University of Technology
Electromechanical Department
Energy Branch

Advanced Mathematics

Partial Differentiation

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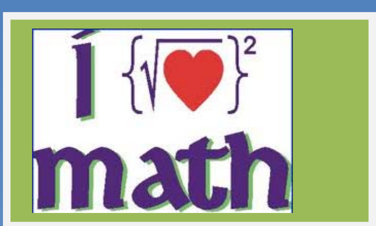


2nd Class

1st Lecture

Partial Derivatives

Recall that given a function of one variable, $f(x)$, the derivative, $f'(x)$, represents the rate of change of the function as *x* changes. This is an important interpretation of derivatives and we are not going to want to lose it with functions of more than one variable. The problem with functions of more than one variable is that there is more than one variable. In other words, what do we do if we only want one of the variables to change, or if we want more than one of them to change? In fact, if we're going to allow more than one of the variables to change there are then going to be an infinite amount of ways for them to change. For instance, one variable could be changing faster than the other variable(s) in the function. Notice as well that it will be completely possible for the function to be changing differently depending on how we allow one or more of the variables to change.

Let's start with the function $f(x,y)=2x^2y^3$ the partial derivatives from above will more commonly be written as,

$$f_x(x,y)=4xy^3 \qquad \text{and} \qquad f_y(x,y)=6x^2y^2$$

Since we can think of the two partial derivatives above as derivatives of single variable functions it shouldn't be too surprising that the definition of each is very similar to the definition of the derivative for single variable functions. Here are the formal definitions of the two partial derivatives we looked at above.

$$f_x(x,y)=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} \qquad f_y(x,y)=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

Now let's take a quick look at some of the possible alternate notations for partial derivatives. Given the function $z=f(x,y)$ the following are all equivalent notations,

$$f_x(x,y)=f_x=\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\big(f(x,y)\big)=z_x=\frac{\partial z}{\partial x}=D_xf$$

$$f_y(x,y)=f_y=\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\big(f(x,y)\big)=z_y=\frac{\partial z}{\partial y}=D_yf$$

For the fractional notation for the partial derivative notice the difference between the partial derivative and the ordinary derivative from single variable calculus.

$$f(x) \qquad \Rightarrow \qquad f'(x)=\frac{df}{dx}$$

$$f(x,y) \qquad \Rightarrow \qquad f_x(x,y)=\frac{\partial f}{\partial x} \ \ \& \ \ f_y(x,y)=\frac{\partial f}{\partial y}$$

Example 1 Find all of the first order partial derivatives for the following functions.

(a) $f(x,y)=x^4+6\sqrt{y}-10$

(b) $w=x^2y-10y^2z^3+43x-7\tan(4y)$

(c) $h(s,t)=t^7\ln(s^2)+\dfrac{9}{t^3}-\sqrt[7]{s^4}$

(d) $f(x,y)=\cos\left(\dfrac{4}{x}\right)\mathbf{e}^{x^2y-5y^3}$

Solution

a) $f(x,y) = x^4 + 6\sqrt{y} - 10$

$$f_x(x,y) = 4x^3 \qquad f_y(x,y) = \frac{3}{\sqrt{y}}$$

(b) $w = x^2 y - 10y^2 z^3 + 43x - 7\tan(4y)$

$$\frac{\partial w}{\partial x} = 2xy + 43 \qquad \frac{\partial w}{\partial y} = x^2 - 20yz^3 - 28\sec^2(4y) \qquad \frac{\partial w}{\partial z} = -30y^2 z^2$$

(c) $h(s,t) = t^7 \ln(s^2) + \frac{9}{t^3} - \sqrt[7]{s^4} \quad = \quad h(s,t) = t^7 \ln(s^2) + 9t^{-3} - s^{\frac{4}{7}}$

$$h_s(s,t) = \frac{\partial h}{\partial s} = t^7\left(\frac{2s}{s^2}\right) - \frac{4}{7}s^{-\frac{3}{7}} = \frac{2t^7}{s} - \frac{4}{7}s^{-\frac{3}{7}}$$

$$h_t(s,t) = \frac{\partial h}{\partial t} = 7t^6 \ln(s^2) - 27t^{-4}$$

Remember how to differentiate natural logarithms.

$$\frac{d}{dx}\left(\ln g(x)\right) = \frac{g'(x)}{g(x)}$$

(d) $f(x,y) = \cos\left(\frac{4}{x}\right)\mathbf{e}^{x^2 y - 5y^3}$

$$f_x(x,y) = -\sin\left(\frac{4}{x}\right)\left(-\frac{4}{x^2}\right)\mathbf{e}^{x^2 y - 5y^3} + \cos\left(\frac{4}{x}\right)\mathbf{e}^{x^2 y - 5y^3}(2xy)$$

$$= \frac{4}{x^2}\sin\left(\frac{4}{x}\right)\mathbf{e}^{x^2 y - 5y^3} + 2xy\cos\left(\frac{4}{x}\right)\mathbf{e}^{x^2 y - 5y^3}$$

Also, don't forget how to differentiate exponential functions,

$$\frac{d}{dx}\left(\mathbf{e}^{f(x)}\right) = f'(x)\mathbf{e}^{f(x)}$$

$$f_y(x,y) = \left(x^2 - 15y^2\right)\cos\left(\frac{4}{x}\right)\mathbf{e}^{x^2 y - 5y^3}$$

Example 2 Find all of the first order partial derivatives for the following functions.

(a) $z = \frac{9u}{u^2 + 5v}$

(b) $g(x,y,z) = \frac{x\sin(y)}{z^2}$

(c) $z = \sqrt{x^2 + \ln(5x - 3y^2)}$

Solution

(a) $z = \frac{9u}{u^2 + 5v}$

$$z_u = \frac{9(u^2 + 5v) - 9u(2u)}{(u^2 + 5v)^2} = \frac{-9u^2 + 45v}{(u^2 + 5v)^2}$$

$$z_v = \frac{(0)(u^2 + 5v) - 9u(5)}{(u^2 + 5v)^2} = \frac{-45u}{(u^2 + 5v)^2}$$

(b) $g(x,y,z)=\dfrac{x\sin(y)}{z^{2}}$

$g_x(x,y,z)=\dfrac{\sin(y)}{z^2}$ $\qquad$ $g_y(x,y,z)=\dfrac{x\cos(y)}{z^2}$

$$g(x,y,z)=x\sin(y)z^{-2}$$

$$g_z(x,y,z)=-2x\sin(y)z^{-3}=-\frac{2x\sin(y)}{z^3}$$

(c) $z=\sqrt{x^2+\ln\left(5x-3y^2\right)}$

$$z_x=\frac{1}{2}\left(x^2+\ln\left(5x-3y^2\right)\right)^{-\frac{1}{2}}\frac{\partial}{\partial x}\left(x^2+\ln\left(5x-3y^2\right)\right)$$

$$=\frac{1}{2}\left(x^2+\ln\left(5x-3y^2\right)\right)^{-\frac{1}{2}}\left(2x+\frac{5}{5x-3y^2}\right)$$

$$=\left(x+\frac{5}{2\left(5x-3y^2\right)}\right)\left(x^2+\ln\left(5x-3y^2\right)\right)^{-\frac{1}{2}}$$

$$z_y=\frac{1}{2}\left(x^2+\ln\left(5x-3y^2\right)\right)^{-\frac{1}{2}}\frac{\partial}{\partial y}\left(x^2+\ln\left(5x-3y^2\right)\right)$$

$$=\frac{1}{2}\left(x^2+\ln\left(5x-3y^2\right)\right)^{-\frac{1}{2}}\left(\frac{-6y}{5x-3y^2}\right)$$

$$=-\frac{3y}{5x-3y^2}\left(x^2+\ln\left(5x-3y^2\right)\right)^{-\frac{1}{2}}$$

Example 3 Find $\dfrac{dy}{dx}$ for $3y^4+x^7=5x$.

Solution

The first step is to differentiate both sides with respect to *x*. $\quad 12y^3\dfrac{dy}{dx}+7x^6=5$

$$\frac{dy}{dx}=\frac{5-7x^6}{12y^3}$$

Example 4 Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for each of the following functions.

(a) $x^3z^2-5xy^5z=x^2+y^3$

(b) $x^2\sin(2y-5z)=1+y\cos(6zx)$

Solution

(a) $x^3z^2-5xy^5z=x^2+y^3$

Let's start with finding $\dfrac{\partial z}{\partial x}$. We first will differentiate both sides with respect to *x* and remember to add on a $\dfrac{\partial z}{\partial x}$ whenever we differentiate a *z*.

$$3x^2z^2+2x^3z\frac{\partial z}{\partial x}-5y^5z-5xy^5\frac{\partial z}{\partial x}=2x$$

Remember that since we are assuming $z=z(x,y)$ then any product of *x*'s and *z*'s will be a product and so will need the product rule!

Now, solve for $\dfrac{\partial z}{\partial x}$.

$$\left(2x^3z - 5xy^5\right)\frac{\partial z}{\partial x} = 2x - 3x^2z^2 + 5y^5z$$

$$\frac{\partial z}{\partial x} = \frac{2x - 3x^2z^2 + 5y^5z}{2x^3z - 5xy^5}$$

w we'll do the same thing for $\frac{\partial z}{\partial y}$ except this time we'll need to remember to add on a $\frac{\partial z}{\partial y}$ henever we differentiate a z.

$$2x^3z\frac{\partial z}{\partial y} - 25xy^4z - 5xy^5\frac{\partial z}{\partial y} = 3y^2$$

$$\left(2x^3z - 5xy^5\right)\frac{\partial z}{\partial y} = 3y^2 + 25xy^4z$$

$$\frac{\partial z}{\partial y} = \frac{3y^2 + 25xy^4z}{2x^3z - 5xy^5}$$

(b) $x^2\sin\left(2y - 5z\right) = 1 + y\cos\left(6zx\right)$

We'll do the same thing for this function as we did in the previous part. First let's find $\frac{\partial z}{\partial x}$.

$$2x\sin\left(2y-5z\right) + x^2\cos\left(2y-5z\right)\left(-5\frac{\partial z}{\partial x}\right) = -y\sin\left(6zx\right)\left(6z + 6x\frac{\partial z}{\partial x}\right)$$

Don't forget to do the chain rule on each of the trig functions and when we are differentiating the inside function on the cosine we will need to also use the product rule. Now let's solve for $\frac{\partial z}{\partial x}$.

