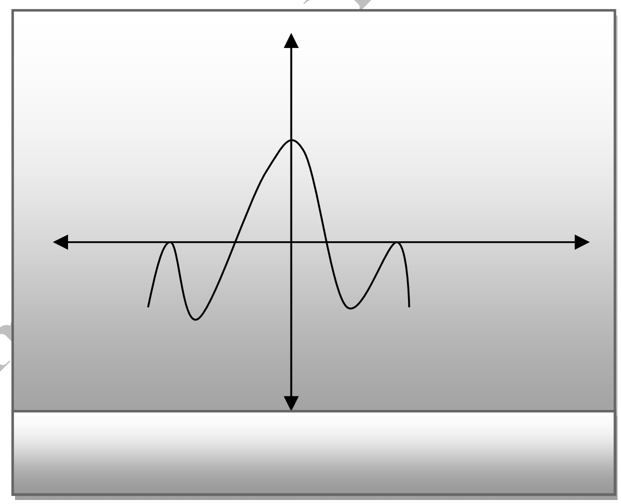
Even Function

f is an even function on [-L , L] if f(-x) = f(x) for $-L \le x \le L$

Odd Function

f is an odd function on [-L , L] if f(-x) = - f (x) $-L \le x \le L$





+4

-2 +2

-4

Fig (4) Odd function symmetric through the origin

If the function is even then

$$\int_{-\pi/2}^{\pi/2} f(x)dx = 2\int_{0}^{\pi/2} f(x)dx$$

While if is odd then

$$\int_{-\pi/2}^{\pi/2} f(x)dx = 0$$

Also even function

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\ dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx \qquad b_n = 0$$

Then the function will be

$$f(x) = \frac{a_0}{2} + \sum a_n \cos(nx)$$

$$a_0 = 0 \qquad a_n = 0$$

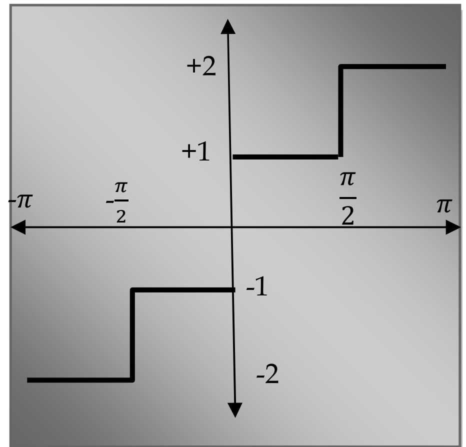
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx$$

$$f(x) = \int_0^{\infty} b_n \sin nx$$

Symmetric a bout the origin

Example

Find the fourier series of the function of the



$$f(x)\begin{cases} -2 & -\pi < x < -\frac{\pi}{2} \\ -1 & -\pi < x < 0 \\ +1 & 0 < x < \frac{\pi}{2} \\ +2 & \frac{\pi}{2} < x < \pi \end{cases}$$

Solution

Its an odd-function

$$so \quad a_0 = 0 \quad and \;\; a_n = 0$$

$$b_n = \frac{2}{\pi}\int f(x)\sin(nx)dx$$

$$b_n = \frac{2}{\pi}\Big[\int_0^{\frac{\pi}{2}} (1)\sin(nx)dx + \int_{\frac{\pi}{2}}^{\pi} 2\sin(nx)dx\,\Big]$$

$$b_n = \frac{2}{\pi}\Big[-\frac{1}{n}\cos(nx)\Big]_0^{\frac{\pi}{2}} + \Big[-\frac{2}{n}\cos(nx)\Big]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{2}{\pi n} - \frac{4}{\pi n}\cos n\pi + \frac{2}{\pi n}\cos\frac{n\pi}{2}$$

$$\therefore\; b_n = \frac{2}{\pi n}\,(1 - 2(-1)^n + \frac{2}{\pi n}\cos\frac{n\pi}{2}$$

$$\therefore f(x) = \sum_1^{\infty}\frac{2}{n\pi}\,[\,1 - 2(-1)^n + \cos\frac{n\pi}{2}]\,\sin(nx)$$

Example

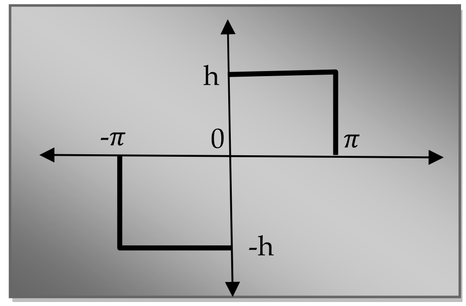
Find fourier series of the function

$$f(t)\begin{cases} -h & -\pi < t < 0 \\ h & 0 < t < \pi \end{cases}$$

The function is <u>odd</u> then

$$a_0 = 0 \;\; ; \;\; a_n = 0$$

$b_n = \frac{2}{\pi}\int_0^\pi h\sin(nt)dt \ = \frac{2}{\pi n}[-h\cos(nt)]_0^\pi$



$= \frac{2h}{\pi n}[\,-1\,(-1)^n]\begin{cases} b_n = 0 \ \ for\ eve\ value\ of\ n \\ b_n = \frac{4h}{\pi n}\ \ for\ odd\ value\ of\ n \end{cases}$

$f(x) = \sum_1^\infty \frac{4h}{n\pi}\sin(nt)$

Example

Let $f(x) = \ x^4 \ \ on\ [-1,1]$ find Fourier series

Solution

$f(x) = \ x^4$ is an even function because f(-x) = f (x) (*i.e.* example f(-3) = f (3) on so on) then $b_n = 0$

$$a_0 = \int_{-1}^{1} x^4\,dx = 2\int_{0}^{1} x^4\cos(n\pi x)dx$$

$$a_n = 2\int_{0}^{-1} x^4\cos(n\pi x)dx = \frac{8\,n^2\pi^2 - 6}{\pi^4 n^4}\ \ (-1)^n$$

∴ the Fourier series

$$\frac{1}{5} + \sum_{n=1}^{\infty} 8\frac{n^2\pi^2 - 6}{\pi^4 n^4}\ (-1)^n\cos(n\pi x)$$

Example

$$f(x) = \ x^3\ for\ -4 \le x \le 4$$

Solution

$$f(x) = \ x^3\ \ odd\ \ \therefore\ a_n = 0\,\big(f(-x) = \ -f(x)\big)$$

$$b_n = \frac{1}{4}\int_{-4}^{4} x^3\sin\left(\frac{n\pi x}{4}\right)dx$$

$$b_n = \frac{1}{2}\int_{0}^{4} x^3\sin\left(\frac{n\pi x}{4}\right)dx = (-1)^{n+1}\ \ 128\,n^2\pi^2 - 6/n^3\,\pi^3$$

The Fourier series will be

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{128\,n^2\pi^2-6}{n^3\pi^3}\,\sin\left(\frac{n\pi x}{4}\right)$$

Conclusion

Even Function

Fourier series will be

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right)$$

$$a_n=\frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})\,dx \qquad n=0\,,1\,,2$$

Odd Function

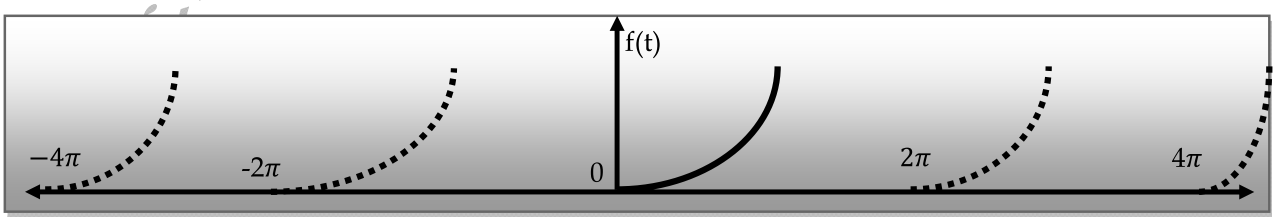
Fourier series will be

$$\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{L}\right)$$

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n=1\,,2$$

Example

Find the Fourier series if f(x) = x^2 $\quad 0 < x < 2\pi$



Neither even nor odd

Period = 2L = 2π $\quad \therefore$L = π

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2 \; dx = \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos(nx) dx$$