$$2x\sin\left(2y-5z\right) - 5\frac{\partial z}{\partial x}x^2\cos\left(2y-5z\right) = -6zy\sin\left(6zx\right) - 6yx\sin\left(6zx\right)\frac{\partial z}{\partial x}$$

$$2x\sin\left(2y-5z\right) + 6zy\sin\left(6zx\right) = \left(5x^2\cos\left(2y-5z\right) - 6yx\sin\left(6zx\right)\right)\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{2x\sin\left(2y-5z\right) + 6zy\sin\left(6zx\right)}{5x^2\cos\left(2y-5z\right) - 6yx\sin\left(6zx\right)}$$

Now let's take care of $\frac{\partial z}{\partial y}$. This one will be slightly easier than the first one.

$$x^2\cos\left(2y-5z\right)\left(2 - 5\frac{\partial z}{\partial y}\right) = \cos\left(6zx\right) - y\sin\left(6zx\right)\left(6x\frac{\partial z}{\partial y}\right)$$

$$2x^2\cos\left(2y-5z\right) - 5x^2\cos\left(2y-5z\right)\frac{\partial z}{\partial y} = \cos\left(6zx\right) - 6xy\sin\left(6zx\right)\frac{\partial z}{\partial y}$$

$$\left(6xy\sin\left(6zx\right) - 5x^2\cos\left(2y-5z\right)\right)\frac{\partial z}{\partial y} = \cos\left(6zx\right) - 2x^2\cos\left(2y-5z\right)$$

$$\frac{\partial z}{\partial y} = \frac{\cos\left(6zx\right) - 2x^2\cos\left(2y-5z\right)}{6xy\sin\left(6zx\right) - 5x^2\cos\left(2y-5z\right)}$$

Interpretations of Partial Derivatives

This is a fairly short section and is here so we can acknowledge that the two main interpretations of derivatives of functions of a single variable still hold for partial derivatives, with small modifications of course to account of the fact that we now have more than one variable.

The first interpretation we've already seen and is the more important of the two. As with functions of single variables partial derivatives represent the rates of change of the functions as the variables change. As we saw in the previous section, $f_x(x, y)$ represents the rate of change of the function $f(x, y)$ as we change x and hold y fixed while $f_y(x, y)$ represents the rate of change of $f(x, y)$ as we change y and hold x fixed.

Example 1 Determine if $f(x, y) = \dfrac{x^2}{y^3}$ is increasing or decreasing at $(2,5)$,

> **(a)** if we allow x to vary and hold y fixed.
> **(b)** if we allow y to vary and hold x fixed.

Solution
(a) If we allow *x* to vary and hold *y* fixed.

In this case we will first need $f_x(x, y)$ and its value at the point.

$$f_x(x, y) = \frac{2x}{y^3} \qquad \Rightarrow \qquad f_x(2,5) = \frac{4}{125} > 0$$

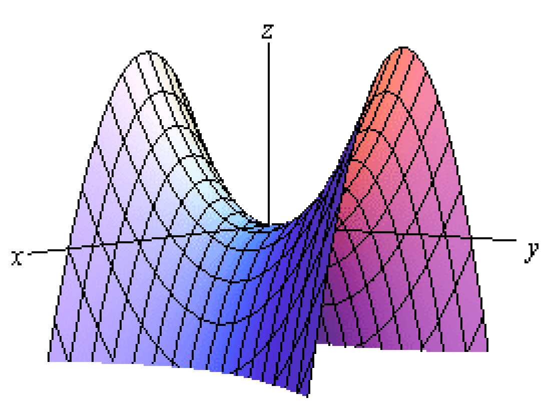
So, the partial derivative with respect to x is positive and so if we hold y fixed the function is increasing at $(2,5)$ as we vary x.

(b) If we allow *y* to vary and hold *x* fixed.

For this part we will need $f_y(x, y)$ and its value at the point.

$$f_y(x, y) = -\frac{3x^2}{y^4} \qquad \Rightarrow \qquad f_y(2,5) = -\frac{12}{625} < 0$$

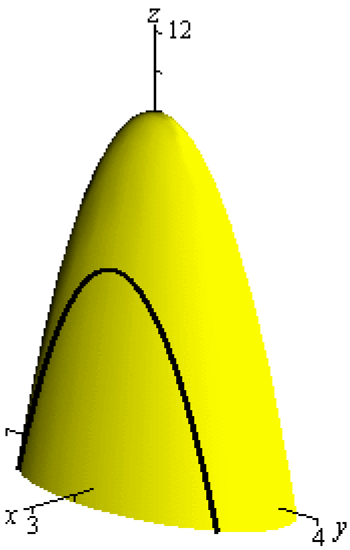
Here the partial derivative with respect to y is negative and so the function is decreasing at $(2,5)$ as we vary y and hold x fixed



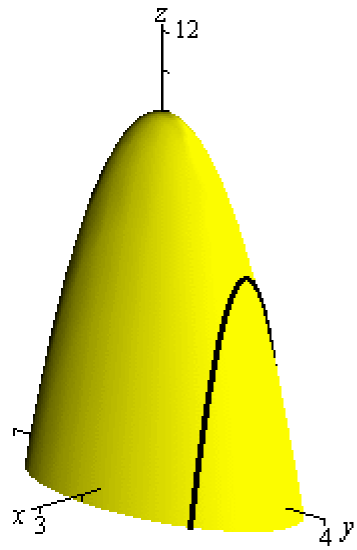
Example 2 Find the slopes of the traces to $z = 10 - 4x^2 - y^2$ at the point $(1, 2)$.

Solution

We sketched the traces for the planes $x = 1$ and $y = 2$ in a previous section and these are the two traces for this point. For reference purposes here are the graphs of the traces.



Trace for $x = 1$



Trace for $y = 2$

Next we'll need the two partial derivatives so we can get the slopes.

$$f_x(x, y) = -8x \qquad\qquad f_y(x, y) = -2y$$

To get the slopes all we need to do is evaluate the partial derivatives at the point in question.

$$f_x(1, 2) = -8 \qquad\qquad f_y(1, 2) = -4$$

So, the tangent line at $(1, 2)$ for the trace to $z = 10 - 4x^2 - y^2$ for the plane $y = 2$ has a slope of -8. Also the tangent line at $(1, 2)$ for the trace to $z = 10 - 4x^2 - y^2$ for the plane $x = 1$ has a slope of -4.

Higher Order Partial Derivatives

Just as we had higher order derivatives with functions of one variable we will also have higher order derivatives of functions of more than one variable. However, this time we will have more options since we do have more than one variable..

Consider the case of a function of two variables, $f(x,y)$ since both of the first order partial derivatives are also functions of *x* and *y* we could in turn differentiate each with respect to *x* or *y*. This means that for the case of a function of two variables there will be a total of four possible second order derivatives. Here they are and the notations that we'll use to denote them.

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Example 1 Find all the second order derivatives for $f(x,y) = \cos(2x) - x^2\mathbf{e}^{5y} + 3y^2$

Solution

We'll first need the first order derivatives so here they are.

$$f_x(x,y) = -2\sin(2x) - 2x\mathbf{e}^{5y}$$

$$f_y(x,y) = -5x^2\mathbf{e}^{5y} + 6y$$

Now, let's get the second order derivatives.

$$f_{xx} = -4\cos(2x) - 2\mathbf{e}^{5y}$$

$$f_{xy} = -10x\mathbf{e}^{5y}$$

$$f_{yx} = -10x\mathbf{e}^{5y}$$

$$f_{yy} = -25x^2\mathbf{e}^{5y} + 6$$

Clairaut's Theorem

Suppose that *f* is defined on a disk *D* that contains the point (a,b). If the functions f_{xy} and f_{yx} are continuous on this disk then,

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Example 2 Verify Clairaut's Theorem for $f(x,y) = x\mathbf{e}^{-x^2y^2}$.

Solution

We'll first need the two first order derivatives.

$$f_x(x,y) = \mathbf{e}^{-x^2y^2} - 2x^2y^2\mathbf{e}^{-x^2y^2}$$

$$f_y(x,y) = -2yx^3\mathbf{e}^{-x^2y^2}$$

Now, compute the two fixed second order partial derivatives.

$$f_{xy}(x,y) = -2yx^2\mathbf{e}^{-x^2y^2} - 4x^2y\mathbf{e}^{-x^2y^2} + 4x^4y^3\mathbf{e}^{-x^2y^2} = -6x^2y\mathbf{e}^{-x^2y^2} + 4x^4y^3\mathbf{e}^{-x^2y^2}$$

$$f_{yx}(x,y) = -6yx^2\mathbf{e}^{-x^2y^2} + 4y^3x^4\mathbf{e}^{-x^2y^2}$$

Sure enough they are the same

$$f_{xyx} = \left(f_{xy}\right)_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial x \partial y \partial x}$$

$$f_{yxx} = \left(f_{yx}\right)_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial^3 f}{\partial x^2 \partial y}$$

an extension to Clairaut's Theorem that says if all three of these are continuous then they should all be equal,

$$f_{xxy} = f_{xyx} = f_{yxx}$$

Example 3 Find the indicated derivative for each of the following functions.

(a) Find f_{xxyzz} for $f(x,y,z) = z^3y^2\ln(x)$

(b) Find $\dfrac{\partial^3 f}{\partial y \partial x^2}$ for $f(x,y) = \mathbf{e}^{xy}$

Solution

(a) Find f_{xxyzz} for $f(x,y,z) = z^3y^2\ln(x)$

In this case remember that we differentiate from left to right. Here are the derivatives for this part.

$$f_x = \frac{z^3y^2}{x}$$

$$f_{xx} = -\frac{z^3y^2}{x^2}$$

$$f_{xxy} = -\frac{2z^3y}{x^2}$$

$$f_{xxyz} = -\frac{6z^2y}{x^2}$$

$$f_{xxyzz} = -\frac{12zy}{x^2}$$

(b) Find $\dfrac{\partial^3 f}{\partial y \partial x^2}$ for $f(x,y) = \mathbf{e}^{xy}$

Here we differentiate from right to left. Here are the derivatives for this function.

$$\frac{\partial f}{\partial x} = y\mathbf{e}^{xy}$$

$$\frac{\partial^2 f}{\partial x^2} = y^2\mathbf{e}^{xy}$$

$$\frac{\partial^3 f}{\partial y \partial x^2} = 2y\mathbf{e}^{xy} + xy^2\mathbf{e}^{xy}$$

Differentials

This is a very short section and is here simply to acknowledge that just like we had differentials for functions of one variable we also have them for functions of more than one variable. Also, as we've already seen in previous sections, when we move up to more than one variable things work pretty much the same, but there are some small differences.

Given the function $z = f(x,y)$ the differential dz or df is given by,

$$dz = f_x\,dx + f_y\,dy \qquad \text{or} \qquad df = f_x\,dx + f_y\,dy$$

There is a natural extension to functions of three or more variables. For instance, given the function $w = g(x,y,z)$ the differential is given by,

$$dw = g_x\,dx + g_y\,dy + g_z\,dz$$

Let's do a couple of quick examples.

Example 1 Compute the differentials for each of the following functions.

(a) $z = \mathbf{e}^{x^2+y^2}\tan(2x)$

(b) $u = \dfrac{t^3 r^6}{s^2}$

Solution

(a) $z = \mathbf{e}^{x^2+y^2}\tan(2x)$

There really isn't a whole lot to these outside of some quick differentiation. Here is the differential for the function.

$$dz = \left(2x\mathbf{e}^{x^2+y^2}\tan(2x) + 2\mathbf{e}^{x^2+y^2}\sec^2(2x)\right)dx + 2y\mathbf{e}^{x^2+y^2}\tan(2x)\,dy$$

(b) $u = \dfrac{t^3 r^6}{s^2}$

Here is the differential for this function.

$$du = \frac{3t^2 r^6}{s^2}dt + \frac{6t^3 r^5}{s^2}dr - \frac{2t^3 r^6}{s^3}ds$$

Chain Rule

We've been using the standard chain rule for functions of one variable throughout the last couple of sections. It's now time to extend the chain rule out to more complicated situations. Before we actually do that let's first review the notation for the chain rule for functions of one variable.

The notation that's probably familiar to most people is the following.

$$F(x) = f(g(x)) \qquad\qquad F'(x) = f'(g(x))\,g'(x)$$

we are going to be using in this section. Here it is.

$$\text{If} \qquad y = f(x) \qquad \text{and} \qquad x = g(t) \qquad \text{then} \qquad \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

<u>Case 1 :</u> $z = f(x,y)$, $x = g(t)$, $y = h(t)$ and compute $\dfrac{dz}{dt}$.

This case is analogous to the standard chain rule from Calculus I that we looked at above. In this case we are going to compute an ordinary derivative since *z* really would be a function of *t* only if we were to substitute in for *x* and *y*.

The chain rule for this case is,

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example 1 Compute $\dfrac{dz}{dt}$ for each of the following.

(a) $z = x\mathbf{e}^{xy}$, $x = t^2$, $y = t^{-1}$

(b) $z = x^2 y^3 + y\cos x$, $x = \ln(t^2)$, $y = \sin(4t)$

Solution

(a) $z = x\mathbf{e}^{xy}$, $x = t^2$, $y = t^{-1}$

There really isn't all that much to do here other than using the formula.

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\\ &= \left(\mathbf{e}^{xy} + yx\mathbf{e}^{xy}\right)(2t) + x^2\mathbf{e}^{xy}\left(-t^{-2}\right)\\ &= 2t\left(\mathbf{e}^{xy} + yx\mathbf{e}^{xy}\right) - t^{-2}x^2\mathbf{e}^{xy}\end{aligned}$$

So, technically we've computed the derivative. However, we should probably go ahead and substitute in for *x* and *y* as well at this point since we've already got *t*'s in the derivative. Doing this gives,

$$\frac{dz}{dt} = 2t\left(\mathbf{e}^{t} + t\mathbf{e}^{t}\right) - t^{-2}t^4\mathbf{e}^{t} = 2t\mathbf{e}^{t} + t^2\mathbf{e}^{t}$$

$$z = t^2\mathbf{e}^t \qquad \Rightarrow \qquad \frac{dz}{dt} = 2t\mathbf{e}^t + t^2\mathbf{e}^t$$

The same result for less work. Note however, that often it will actually be more work to do the substitution first.

(b) $z = x^2y^3 + y\cos x$, $x = \ln\left(t^2\right)$, $y = \sin\left(4t\right)$

Okay, in this case it would almost definitely be more work to do the substitution first so we'll use the chain rule first and then substitute.

$$\begin{aligned}\frac{dz}{dt} &= \left(2xy^3 - y\sin x\right)\left(\frac{2}{t}\right) + \left(3x^2y^2 + \cos x\right)\left(4\cos\left(4t\right)\right)\\ &= \frac{4\sin^3\left(4t\right)\ln t^2 - 2\sin\left(4t\right)\sin\left(\ln t^2\right)}{t} + 4\cos\left(4t\right)\left(3\sin^2\left(4t\right)\left[\ln t^2\right]^2 + \cos\left(\ln t^2\right)\right)\end{aligned}$$

Now, there is a special case that we should take a quick look at before moving on to the next case. Let's suppose that we have the following situation,

$$z = f\left(x,y\right) \qquad y = g\left(x\right)$$

In this case the chain rule for $\frac{dz}{dx}$ becomes,

$$\frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

In the first term we are using the fact that,

$$\frac{dx}{dx} = \frac{d}{dx}\left(x\right) = 1$$

Example 2 Compute $\frac{dz}{dx}$ for $z = x\ln\left(xy\right) + y^3$, $y = \cos\left(x^2+1\right)$

Solution

We'll just plug into the formula.

$$\begin{aligned}\frac{dz}{dx} &= \left(\ln\left(xy\right) + x\frac{y}{xy}\right) + \left(x\frac{x}{xy} + 3y^2\right)\left(-2x\sin\left(x^2+1\right)\right)\\ &= \ln\left(x\cos\left(x^2+1\right)\right) + 1 - 2x\sin\left(x^2+1\right)\left(\frac{x}{\cos\left(x^2+1\right)} + 3\cos^2\left(x^2+1\right)\right)\\ &= \ln\left(x\cos\left(x^2+1\right)\right) + 1 - 2x^2\tan\left(x^2+1\right) - 6x\sin\left(x^2+1\right)\cos^2\left(x^2+1\right)\end{aligned}$$

Now let's take a look at the second case.

<u>**Case 2 :**</u> $z = f\left(x,y\right)$, $x = g\left(s,t\right)$, $y = h\left(s,t\right)$ and compute $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

In this case if we were to substitute in for x and y we would get that z is a function of s and t and so it makes sense that we would be computing partial derivatives here and that there would be two of them.

Here is the chain rule for both of these cases.

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \qquad\qquad \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Example 3 Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ for $z = \mathbf{e}^{2r}\sin(3\theta)$, $r = st - t^2$, $\theta = \sqrt{s^2+t^2}$.

Solution

Here is the chain rule for $\dfrac{\partial z}{\partial s}$.

$$\begin{aligned}\frac{\partial z}{\partial s} &= \left(2\mathbf{e}^{2r}\sin(3\theta)\right)(t) + \left(3\mathbf{e}^{2r}\cos(3\theta)\right)\frac{s}{\sqrt{s^2+t^2}} \\ &= t\left(2\mathbf{e}^{2(st-t^2)}\sin\left(3\sqrt{s^2+t^2}\right)\right) + \frac{3s\mathbf{e}^{2(st-t^2)}\cos\left(3\sqrt{s^2+t^2}\right)}{\sqrt{s^2+t^2}}\end{aligned}$$

Now the chain rule for $\dfrac{\partial z}{\partial t}$.

$$\begin{aligned}\frac{\partial z}{\partial t} &= \left(2\mathbf{e}^{2r}\sin(3\theta)\right)(s-2t) + \left(3\mathbf{e}^{2r}\cos(3\theta)\right)\frac{t}{\sqrt{s^2+t^2}} \\ &= (s-2t)\left(2\mathbf{e}^{2(st-t^2)}\sin\left(3\sqrt{s^2+t^2}\right)\right) + \frac{3t\mathbf{e}^{2(st-t^2)}\cos\left(3\sqrt{s^2+t^2}\right)}{\sqrt{s^2+t^2}}\end{aligned}$$

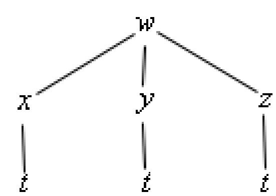
Example 4 Use a tree diagram to write down the chain rule for the given derivatives.

(a) $\dfrac{dw}{dt}$ for $w = f(x,y,z)$, $x = g_1(t)$, $y = g_2(t)$, and $z = g_3(t)$

(b) $\dfrac{\partial w}{\partial r}$ for $w = f(x,y,z)$, $x = g_1(s,t,r)$, $y = g_2(s,t,r)$, and $z = g_3(s,t,r)$

Solution

(a) $\dfrac{dw}{dt}$ for $w = f(x,y,z)$, $x = g_1(t)$, $y = g_2(t)$, and $z = g_3(t)$



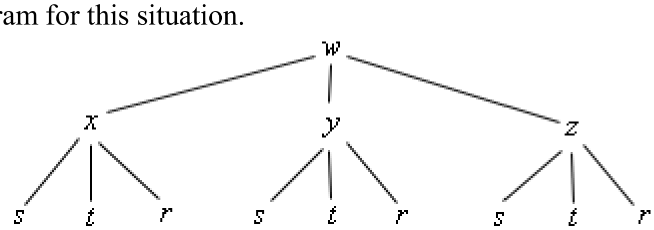
So, we'll first need the tree diagram so let's get that.

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

which is really just a natural extension to the two variable case that we saw above.

(b) $\dfrac{\partial w}{\partial r}$ for $w = f(x,y,z)$, $x = g_1(s,t,r)$, $y = g_2(s,t,r)$, and $z = g_3(s,t,r)$

Here is the tree diagram for this situation.



From this it looks like the derivative will be, $\dfrac{\partial w}{\partial r} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial r}$

Example 5 Compute $\frac{\partial^2 f}{\partial \theta^2}$ for $f(x,y)$ if $x = r\cos\theta$ and $y = r\sin\theta$.

Solution

We will need the first derivative before we can even think about finding the second derivative so let's get that. This situation falls into the second case that we looked at above so we don't need a new tree diagram. Here is the first derivative.

$$\begin{aligned}\frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} \\ &= -r\sin(\theta)\frac{\partial f}{\partial x} + r\cos(\theta)\frac{\partial f}{\partial y}\end{aligned}$$

Okay, now we know that the second derivative is,

$$\frac{\partial^2 f}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial \theta}\right) = \frac{\partial}{\partial \theta}\left(-r\sin(\theta)\frac{\partial f}{\partial x} + r\cos(\theta)\frac{\partial f}{\partial y}\right)$$

The issue here is to correctly deal with this derivative. Since the two first order derivatives, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, are both functions of *x* and *y* which are in turn functions of *r* and θ both of these terms are products. So, the using the product rule gives the following,

$$\frac{\partial^2 f}{\partial \theta^2} = -r\cos(\theta)\frac{\partial f}{\partial x} - r\sin(\theta)\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right) - r\sin(\theta)\frac{\partial f}{\partial y} + r\cos(\theta)\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial y}\right)$$

We now need to determine what $\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right)$ and $\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial y}\right)$ will be. These are both chain rule problems again since both of the derivatives are functions of *x* and *y* and we want to take the derivative with respect to θ.