$$a_n = \frac{1}{\pi}[x^2 \frac{\sin nx}{n} - \; 2x \; \frac{-\cos nx}{n^2} + 2\frac{-\sin nx}{n^3}]_0^{2\pi} = \frac{4}{n^2} \qquad n \neq 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin(nx) dx$$

$$b_n = \frac{1}{\pi}\;[\, x^2(\frac{-\cos nx}{n} - \; 2x - \frac{\cos nx}{n^2} + 2 - \frac{\sin nx}{n^3}]_0^{2\pi} = \frac{4}{n^2} \qquad n \neq 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin(nx) dx$$

$$b_n = \frac{1}{\pi}\;[\, x^2\left(-\frac{\cos nx}{n}\right) - 2x\,\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)]_0^{2\pi} = -\frac{4\pi}{n}$$

The Fourier series will be

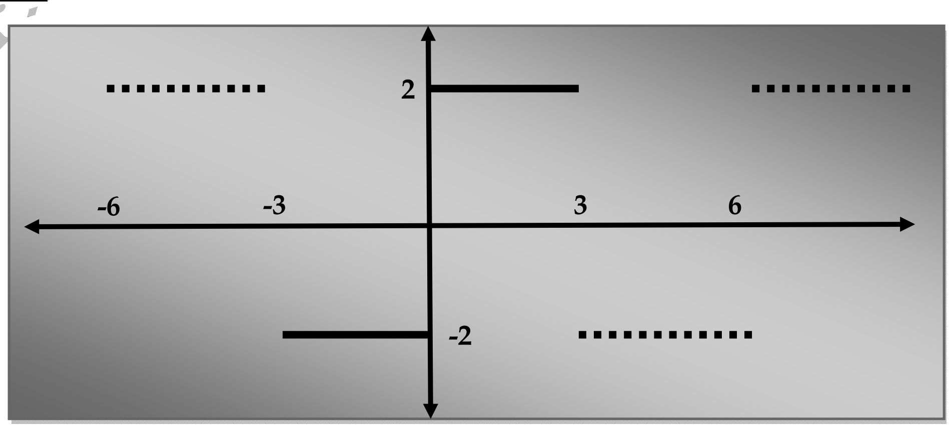
$$f(x) = \; x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}(\frac{4}{n^2}\cos 4x - \frac{4\pi}{n}\sin nx)$$

#Example

"a" odd $\longrightarrow$ f (-x) = - f (x)

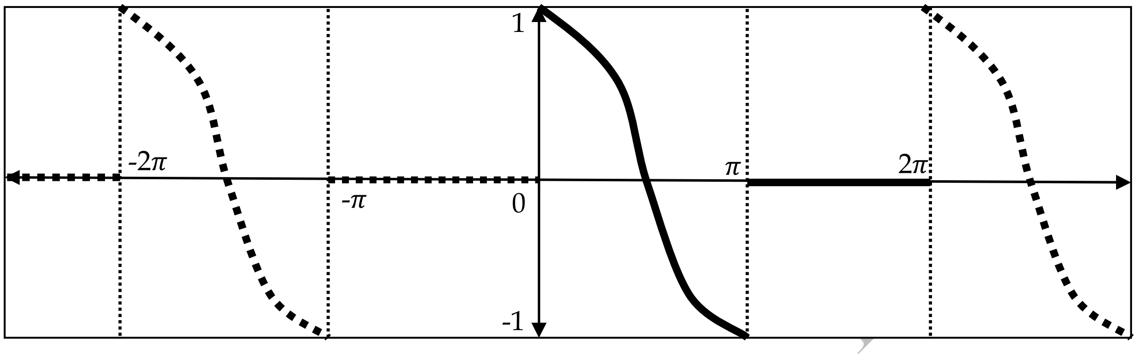
$$f(x)\begin{cases} 2 & 0 < x < 3 \\ -2 & -3 < x < 0 \end{cases}$$