Before we do these let's rewrite the first chain rule that we did above a little.

$$\frac{\partial}{\partial \theta}(f) = -r\sin(\theta)\frac{\partial}{\partial x}(f) + r\cos(\theta)\frac{\partial}{\partial y}(f) \qquad (1)$$

Note that all we've done is change the notation for the derivative a little. With the first chain rule written in this way we can think of (1) as a formula for differentiating any function of *x* and *y* with respect to θ provided we have $x = r\cos\theta$ and $y = r\sin\theta$.

This however is exactly what we need to do the two new derivatives we need above. Both of the first order partial derivatives, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, are functions of *x* and *y* and $x = r\cos\theta$ and $y = r\sin\theta$ so we can use (1) to compute these derivatives.

Here is the use of (1) to compute $\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right)$.

$$\begin{aligned}\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right) &= -r\sin(\theta)\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + r\cos(\theta)\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \\ &= -r\sin(\theta)\frac{\partial^2 f}{\partial x^2} + r\cos(\theta)\frac{\partial^2 f}{\partial y \partial x}\end{aligned}$$

Here is the computation for $\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial y}\right)$.

$$\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial y}\right)=-r\sin(\theta)\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)+r\cos(\theta)\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$
$$=-r\sin(\theta)\frac{\partial^2 f}{\partial x\partial y}+r\cos(\theta)\frac{\partial^2 f}{\partial y^2}$$

The final step is to plug these back into the second derivative and do some simplifying.

$$\frac{\partial^2 f}{\partial\theta^2}=-r\cos(\theta)\frac{\partial f}{\partial x}-r\sin(\theta)\left(-r\sin(\theta)\frac{\partial^2 f}{\partial x^2}+r\cos(\theta)\frac{\partial^2 f}{\partial y\partial x}\right)-$$
$$r\sin(\theta)\frac{\partial f}{\partial y}+r\cos(\theta)\left(-r\sin(\theta)\frac{\partial^2 f}{\partial x\partial y}+r\cos(\theta)\frac{\partial^2 f}{\partial y^2}\right)$$
$$=-r\cos(\theta)\frac{\partial f}{\partial x}+r^2\sin^2(\theta)\frac{\partial^2 f}{\partial x^2}-r^2\sin(\theta)\cos(\theta)\frac{\partial^2 f}{\partial y\partial x}-$$
$$r\sin(\theta)\frac{\partial f}{\partial y}-r^2\sin(\theta)\cos(\theta)\frac{\partial^2 f}{\partial x\partial y}+r^2\cos^2(\theta)\frac{\partial^2 f}{\partial y^2}$$
$$=-r\cos(\theta)\frac{\partial f}{\partial x}-r\sin(\theta)\frac{\partial f}{\partial y}+r^2\sin^2(\theta)\frac{\partial^2 f}{\partial x^2}-$$
$$2r^2\sin(\theta)\cos(\theta)\frac{\partial^2 f}{\partial y\partial x}+r^2\cos^2(\theta)\frac{\partial^2 f}{\partial y^2}$$

implicit differentiation

$$F_x+F_y\frac{dy}{dx}=0\qquad\Rightarrow\qquad\frac{dy}{dx}=-\frac{F_x}{F_y}$$

Example 6 Find $\frac{dy}{dx}$ for $x\cos(3y)+x^3y^5=3x-\mathbf{e}^{xy}$.

Solution

The first step is to get a zero on one side of the equal sign and that's easy enough to do.

$$x\cos(3y)+x^3y^5-3x+\mathbf{e}^{xy}=0$$

Now, the function on the left is $F(x,y)$ in our formula so all we need to do is use the formula to find the derivative.

$$\frac{dy}{dx}=-\frac{\cos(3y)+3x^2y^5-3+y\mathbf{e}^{xy}}{-3x\sin(3y)+5x^3y^4+x\mathbf{e}^{xy}}$$

Example 7 Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $x^2\sin(2y-5z)=1+y\cos(6zx)$.

Solution

This was one of the functions that we used the old implicit differentiation on back in the Partial Derivatives section. You might want to go back and see the difference between the two.

First let's get everything on one side.

$$x^2\sin(2y-5z)-1-y\cos(6zx)=0$$

Now, the function on the left is $F(x,y,z)$ and so all that we need to do is use the formulas developed above to find the derivatives.

$$\frac{\partial z}{\partial x}=-\frac{2x\sin(2y-5z)+6yz\sin(6zx)}{-5x^2\cos(2y-5z)+6yx\sin(6zx)}$$
$$\frac{\partial z}{\partial y}=-\frac{2x^2\cos(2y-5z)-\cos(6zx)}{-5x^2\cos(2y-5z)+6yx\sin(6zx)}$$

Directional Derivatives

To this point we've only looked at the two partial derivatives $f_x(x,y)$ and $f_y(x,y)$. Recall that these derivatives represent the rate of change of *f* as we vary *x* (holding *y* fixed) and as we vary *y* (holding *x* fixed) respectively. We now need to discuss how to find the rate of change of *f* if we allow both *x* and *y* to change simultaneously. The problem here is that there are many ways to allow both *x* and *y* to change. For instance one could be changing faster than the other and then there is also the issue of whether or not each is increasing or decreasing. So, before we get into finding the rate of change we need to get a couple of preliminary ideas taken care of first. The main idea that we need to look at is just how are we going to define the changing of *x* and/or *y*.

Definition

The rate of change of $f(x,y)$ in the direction of the unit vector $\vec{u} = \langle a,b \rangle$ is called the **directional derivative** and is denoted by $D_{\vec{u}} f(x,y)$. The definition of the directional derivative is,

$$D_{\vec{u}} f(x,y) = \lim_{h\to 0} \frac{f(x+ah, y+bh) - f(x,y)}{h}$$

If we now go back to allowing *x* and *y* to be any number we get the following formula for computing directional derivatives.

$$D_{\vec{u}} f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b$$

$$D_{\vec{u}} f(x,y,z) = f_x(x,y,z)\,a + f_y(x,y,z)\,b + f_z(x,y,z)\,c$$

Example 1 Find each of the directional derivatives.

(a) $D_{\vec{u}} f(2,0)$ where $f(x,y) = x\mathbf{e}^{xy} + y$ and $\vec{u}$ is the unit vector in the direction of $\theta = \dfrac{2\pi}{3}$.

(b) $D_{\vec{u}} f(x,y,z)$ where $f(x,y,z) = x^2 z + y^3 z^2 - xyz$ in the direction of $\vec{v} = \langle -1,0,3 \rangle$.

Solution

(a) $D_{\vec{u}} f(2,0)$ **where** $f(x,y) = x\mathbf{e}^{xy} + y$ **and** $\vec{u}$ **is the unit vector in the direction of** $\theta = \dfrac{2\pi}{3}$**.**

We'll first find $D_{\vec{u}} f(x,y)$ and then use this a formula for finding $D_{\vec{u}} f(2,0)$. The unit vector giving the direction is,

$$\vec{u} = \left\langle \cos\left(\frac{2\pi}{3}\right), \sin\left(\frac{2\pi}{3}\right) \right\rangle = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

So, the directional derivative is,

$$D_{\vec{u}} f(x,y) = \left(-\frac{1}{2}\right)\left(\mathbf{e}^{xy} + xy\mathbf{e}^{xy}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(x^2\mathbf{e}^{xy} + 1\right)$$

Now, plugging in the point in question gives,

$$D_{\vec{u}} f(2,0) = \left(-\frac{1}{2}\right)(1) + \left(\frac{\sqrt{3}}{2}\right)(5) = \frac{5\sqrt{3}-1}{2}$$

(b) $D_{\vec{u}} f(x,y,z)$ **where** $f(x,y,z) = x^2 z + y^3 z^2 - xyz$ **in the direction of** $\vec{v} = \langle -1,0,3 \rangle$**.**

In this case let's first check to see if the direction vector is a unit vector or not and if it isn't convert it into one. To do this all we need to do is compute its magnitude.

$$\|\vec{v}\| = \sqrt{1+0+9} = \sqrt{10} \neq 1$$

So, it's not a unit vector. Recall that we can convert any vector into a unit vector that points in the same direction by dividing the vector by its magnitude. So, the unit vector that we need is,

$$\vec{u} = \frac{1}{\sqrt{10}}\langle -1,0,3 \rangle = \left\langle -\frac{1}{\sqrt{10}}, 0, \frac{3}{\sqrt{10}} \right\rangle$$

The directional derivative is then,

$$\begin{aligned} D_{\vec{u}} f(x,y,z) &= \left(-\frac{1}{\sqrt{10}}\right)(2xz - yz) + (0)(3y^2z^2 - xz) + \left(\frac{3}{\sqrt{10}}\right)(x^2 + 2y^3 z - xy) \\ &= \frac{1}{\sqrt{10}}(3x^2 + 6y^3 z - 3xy - 2xz + yz) \end{aligned}$$

$$\begin{aligned} D_{\vec{u}} f(x,y,z) &= f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)c \\ &= \langle f_x, f_y, f_z \rangle \bullet \langle a,b,c \rangle \end{aligned}$$

Now let's give a name and notation to the first vector in the dot product since this vector will show up fairly regularly throughout this course (and in other courses). The **gradient of *f*** or **gradient vector of *f*** is defined to be,

$$\nabla f = \langle f_x, f_y, f_z \rangle \qquad \text{or} \qquad \nabla f = \langle f_x, f_y \rangle$$

Or, if we want to use the standard basis vectors the gradient is,

$$\nabla f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k} \qquad \text{or} \qquad \nabla f = f_x \vec{i} + f_y \vec{j}$$

The definition is only shown for functions of two or three variables, however there is a natural extension to functions of any number of variables that we'd like.

With the definition of the gradient we can now say that the directional derivative is given by,

$$D_{\vec{u}} f = \nabla f \bullet \vec{u}$$

where we will no longer show the variable and use this formula for any number of variables. Note as well that we will sometimes use the following notation,

$$D_{\vec{u}} f(\vec{x}) = \nabla f \bullet \vec{u}$$

Example 2 Find each of the directional derivative.

(a) $D_{\vec{u}} f(\vec{x})$ for $f(x,y) = x\cos(y)$ in the direction of $\vec{v} = \langle 2,1 \rangle$.

(b) $D_{\vec{u}} f(\vec{x})$ for $f(x,y,z) = \sin(yz) + \ln(x^2)$ at $(1,1,\pi)$ in the direction of $\vec{v} = \langle 1,1,-1 \rangle$.

Solution

(a) $D_{\vec{u}} f(\vec{x})$ **for** $f(x,y) = x\cos(y)$ **in the direction of** $\vec{v} = \langle 2,1 \rangle$.

Let's first compute the gradient for this function.

$$\nabla f = \langle \cos(y), -x\sin(y) \rangle$$

Also, as we saw earlier in this section the unit vector for this direction is,

$$\vec{u} = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

The directional derivative is then,

$$D_{\vec{u}} f(\vec{x}) = \langle \cos(y), -x\sin(y) \rangle \bullet \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$
$$= \frac{1}{\sqrt{5}}\left(2\cos(y) - x\sin(y)\right)$$

(b) $D_{\vec{u}} f(\vec{x})$ **for** $f(x,y,z) = \sin(yz) + \ln(x^2)$ **at** $(1,1,\pi)$ **in the direction of** $\vec{v} = \langle 1,1,-1 \rangle$.

In this case are asking for the directional derivative at a particular point. To do this we will first compute the gradient, evaluate it at the point in question and then do the dot product. So, let's get the gradient.

$$\nabla f(x,y,z) = \left\langle \frac{2}{x}, z\cos(yz), y\cos(yz) \right\rangle$$
$$\nabla f(1,1,\pi) = \left\langle \frac{2}{1}, \pi\cos(\pi), \cos(\pi) \right\rangle = \langle 2, -\pi, -1 \rangle$$

Next, we need the unit vector for the direction,

$$\|\vec{v}\| = \sqrt{3} \qquad \vec{u} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle$$

Finally, the directional derivative at the point in question is,

$$D_{\vec{u}} f(1,1,\pi) = \langle 2,-\pi,-1 \rangle \bullet \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle = \frac{1}{\sqrt{3}}(2 - \pi + 1) = \frac{3-\pi}{\sqrt{3}}$$

Theorem

The maximum value of $D_{\vec{u}} f(\vec{x})$ (and hence then the maximum rate of change of the function $f(\vec{x})$) is given by $\|\nabla f(\vec{x})\|$ and will occur in the direction given by $\nabla f(\vec{x})$.

Example 3 Suppose that the height of a hill above sea level is given by $z = 1000 - 0.01x^2 - 0.02y^2$. If you are at the point $(60,100)$ in what direction is the elevation changing fastest? What is the maximum rate of change of the elevation at this point?

Solution

Now on to the problem. There are a couple of questions to answer here, but using the theorem makes answering them very simple. We'll first need the gradient vector.

$$\nabla f(\vec{x}) = \langle -0.02x, -0.04y \rangle$$

The maximum rate of change of the elevation will then occur in the direction of

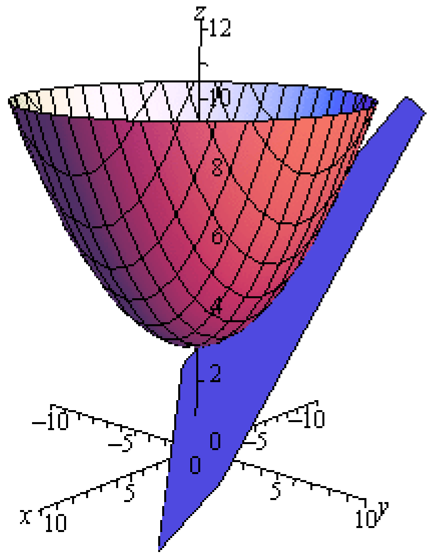
$$\nabla f(60,100) = \langle -1.2, -4 \rangle$$

The maximum rate of change of the elevation at this point is,

$$\|\nabla f(60,100)\| = \sqrt{(-1.2)^2 + (4)^2} = \sqrt{17.44} = 4.176$$

Applications of Partial Derivatives

Tangent Planes and Linear Approximations



The equation of the tangent plane to the surface given by $z = f(x, y)$ at (x_0, y_0) is then,

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Also, if we use the fact that $z_0 = f(x_0, y_0)$ we can rewrite the equation of the tangent plane as,

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example 1 Find the equation of the tangent plane to $z = \ln\ 2x + y$ at $-1, 3$.

Solution

$$f(x, y) = \ln(2x + y) \qquad z_0 = f(-1, 3) = \ln(1) = 0$$

$$f_x(x, y) = \frac{2}{2x + y} \qquad f_x(-1, 3) = 2$$

$$f_y(x, y) = \frac{1}{2x + y} \qquad f_y(-1, 3) = 1$$

The equation of the plane is then,

$$z - 0 = 2(x + 1) + (1)(y - 3)$$

$$z = 2x + y - 1$$

linear approximation to be,

One nice use of tangent planes is they give us a way to approximate a surface near a point. As long as we are near to the point (x_0, y_0) then the tangent plane should nearly approximate the function at that point. Because of this we define the **linear approximation** to be,

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

and as long as we are "near" (x_0, y_0) then we should have that,

$$f(x, y) \approx L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example 2 Find the linear approximation to $z = 3 + \frac{x^2}{16} + \frac{y^2}{9}$ at $(-4,3)$.

Solution
So, we're really asking for the tangent plane so let's find that.

$$f(x,y) = 3 + \frac{x^2}{16} + \frac{y^2}{9} \qquad f(-4,3) = 3+1+1 = 5$$

$$f_x(x,y) = \frac{x}{8} \qquad f_x(-4,3) = -\frac{1}{2}$$

$$f_y(x,y) = \frac{2y}{9} \qquad f_y(-4,3) = \frac{2}{3}$$

The tangent plane, or linear approximation, is then,

$$L(x,y) = 5 - \frac{1}{2}(x+4) + \frac{2}{3}(y-3)$$

Gradient Vector, Tangent Planes and Normal Lines

The equation of the tangent plane is then,

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0$$

Or, upon solving for *z*, we get,

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

the equation of the normal line is,

$$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t\nabla f(x_0,y_0,z_0)$$

Example 1 Find the tangent plane and normal line to $x^2 + y^2 + z^2 = 30$ at the point $(1,-2,5)$.

Solution
For this case the function that we're going to be working with is,

$$F(x,y,z) = x^2 + y^2 + z^2$$

and note that we don't have to have a zero on one side of the equal sign. All that we need is a constant. To finish this problem out we simply need the gradient evaluated at the point.

$$\nabla F = \langle 2x, 2y, 2z \rangle$$

$$\nabla F(1,-2,5) = \langle 2,-4,10 \rangle$$

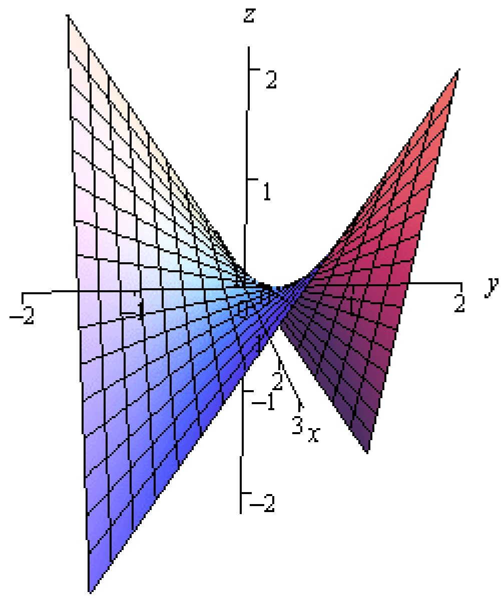
The tangent plane is then,

$$2(x-1) - 4(y+2) + 10(z-5) = 0$$

The normal line is,

$$\vec{r}(t) = \langle 1,-2,5 \rangle + t\langle 2,-4,10 \rangle = \langle 1+2t, -2-4t, 5+10t \rangle$$

Relative Minimums and Maximums



Definition

1. A function $f(x,y)$ has a **relative minimum** at the point (a,b) if $f(x,y) \ge f(a,b)$ for all points (x,y) in some region around (a,b).

2. A function $f(x,y)$ has a **relative maximum** at the point (a,b) if $f(x,y) \le f(a,b)$ for all points (x,y) in some region around (a,b).

Definition

The point (a,b) is a **critical point** (or a **stationary point**) of $f(x,y)$ provided one of the following is true,

1. $\nabla f(a,b) = \vec{0}$ (this is equivalent to saying that $f_x(a,b) = 0$ and $f_y(a,b) = 0$),
2. $f_x(a,b)$ and/or $f_y(a,b)$ doesn't exist.

Fact

If the point (a,b) is a relative extrema of the function $f(x,y)$ then (a,b) is also a critical point of $f(x,y)$ and in fact we'll have $\nabla f(a,b) = \vec{0}$.

Fact

Suppose that (a,b) is a critical point of $f(x,y)$ and that the second order partial derivatives are continuous in some region that contains (a,b). Next define,

$$D = D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - \left[f_{xy}(a,b) \right]^2$$

We then have the following classifications of the critical point.

1. If $D > 0$ and $f_{xx}(a,b) > 0$ then there is a relative minimum at (a,b).
2. If $D > 0$ and $f_{xx}(a,b) < 0$ then there is a relative maximum at (a,b).
3. If $D < 0$ then the point (a,b) is a saddle point.
4. If $D = 0$ then the point (a,b) may be a relative minimum, relative maximum or a saddle point. Other techniques would need to be used to classify the critical point.

Example 1 Find and classify all the critical points of $f(x,y) = 4 + x^3 + y^3 - 3xy$.

Solution
We first need all the first order (to find the critical points) and second order (to classify the critical points) partial derivatives so let's get those.

$$f_x = 3x^2 - 3y \qquad f_y = 3y^2 - 3x$$
$$f_{xx} = 6x \qquad f_{yy} = 6y \qquad f_{xy} = -3$$

Let's first find the critical points. Critical points will be solutions to the system of equations,

$$f_x = 3x^2 - 3y = 0$$
$$f_y = 3y^2 - 3x = 0$$

This is a non-linear system of equations and these can, on occasion, be difficult to solve. However, in this case it's not too bad. We can solve the first equation for *y* as follows,

$$3x^2 - 3y = 0 \qquad \Rightarrow \qquad y = x^2$$

Plugging this into the second equation gives,

$$3\left(x^2\right)^2 - 3x = 3x\left(x^3 - 1\right) = 0$$

From this we can see that we must have $x = 0$ or $x = 1$. Now use the fact that $y = x^2$ to get the critical points.

$$x = 0: \quad y = 0^2 = 0 \qquad \Rightarrow \qquad (0,0)$$
$$x = 1: \quad y = 1^2 = 1 \qquad \Rightarrow \qquad (1,1)$$

So, we get two critical points. All we need to do now is classify them. To do this we will need *D*. Here is the general formula for *D*.

$$\begin{aligned} D(x,y) &= f_{xx}(x,y)\, f_{yy}(x,y) - \left[f_{xy}(x,y)\right]^2 \\ &= (6x)(6y) - (-3)^2 \\ &= 36xy - 9 \end{aligned}$$

To classify the critical points all that we need to do is plug in the critical points and use the fact above to classify them.