$\boldsymbol{period = 6}$



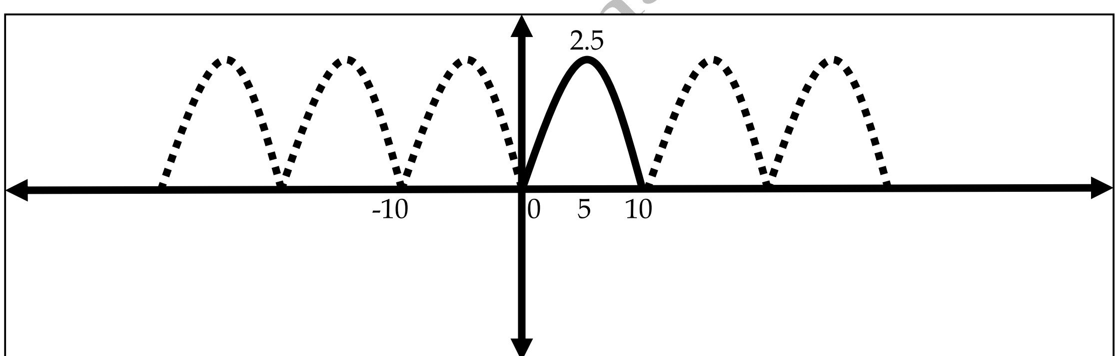
“b” Neither EVEN nor odd

$$f(x)\begin{cases} cosx & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases} \qquad period = 2\pi$$



“C” EVEN ➡ f (-x) = f (x)

$$f(x) = x(10 - x) \qquad 0 < x < 10 \qquad period = 10$$



Half Range Expansions

Half range fourier series if function f (x) is defined only in the half fourier interval (0 ➡ π) the equation of such function can be problem into other half of period (-π ➡ 0) infinite way .

a) An odd
b) An even
c) Neither odd nor even

Example

Give f(x) = x in the interval 0 ➡ π $\qquad 0 < x < \pi$

a) Find the Fourier series an a even function (cos function)
b) Find the Fourier series an odd function (sin function)

a- <u>Even function</u> $b_n = 0$

$$a_o = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\ dx \ \rightarrow\ integration\ by\ parts$$

U = x ➡ du = dx

dv = cos (nx) dx ➡ v = $\frac{1}{n}$ sin (nx)

$$\frac{2}{\pi}\ \frac{x}{n}\ (\sin(nx))_0^{\pi} - \frac{1}{n}\int_0^{\pi} \sin(nx)dx$$

$$\frac{2}{\pi}\{\ \frac{x}{n}\ \sin(nx) - \frac{x}{n}\sin 0\ \} + \ \{\frac{1}{n^2}\cos(nx)\}_0^{\pi}$$

$$\frac{2}{\pi}\ \frac{1}{n^2}\ (\cos(nx) - \cos(0)$$

$$\frac{2}{\pi n^2}((-1)^n - 1) \rightarrow for\ even\ value$$

$$\rightarrow -\frac{4}{n^2\pi}\quad for\ odd\ value$$

$$\therefore\ f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{1,2,3}^{\infty}\cos\frac{nx}{n^2}$$

b – <u>odd function</u> $a_n = 0 \qquad a_0 = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)dx \;\rightarrow integralion\ by\ parts$$

$$u = x \;\rightarrow du = dx$$

$$dv = \sin(nx)dx \;\rightarrow v = \;-\frac{1}{n}\cos(nx)$$

$$\frac{2}{\pi}[\,-\frac{x}{n}\cos(nx)]_0^{\pi} + \frac{1}{n}\int_0^{\pi} \cos(nx0\ dx$$

$$= \frac{2}{\pi}\left[-\frac{x}{n}\cos(nx) + \;0\cos(0)\right] + \frac{2}{\pi}[\frac{1}{n^2}\ sin\ (nx)]_0^{\pi}$$

$$\frac{2}{n}[\,-\pi\cos n\pi]$$

$$= -\frac{2\pi}{\pi n}(-1)^n \;= \frac{2}{n}\,for\ odd$$

$$= -\frac{2}{n}\,for\ even$$

$$or\,\frac{2}{n}\cos(n\pi)$$

$$f(x) = -2\sum_{n=1}^{\infty}\cos\frac{n\pi}{n}\;(\sin(nx))$$

<u>Example</u>

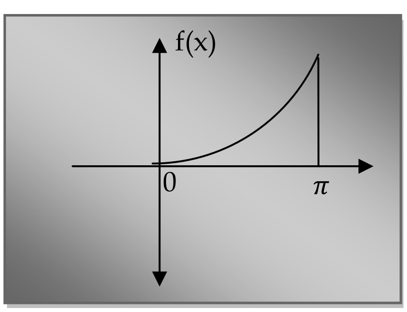
Find the sine and the cosine half range series of the function series