$(0,0)$:

$$D = D(0,0) = -9 < 0$$

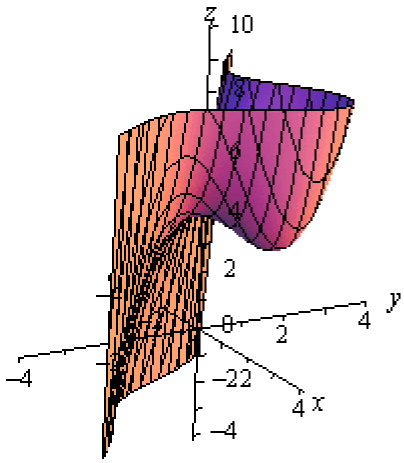
So, for $(0,0)$ *D* is negative and so this must be a saddle point.

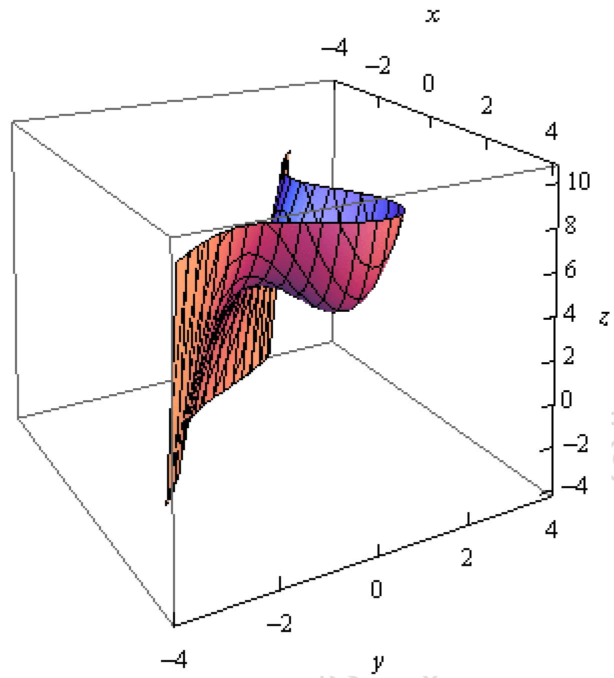
$(1,1)$:

$$D = D(1,1) = 36 - 9 = 27 > 0 \qquad f_{xx}(1,1) = 6 > 0$$

For $(1,1)$ *D* is positive and f_{xx} is positive and so we must have a relative minimum.

For the sake of completeness here is a graph of this function.





Notice that in order to get a better visual we used a somewhat nonstandard orientation. We can see that there is a relative minimum at $(1,1)$ and (hopefully) it's clear that at $(0,0)$ we do get a saddle point.

Example 2 Find and classify all the critical points for $f(x,y) = 3x^2 y + y^3 - 3x^2 - 3y^2 + 2$

Solution
As with the first example we will first need to get all the first and second order derivatives.

$$f_x = 6xy - 6x \qquad f_y = 3x^2 + 3y^2 - 6y$$
$$f_{xx} = 6y - 6 \qquad f_{yy} = 6y - 6 \qquad f_{xy} = 6x$$

We'll first need the critical points. The equations that we'll need to solve this time are,

$$6xy - 6x = 0$$
$$3x^2 + 3y^2 - 6y = 0$$

These equations are a little trickier to solve than the first set, but once you see what to do they really aren't terribly bad.

First, let's notice that we can factor out a $6x$ from the first equation to get,

$$6x(y-1) = 0$$

So, we can see that the first equation will be zero if $x = 0$ or $y = 1$. Be careful to not just cancel the x from both sides. If we had done that we would have missed $x = 0$.

To find the critical points we can plug these (individually) into the second equation and solve for the remaining variable

$x = 0$:

$$3y \ -6y = 3y(y-2) = 0 \qquad \Rightarrow \qquad y = 0,\ y = 2$$

$y = 1$:

$$3x^2 - 3 = 3(x^2 - 1) = 0 \qquad \Rightarrow \qquad x = -1,\ x = 1$$

So, if $x = 0$ we have the following critical points,

$$(0,0) \quad (0,2)$$

and if $y = 1$ the critical points are,

$$(1,1) \quad (-1,1)$$

$$D = D(0,0) = 36 > 0 \qquad f_{xx}(0,0) = -6 < 0$$

$(0,2)$:

$$D = D(0,2) = 36 > 0 \qquad f_{xx}(0,2) = 6 > 0$$

$(1,1)$:

$$D = D(1,1) = -36 < 0$$

$(-1,1)$:

$$D = D(-1,1) = -36 < 0$$

So, it looks like we have the following classification of each of these critical points.

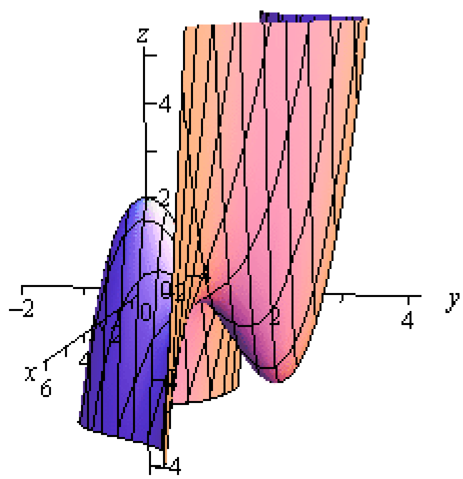
$(0,0)$: Relative Maximum

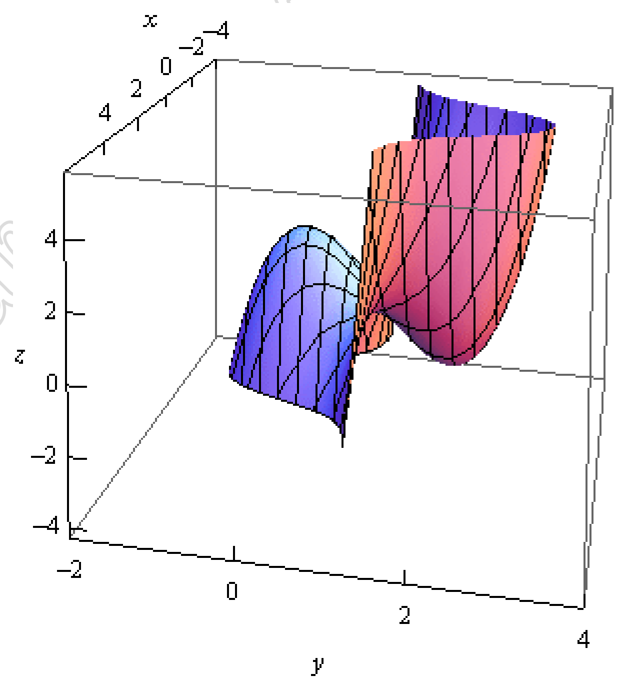
$(0,2)$: Relative Minimum

$(1,1)$: Saddle Point

$(-1,1)$: Saddle Point

Here is a graph of the surface for the sake of completeness.





Example 3 Determine the point on the plane $4x - 2y + z = 1$ that is closest to the point $(-2,-1,5)$.

Solution

First, let's suppose that (x, y, z) is any point on the plane. The distance between this point and the point in question, $(-2,-1,5)$, is given by the formula,

$$d = \sqrt{(x+2)^2 + (y+1)^2 + (z-5)^2}$$

equation of the plane to see that,

$$z = 1 - 4x + 2y$$

Plugging this into the distance formula gives,

$$d = \sqrt{(x+2)^2 + (y+1)^2 + (1 - 4x + 2y - 5)^2}$$
$$= \sqrt{(x+2)^2 + (y+1)^2 + (-4 - 4x + 2y)^2}$$

So, let's instead find the minimum value of

$$f(x,y)=d^2=(x+2)^2+(y+1)^2+(-4-4x+2y)^2$$

that is closest to $(-2,-1,5)$.

We'll need the derivatives first.

$$f_x=2(x+2)+2(-4)(-4-4x+2y)=36+34x-16y$$
$$f_y=2(y+1)+2(2)(-4-4x+2y)=-14-16x+10y$$
$$f_{xx}=34$$
$$f_{yy}=10$$
$$f_{xy}=-16$$

Now, before we get into finding the critical point let's compute D quickly.

$$D=34(10)-(-16)^2=84>0$$

So, in this case D will always be positive and also notice that $f_{xx}=34>0$ is always positive and so any critical points that we get will be guaranteed to be relative minimums.

Now let's find the critical point(s). This will mean solving the system.

$$36+34x-16y=0$$
$$-14-16x+10y=0$$

To do this we can solve the first equation for x.

$$x=\frac{1}{34}(16y-36)=\frac{1}{17}(8y-18)$$

Now, plug this into the second equation and solve for y.

$$-14-\frac{16}{17}(8y-18)+10y=0 \qquad \Rightarrow \qquad y=-\frac{25}{21}$$

Back substituting this into the equation for x gives $x=-\frac{34}{21}$.

So, it looks like we get a single critical point : $\left(-\frac{34}{21},-\frac{25}{21}\right)$.

$$z=1-4\left(-\frac{34}{21}\right)+2\left(-\frac{25}{21}\right)=\frac{107}{21}$$

So, the point on the plane that is closest to $(-2,-1,5)$ is $\left(-\frac{34}{21},-\frac{25}{21},\frac{107}{21}\right)$.

Absolute Minimums and Maximums

In this section we are going to extend the work from the previous section. In the previous section we were asked to find and classify all critical points as relative minimums, relative maximums and/or saddle points. In this section we want to optimize a function, that is identify the absolute minimum and/or the absolute maximum of the function, on a given region in $\mathbb{R}^2$. Note that when we say we are going to be working on a region in $\mathbb{R}^2$ we mean that we're going to be looking at some region in the *xy*-plane.

In order to optimize a function in a region we are going to need to get a couple of definitions out of the way and a fact. Let's first get the definitions out of the way.

Definitions

1. A region in $\mathbb{R}^2$ is called **closed** if it includes its boundary. A region is called **open** if it doesn't include any of its boundary points.
2. A region in $\mathbb{R}^2$ is called **bounded** if it can be completely contained in a disk. In other words, a region will be bounded if it is finite.

Let's think a little more about the definition of closed. We said a region is closed if it includes its boundary. Just what does this mean? Let's think of a rectangle. Below are two definitions of a rectangle, one is closed and the other is open.

Open	Closed
$-5 < x < 3$	$-5 \le x \le 3$
$1 < y < 6$	$1 \le y \le 6$

In this first case we don't allow the ranges to include the endpoints (*i.e.* we aren't including the edges of the rectangle) and so we aren't allowing the region to include any points on the edge of the rectangle. In other words, we aren't allowing the region to include its boundary and so it's open.

In the second case we are allowing the region to contain points on the edges and so will contain its entire boundary and hence will be closed.

This is an important idea because of the following fact.

Extreme Value Theorem

If $f(x, y)$ is continuous in some closed, bounded set *D* in $\mathbb{R}^2$ then there are points in *D*, (x_1, y_1) and (x_2, y_2) so that $f(x_1, y_1)$ is the absolute maximum and $f(x_2, y_2)$ is the absolute minimum of the function in *D*.

Note that this theorem does NOT tell us where the absolute minimum or absolute maximum will occur. It only tells us that they will exist. Note as well that the absolute minimum and/or absolute maximum may occur in the interior of the region or it may occur on the boundary of the region.

Finding Absolute Extrema

1. Find all the critical points of the function that lie in the region D and determine the function value at each of these points.

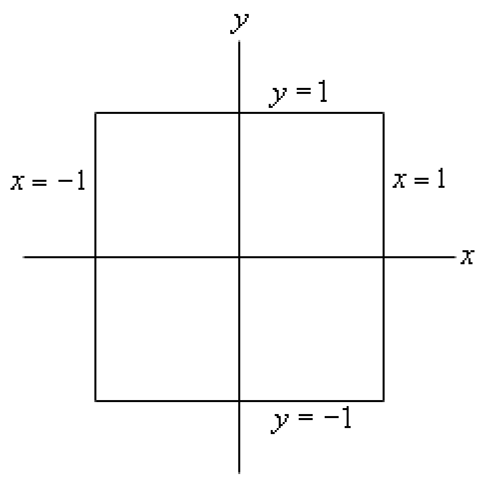
2. Find all extrema of the function on the boundary.

3. The largest and smallest values found in the first two steps are the absolute minimum and the absolute maximum of the function.

The main difference between this process and the process that we used in Calculus I is that the "boundary" in Calculus I was just two points and so there really wasn't a lot to do in the second step. For these problems the majority of the work is often in the second step as we will often end up doing a Calculus I absolute extrema problem one or more times.

Example 1 Find the absolute minimum and absolute maximum of $f(x,y) = x^2 + 4y^2 - 2x^2y + 4$ on the rectangle given by $-1 \le x \le 1$ and $-1 \le y \le 1$.

Solution
Let's first get a quick picture of the rectangle for reference purposes.



The boundary of this rectangle is given by the following conditions

$$\begin{aligned}
&\text{right side}: && x = 1,\ -1 \le y \le 1\\
&\text{left side}: && x = -1,\ -1 \le y \le 1\\
&\text{upper side}: && y = 1,\ -1 \le x \le 1\\
&\text{lower side}: && y = -1,\ -1 \le x \le 1
\end{aligned}$$

These will be important in the second step of our process.

We'll start this off by finding all the critical points that lie inside the given rectangle. To do this we'll need the two first order derivatives.

$$f_x = 2x - 4xy \qquad\qquad f_y = 8y - 2x^2$$

Note that since we aren't going to be classifying the critical points we don't need the second order derivatives. To find the critical points we will need to solve the system,

$$\begin{aligned} 2x - 4xy &= 0\\ 8y - 2x^2 &= 0 \end{aligned}$$

We can solve the second equation for y to get,

$$y = \frac{x^2}{4}$$

Plugging this into the first equation gives us,

$$2x-4x\left(\frac{x^2}{4}\right)=2x-x^3=x\left(2-x^2\right)=0$$

This tells us that we must have $x=0$ or $x=\pm\sqrt{2}=\pm1.414\ldots$. Now, recall that we only want critical points in the region that we're given. That means that we only want critical points for which $-1\le x\le 1$. The only value of x that will satisfy this is the first one so we can ignore the last two for this problem. Note however that a simple change to the boundary would include these two so don't forget to always check if the critical points are in the region (or on the boundary since that can also happen).

Plugging $x=0$ into the equation for y gives us,

$$y=\frac{0^2}{4}=0$$

The single critical point, in the region (and again, that's important), is $\left(0,0\right)$. We now need to get the value of the function at the critical point.

$$f\left(0,0\right)=4$$

Eventually we will compare this to values of the function found in the next step and take the largest and smallest as the absolute extrema of the function in the rectangle.

Now we have reached the long part of this problem. We need to find the absolute extrema of the function along the boundary of the rectangle. What this means is that we're going to need to look at what the function is doing along each of the sides of the rectangle listed above.

Let's first take a look at the right side. As noted above the right side is defined by

$$x=1,\ -1\le y\le 1$$

Notice that along the right side we know that $x=1$. Let's take advantage of this by defining a new function as follows,

$$g\left(y\right)=f\left(1,y\right)=1^2+4y^2-2\left(1^2\right)y+4=5+4y^2-2y$$

Now, finding the absolute extrema of $f\left(x,y\right)$ along the right side will be equivalent to finding the absolute extrema of $g\left(y\right)$ in the range $-1\le y\le 1$. Hopefully you recall how to do this from Calculus I. We find the critical points of $g\left(y\right)$ in the range $-1\le y\le 1$ and then evaluate $g\left(y\right)$ at the critical points and the end points of the range of y's.

Let's do that for this problem.

$$g'\left(y\right)=8y-2 \qquad\Rightarrow\qquad y=\frac{1}{4}$$

This is in the range and so we will need the following function evaluations.

$$g\left(-1\right)=11 \qquad g\left(1\right)=7 \qquad g\left(\frac{1}{4}\right)=\frac{19}{4}=4.75$$

Notice that, using the definition of $g\left(y\right)$ these are also function values for $f\left(x,y\right)$.

$$g(-1) = f(1,-1) = 11$$
$$g(1) = f(1,1) = 7$$
$$g\left(\frac{1}{4}\right) = f\left(1, \frac{1}{4}\right) = \frac{19}{4} = 4.75$$

We can now do the left side of the rectangle which is defined by,

$$x = -1, \ -1 \le y \le 1$$

Again, we'll define a new function as follows,

$$g(y) = f(-1, y) = (-1)^2 + 4y^2 - 2(-1)^2 y + 4 = 5 + 4y^2 - 2y$$

Notice however that, for this boundary, this is the same function as we looked at for the right side. This will not always happen, but since it has let's take advantage of the fact that we've already done the work for this function. We know that the critical point is $y = \frac{1}{4}$ and we know that the function value at the critical point and the end points are,

$$g(-1) = 11 \qquad g(1) = 7 \qquad g\left(\frac{1}{4}\right) = \frac{19}{4} = 4.75$$

The only real difference here is that these will correspond to values of $f(x, y)$ at different points than for the right side. In this case these will correspond to the following function values for $f(x, y)$.

$$g(-1) = f(-1,-1) = 11$$
$$g(1) = f(-1,1) = 7$$
$$g\left(\frac{1}{4}\right) = f\left(-1, \frac{1}{4}\right) = \frac{19}{4} = 4.75$$

We can now look at the upper side defined by,

$$y = 1, \ -1 \le x \le 1$$

We'll again define a new function except this time it will be a function of x.

$$h(x) = f(x,1) = x^2 + 4(1^2) - 2x^2(1) + 4 = 8 - x^2$$

We need to find the absolute extrema of $h(x)$ on the range $-1 \le x \le 1$. First find the critical point(s).

$$h'(x) = -2x \qquad \Rightarrow \qquad x = 0$$

The value of this function at the critical point and the end points is,

$$h(-1) = 7 \qquad h(1) = 7 \qquad h(0) = 8$$

and these in turn correspond to the following function values for $f(x, y)$

$$h(-1) = f(-1,1) = 7$$
$$h(1) = f(1,1) = 7$$
$$h(0) = f(0,1) = 8$$

Note that there are several "repeats" here. The first two function values have already been computed when we looked at the right and left side. This will often happen.

Finally, we need to take care of the lower side. This side is defined by,

$$y = -1, \ -1 \le x \le 1$$

The new function we'll define in this case is,

$$h(x) = f(x,-1) = x^2 + 4(-1)^2 - 2x^2(-1) + 4 = 8 + 3x^2$$

The critical point for this function is,

$$h'(x) = 6x \qquad \Rightarrow \qquad x = 0$$

The function values at the critical point and the endpoint are,

$$h(-1) = 11 \qquad h(1) = 11 \qquad h(0) = 8$$

and the corresponding values for $f(x,y)$ are,

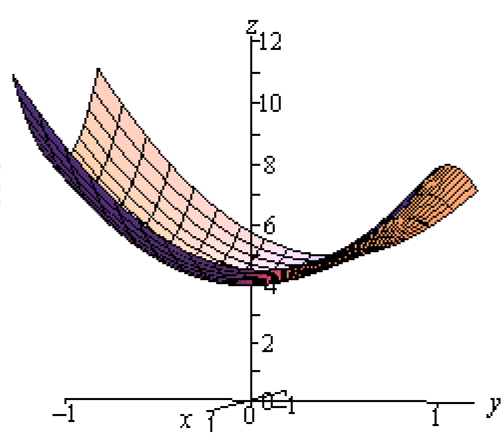
$$h(-1) = f(-1,-1) = 11$$
$$h(1) = f(1,-1) = 11$$
$$h(0) = f(0,-1) = 8$$

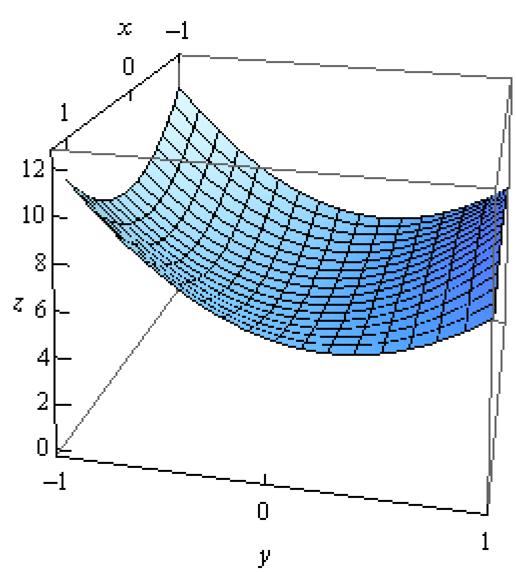
The final step to this (long…) process is to collect up all the function values for $f(x,y)$ that we've computed in this problem. Here they are,

$$f(0,0) = 4 \qquad f(1,-1) = 11 \qquad f(1,1) = 7$$
$$f\left(1,\frac{1}{4}\right) = 4.75 \qquad f(-1,1) = 7 \qquad f(-1,-1) = 11$$
$$f\left(-1,\frac{1}{4}\right) = 4.75 \qquad f(0,1) = 8 \qquad f(0,-1) = 8$$

The absolute minimum is at $(0,0)$ since gives the smallest function value and the absolute maximum occurs at $(1,-1)$ and $(-1,-1)$ since these two points give the largest function value.