$$f(x) = \;x^2 \quad 0 < x < \pi$$

a. **<u>Even function</u>** $\boldsymbol{b_n = 0}$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{2}{3}\pi^2$$



$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos x\,dx \;= \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)dx$$

$$a_n = \frac{4}{n^2}\cos(nx) = \frac{4}{n^2}(-1)^4$$

$$\therefore f(x) = \frac{2}{3}\pi^2 + \frac{4}{n^2}\sum_1^{\infty}(-1)^n\cos\,(nx)$$

b. <u>Odd function</u> $\boldsymbol{a_n = 0 \quad a_0 = 0}$

<u>Solved Examples</u>

The formula for a Fourier series on an interval [c,c+T] is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2n\pi x}{T}\right) + b_n \sin\left(\frac{2n\pi x}{T}\right) \right]$$

$$a_n = \frac{2}{T} \int_c^{c+T} f(x) \cos\left(\frac{2n\pi x}{T}\right)\, dx$$

$$b_n = \frac{2}{T} \int_c^{c+T} f(x) \sin\left(\frac{2n\pi x}{T}\right)\, dx$$

<u>Example (1)</u>

Find the Fourier series for $|x|$, $-\pi < x < \pi$.

Following the rules from the link above,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x|\, dx = \frac{2}{\pi} \int_0^{\pi} x\, dx = \pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos(nx)\, dx = \frac{2}{\pi} \int_0^{\pi} x \cos(nx)\, dx$$

$$= \frac{2}{\pi} \left(\left[\frac{x \sin(nx)}{n} \right]_0^{\pi} - \int_0^{\pi} \frac{\sin(nx)}{n}\, dx \right) = \frac{2}{\pi} \left[\frac{\cos(nx)}{n^2} \right]_0^{\pi}$$

$$= \frac{2(-1)^n}{\pi n^2} - \frac{2}{\pi n^2} = \frac{2((-1)^n - 1)}{\pi n^2}.$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \sin(nx)\, dx = \frac{1}{\pi} \int_{-\pi}^{0} -x \sin(nx)\, dx + \frac{1}{\pi} \int_0^{\pi} x \sin(nx)\, dx$$

$$= \frac{1}{\pi} \left[\left[\frac{x \cos(nx)}{n} \right]_{-\pi}^{0} - \int_{-\pi}^{0} \frac{\cos(nx)}{n}\, dx \right] + \frac{1}{n} \left[\left[\frac{-x \cos(nx)}{n} \right]_0^{\pi} + \int_0^{\pi} \frac{\cos(nx)}{n}\, dx \right]$$

$$= \frac{1}{\pi}\left[\frac{\pi(-1)^n}{n} - \left[\frac{\sin(nx)}{n^2}\right]_{-\pi}^{0}\right] + \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} + \left[\frac{\sin(nx)}{n^2}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\frac{\pi(-1)^n}{n} + \frac{1}{\pi}\frac{-\pi(-1)^n}{n} = 0.$$

So,

$$|x| = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos(nx) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{-2}{(2n+1)^2}\cos\left((2n+1)x\right)$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos(3x) + \frac{1}{25}\cos(5x) + ...\right)$$

Example (2)

Find the Fourier series for . $f(x) = \begin{cases} 0 & -\pi < x < 0, \\ 1 & 0 < x < \pi \end{cases}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx))$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} 0dx + \frac{1}{\pi}\int_{0}^{\pi} 1dx = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} 0dx + \frac{1}{\pi}\int_{0}^{\pi}\cos(nx)dx = \frac{1}{\pi}\left[\frac{\sin(nx)}{n}\right]_{0}^{\pi} = 0$$

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$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} 0dx + \frac{1}{\pi}\int_{0}^{\pi}\sin(nx)dx = \left[\frac{-\cos(nx)}{\pi n}\right]_{0}^{\pi} = \frac{-(-1)^n}{\pi n} + \frac{1}{\pi n}$$

$$f(x) = \frac{1}{2} + \sum_{n=0}^{\infty}\frac{2}{\pi(2n+1)}\sin((2n+1)x)$$

$$= \frac{1}{2} + \frac{2}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + ...\right)$$

Example (3)

Find the Fourier series for $f(x) = 1 + x$ on $[-\pi, \pi]$

The general Fourier series on $[-L, L]$ is:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\ \cos(\frac{k\pi x}{L}) + b_k\ \sin(\frac{k\pi x}{L})$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\,dx,\ \ n = 0, 1, 2, ...$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})\,dx,\ \ n = 1, 2, 3, ...$$

The n = 0 case is not needed since the integrand in the formula for b_0 is $\sin(0)$.

In the present problem,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(1{+}x)\cos(\frac{n\pi x}{\pi})\,dx = \frac{1}{\pi}\left(\left[\frac{(1+x)\sin(nx)}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{(1+x)\cos(nx)}{n^2}\,dx\right.$$

But since the right hand side is not defined if n = 0, the 0 index for a will have to be calculated seperately.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x)\,dx = \frac{1}{\pi}\left[x + \frac{1}{2}x^2\right]_{-\pi}^{\pi} = 2,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(1{+}x)\sin(nx)\,dx = \frac{1}{\pi}\left(-\frac{1}{n}\left[(1+x)\cos(nx)\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos(nx)}{n}\,dx\right)$$

$$= \frac{1}{\pi}\left(-\frac{1}{n}(1+\pi-1+\pi)\cos(nx) + \left[\frac{\sin(nx)}{n^2}\right]_{-\pi}^{\pi}\right) = \frac{2(-1)^{n+1}}{n}$$

So the Fourier series is

$$f(x) = 1 + x \sim 1 + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin(nx)$$ for $[-\pi, \pi]$

Example (4) Find the Fourier series for

$$f(x) = \begin{cases} 1 & -1 \le x < 0 \\ \frac{1}{2} & x = 0 \\ x & 0 < x \le 1 \end{cases}$$ on $[-1, 1]$

The general Fourier series on $[-L, L]$ is:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(\frac{k\pi x}{L}) + b_k \sin(\frac{k\pi x}{L})$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\,dx, \;\; n = 0, 1, 2, ...$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})\,dx, \;\; n = 1, 2, 3, ...$$

In the present problem,

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx = \int_{-1}^{0} f(x)\cos(n\pi x)\,dx + \int_{0}^{1} f(x)\cos(n\pi x)\,dx$$

$$= \int_{-1}^{0} \cos(n\pi x)\,dx + \int_{0}^{1} x\cos(n\pi x)\,dx$$

$$= \left[\frac{\sin(n\pi x)}{n\pi}\right]_{-1}^{0} + \left[\frac{x\sin(n\pi x)}{n\pi} + \frac{\cos(n\pi x)}{n^2\pi^2}\right]_{0}^{1}$$

$$= \frac{\cos(n\pi) - 1}{n^2\pi^2} = \frac{(-1)^n - 1}{n^2\pi^2} \;\; n = 1, 2, 3, ...$$

$$a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx$$

$$= \int_{-1}^{0} dx + \int_{0}^{1} x\,dx = \frac{3}{2}$$

$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx = \int_{-1}^{0} f(x)\sin(n\pi x)\,dx + \int_{0}^{1} f(x)\sin(n\pi x)\,dx$$

$$= \int_{-1}^{0} \sin(n\pi x)\,dx + \int_{0}^{1} x\,\sin(n\pi x)\,dx$$

$$= \left[-\frac{\cos(n\pi x)}{n\pi}\right]_{-1}^{0} + \left[-\frac{x\cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{n^2\pi^2}\right]_{0}^{1}$$

$$= -\frac{1}{n\pi}$$

So the Fourier series is:

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi^2}\cos(n\pi x) - \frac{1}{n\pi}\sin(n\pi x)$$

Setting $x = 0$ gives $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$

Example (5) Find the Fourier series for

$f(x) = \begin{cases} -1 & -3 \le x < 0 \\ 1 & 0 < x \le 3 \end{cases}$ on $[-3, 3]$

$$a_k = \frac{1}{3}\int_{-3}^{3} f(x)\cos(\frac{k\pi x}{3})\,dx = \frac{1}{3}\int_{-3}^{0} -\cos(\frac{k\pi x}{3})\,dx + \frac{1}{3}\int_{0}^{3}\cos(\frac{k\pi x}{3})\,dx$$

$$= \frac{1}{k\pi}\left[\sin(\frac{k\pi x}{3})\right]_{-3}^{0} - \frac{1}{k\pi}\left[\sin(\frac{k\pi x}{3})\right]_{0}^{3} = 0$$

$$a_0 = \frac{1}{3}\int_{-3}^{0} -\,dx + \frac{1}{3}\int_{0}^{3} dx = -1 + 1 = 0$$

$$b_k = \frac{1}{3}\int_{-3}^{3} f(x)\sin(\frac{k\pi x}{3})\,dx = \frac{1}{3}\int_{-3}^{0} -\sin(\frac{k\pi x}{3})\,dx + \frac{1}{3}\int_{0}^{3}\sin(\frac{k\pi x}{3})\,dx$$

$$= \frac{1}{k\pi}\left[\cos(\frac{k\pi x}{3})\right]_{-3}^{0} - \frac{1}{k\pi}\left[\cos(\frac{k\pi x}{3})\right]_{0}^{3}$$

$$= \frac{1}{k\pi} - \frac{(-1)^k}{k\pi} - \frac{(-1)^k}{k\pi} + \frac{1}{k\pi} = \frac{2 - 2(-1)^k}{k\pi} \quad k = 1, 2, 3, \dots$$

$$= \frac{4}{(2k+1)\pi}, \quad k = 0, 1, 2, \dots$$

So the Fourier series is:

$$f(x) = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin\left(\frac{(2k+1)\pi x}{3}\right)$$

<u>Example (6)</u> Find the Fourier series for

$f(x) = x^2$ on $[-\pi, \pi]$

The general Fourier series on $[-L, L]$ is:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(\frac{k\pi x}{L}) + b_k \sin(\frac{k\pi x}{L})$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\,dx, \;\; n = 0, 1, 2, ...$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})\,dx, \;\; n = 1, 2, 3, ...$$

In the present problem, $L = \pi$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(kx)\,dx = \frac{4(-1)^k}{k^2} \;\; k = 1, 2, 3, ...$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{3\pi}[x^3]_{-\pi}^{\pi} = \frac{2\pi^2}{3}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin(kx)\,dx = 0 \;\; k = 1, 2, 3, ...$$

So the Fourier series is:

$x^2 = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2}\cos(kx)$ on $[-\pi, \pi]$

<u>Example (7)</u> Find the Fourier series for a function

$f(x) = f(x+2), f(x) = (x-1)(x-3)$ on $[1, 3]$.

Make the change of variables $z = x - 2$.

Now, look for the Fourier series of the function
$f(z+2) = f(z+4), f(z+2) = (z+1)(z-1)$ on $-1 \le z \le 1$

$$f(z) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\pi x) + b_k \sin(k\pi x)$$

$$a_k = \int_{-1}^{1} (z+1)(z-1)\cos(k\pi(z+2))\,dz = \frac{4(-1)^k}{k^2\pi^2}$$

$$a_0 = \int_{-1}^{1} (z+1)(z-1)dz = \frac{-4}{3}$$

$$b_k = \int_{-1}^{1} (z+1)(z-1)\sin(k\pi(z+2))\,dz = 0$$

Since $f(z) = f(x-2) = f(x)$,

$$f(x) = \frac{-2}{3} + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2\pi^2}\cos(k\pi x)$$

Example (8)

Find the Fourier series for $f(x) = x$ on $[0, 1]$.

A general formula for the Fourier series of a function on an interval $[c, c+T]$ is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{T}\right) + b_n \sin\left(\frac{2n\pi x}{T}\right)$$

$$a_n = \frac{2}{T}\int_{c}^{c+T} f(x)\cos\left(\frac{2n\pi x}{T}\right)\,dx$$

$$b_n = \frac{2}{T}\int_{c}^{c+T} f(x)\sin\left(\frac{2n\pi x}{T}\right)\,dx$$

In the current problem, $c = 0$ and $T = 1$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 2n\pi x + b_n \sin 2n\pi x$$

The function $f(x) = x$ is odd, so the cosine coefficients will all equal zero. Nevertheless, a_0 should still be calculated separately.

$$a_0 = 2\int_0^1 x\,dx = 1$$

$$b_n = 2\int_0^1 x\sin(2n\pi x)\,dx$$

$$= 2\left(\left[\frac{-x\cos 2n\pi x}{2n\pi}\right]_0^1 + \int_0^1 \frac{\cos 2n\pi x}{2n\pi}\,dx\right)$$

$$= \frac{-1}{n\pi} + \left[\frac{\sin 2n\pi x}{(2n\pi)^2}\right]_0^1 = \frac{-1}{n\pi}$$

So the Fourier series for $f(x)$ is

$$f(x) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n\pi}$$

Example (9) Find the Fourier series for

$$f(t) = \begin{cases} \frac{4}{\pi}t & 0 \le t < \frac{\pi}{2}, \\ \frac{-4}{\pi}t & \frac{-\pi}{2} \le t \le 0 \end{cases}$$

This is the general Fourier Series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(\frac{2n\pi x}{T}) + b_n \sin(\frac{2n\pi x}{T})\right]$$

$$a_n = \frac{2}{T}\int_c^{c+T} f(x)\cos(\frac{2n\pi x}{T})\,dx$$

$$b_n = \frac{2}{T}\int_c^{c+T} f(x)\sin(\frac{2n\pi x}{T})\,dx$$

So the given function can be replaced by its Fourier expansion:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2n\pi x}{\pi}\right) + b_n \sin\left(\frac{2n\pi x}{\pi}\right)\right]$$

$$a_0 = \frac{2}{\pi}\int_{\frac{-\pi}{2}}^{0} \frac{-4}{\pi}t\,dt + \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{4}{\pi}t\,dt = 2$$

$$a_n = \frac{2}{\pi}\int_{\frac{-\pi}{2}}^{0} \frac{-4}{\pi}t\cos(2nt)\,dt + \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{4}{\pi}t\cos(2nt)\,dt = \frac{4((-1)^n - 1)}{\pi^2 n^2}$$

$$b_n = \frac{2}{\pi}\int_{\frac{-\pi}{2}}^{0} \frac{-4}{\pi}t\sin(2nt)\,dt + \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{4}{\pi}t\sin(2nt)\,dt = 0$$

So the solution is

$$f(t) = 1 + \sum_{n=1}^{\infty} \left[\frac{4}{n^2\pi^2}((-1)^n - 1)\cos(2nt) \right]$$

$$= 1 + \sum_{n=1}^{\infty} \left[\frac{-4 \cdot 2}{(2n-1)^2\pi^2} \cos(2(2n-1)t) \right]$$

$$= 1 - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos[2(2n-1)t]}{(2n-1)^2}$$

Home Work

Problem (1) Find the Fourier series of the function

$$f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ \pi, & 0 \le x \le \pi \end{cases}$$

Answer.

$$f(x) \sim \frac{\pi}{2} + 2\left(\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \dots\right).$$

Problem (2) **.** Find the Fourier series of the function

$$f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ \pi, & 0 \le x \le \pi \end{cases}$$

Answer. We have
Therefore, the Fourier series of $f(x)$ is

$$f(x) \sim \frac{\pi}{2} + 2\left(\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \dots\right).$$

Problem (3) Find the Fourier series of

$$f(x) = \begin{cases} 0, & -2 \le x < 0 \\ x, & 0 \le x \le 2 \end{cases}$$

Answer.

$$f(x) \sim \frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2\pi^2}((-1)^n - 1)\cos\left(n\frac{\pi x}{2}\right) + \frac{2}{n\pi}(-1)^{n+1}\sin\left(n\frac{\pi x}{2}\right)\right].$$