Here is a sketch of the function on the rectangle for reference purposes.





Example 2 Find the absolute minimum and absolute maximum of $f(x, y) = 2x^2 - y^2 + 6y$ on the disk of radius 4, $x^2 + y^2 \le 16$

Solution

First note that a disk of radius 4 is given by the inequality in the problem statement. The "less than" inequality is included to get the interior of the disk and the equal sign is included to get the

As this example has shown these can be very long problems. Let's take a look at an easier problem with a different kind of boundary.

boundary. Of course, this also means that the boundary of the disk is a circle of radius 4.

Let's first find the critical points of the function that lies inside the disk. This will require the following two first order partial derivatives.

$$f_x = 4x \qquad f_y = -2y + 6$$

To find the critical points we'll need to solve the following system.

$$4x = 0$$
$$-2y + 6 = 0$$

This is actually a fairly simple system to solve however. The first equation tells us that $x = 0$ and the second tells us that $y = 3$. So the only critical point for this function is $(0, 3)$ and this is inside the disk of radius 4. The function value at this critical point is,

$$f(0, 3) = 9$$

Now we need to look at the boundary. This one will be somewhat different from the previous example. In this case we don't have fixed values of *x* and *y* on the boundary. Instead we have,

$$x^2 + y^2 = 16$$

We can solve this for x^2 and plug this into the x^2 in $f(x, y)$ to get a function of *y* as follows.

$$x^2 = 16 - y^2$$
$$g(y) = 2(16 - y^2) - y^2 + 6y = 32 - 3y^2 + 6y$$

We will need to find the absolute extrema of this function on the range $-4 \le y \le 4$ (this is the range of *y*'s for the disk….). We'll first need the critical points of this function.

$$g'(y) = -6y + 6 \qquad \Rightarrow \qquad y = 1$$

The value of this function at the critical point and the endpoints are,

$$g(-4) = -40 \qquad g(4) = 8 \qquad g(1) = 35$$

Unlike the first example we will still need to find the values of x that correspond to these. We can do this by plugging the value of y into our equation for the circle and solving for y.

$$\begin{aligned} y = -4 &: \quad x^2 = 16 - 16 = 0 \quad \Rightarrow \quad x = 0 \\ y = 4 &: \quad x^2 = 16 - 16 = 0 \quad \Rightarrow \quad x = 0 \\ y = 1 &: \quad x^2 = 16 - 1 = 15 \Rightarrow \quad x = \pm\sqrt{15} = \pm 3.87 \end{aligned}$$

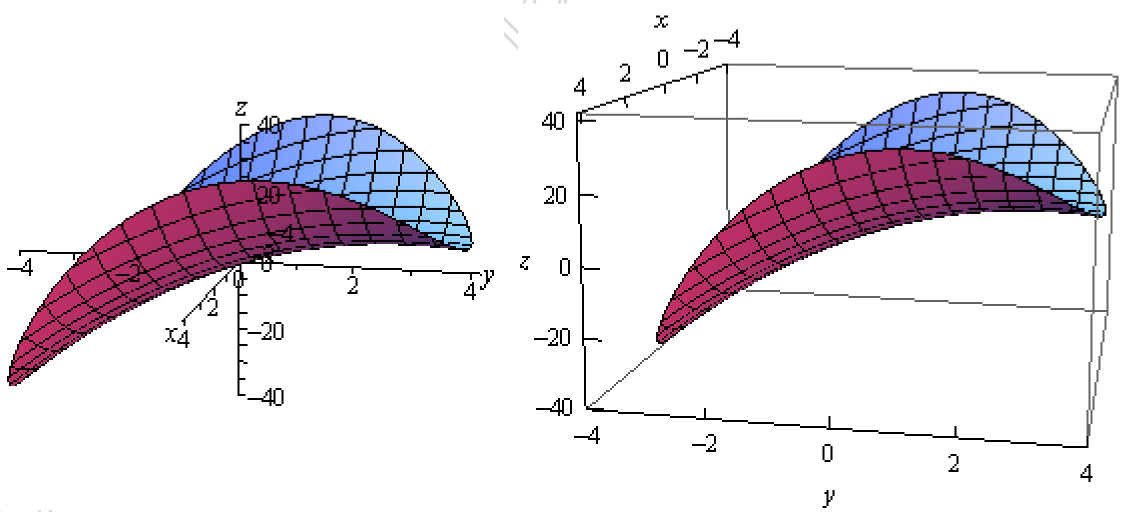
The function values for $g(y)$ then correspond to the following function values for $f(x, y)$.

$$\begin{aligned} g(-4) = -40 &\quad \Rightarrow \quad f(0,-4) = -40 \\ g(4) = 8 &\quad \Rightarrow \quad f(0,4) = 8 \\ g(1) = 35 &\quad \Rightarrow \quad f(-\sqrt{15},1) = 35 \ \text{ and } \ f(\sqrt{15},1) = 35 \end{aligned}$$

Note that the third one actually corresponded to two different values for $f(x, y)$ since that y also produced two different values of x.

So, comparing these values to the value of the function at the critical point of $f(x, y)$ that we found earlier we can see that the absolute minimum occurs at $(0,-4)$ while the absolute maximum occurs twice at $(-\sqrt{15},1)$ and $(\sqrt{15},1)$.

Here is a sketch of the region for reference purposes.



In both of these examples one of the absolute extrema actually occurred at more than one place. Sometimes this will happen and sometimes it won't so don't read too much into the fact that it happened in both examples given here.

Also note that, as we've seen, absolute extrema will often occur on the boundaries of these regions, although they don't have to occur at the boundaries. Had we given much more complicated examples with multiple critical points we would have seen examples where the absolute extrema occurred interior to the region and not on the boundary.

Small changes

If y is a function of x, i.e. $y=f(x)$, and the approximate change in y corresponding to a small change δx in x is required, then:

$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$$

and $\boldsymbol{\delta y \approx \frac{dy}{dx} \cdot \delta x}$ or $\boldsymbol{\delta y \approx f'(x) \cdot \delta x}$

Example1.Given $y=4x^2-x$, determine the approximate change in y if x changes from 1 to 1.02.

Since $y=4x^2-x$, then

$$\frac{dy}{dx} = 8x - 1$$

Approximate change in y,

$$\delta y \approx \frac{dy}{dx} \cdot \delta x \approx (8x-1)\delta x$$

When $x = 1$ and $\delta x = 0.02$, $\boldsymbol{\delta y} \approx [8(1)-1](0.02)$
$\approx \mathbf{0.14}$

[Obviously, in this case, the exact value of dy may be obtained by evaluating y when $x=1.02$, i.e. $y=4(1.02)^2-1.02=3.1416$ and then subtracting from it the value of y when $x=1$, i.e. $y=4(1)^2-1=3$, giving $\boldsymbol{\delta y}=3.1416-3=\mathbf{0.1416}$.

Using $\delta y=\frac{dy}{dx} \cdot \delta x$ above gave 0.14, which shows that the formula gives the approximate change in y for a small change in x.]

Example2.The time of swing T of a pendulum is given by $T=k\sqrt{l}$, where k is a constant. Determine the percentage change in the time of swing if the length of the pendulum l changes from 32.1 cm to 32.0 cm.

If $T=k\sqrt{l}=kl^{\frac{1}{2}}$, then

$$\frac{dT}{dl} = k\left(\frac{1}{2}l^{\frac{-1}{2}}\right) = \frac{k}{2\sqrt{l}}$$

Approximate change in T,

$$\delta t \approx \frac{dT}{dl}\delta l \approx \left(\frac{k}{2\sqrt{l}}\right)\delta l$$
$$\approx \left(\frac{k}{2\sqrt{l}}\right)(-0.1)$$

Further Problem

small changes

1. Determine the change in y if x changes from 2.50 to 2.51 when

   (a) $y=2x-x^2$ (b) $y=\dfrac{5}{x}$

   [(a) -0.03 (b) -0.008]

2. The pressure p and volume v of a mass of gas are related by the equation $pv=50$. If the pressure increases from 25.0 to 25.4, determine the approximate change in the volume of the gas. Find also the percentage change in the volume of the gas. [-0.032, -1.6%]

3. Determine the approximate increase in (a) the volume, and (b) the surface area of a cube of side x cm if x increases from 20.0 cm to 20.05 cm. [(a) 60 cm^3 (b) 12 cm^2]

4. The radius of a sphere decreases from 6.0 cm to 5.96 cm. Determine the approximate change in (a) the surface area, and (b) the volume. [(a) $-6.03\ cm^2$ (b) $-18.10\ cm^3$]

5. The rate of flow of a liquid through a tube is given by Poiseuilles's equation as: $Q=\dfrac{p\pi r^4}{8\eta L}$ where Q is the rate of flow, p is the pressure difference between the ends of the tube, r is the radius of the tube, L is the length of the tube and η is the coefficient of viscosity of the liquid. η is obtained by measuring Q, p, r and L. If Q can be measured accurate to $\pm 0.5\%$, p accurate to $\pm 3\%$, r accurate to $\pm 2\%$ and L accurate to $\pm 1\%$, calculate the maximum possible percentage error in the value of η.

   [12.5%]

rates of change

1. An alternating current, i amperes, is given by $i=10\sin 2\pi ft$, where f is the frequency in hertz and t the time in seconds. Determine the rate of change of current when $t=20$ ms, given that $f=150$ Hz. [3000π A/s]

2. The luminous intensity, I candelas, of a lamp is given by $I=6\times 10^{-4}\,V^2$, where V is the voltage. Find (a) the rate of change of luminous intensity with voltage when $V=200$ volts, and (b) the voltage at which the light is increasing at a rate of 0.3 candelas per volt. [(a) 0.24 cd/V (b) 250 V]

3. The voltage across the plates of a capacitor at any time t seconds is given by $v=V\mathrm{e}^{-t/CR}$, where V, C and R are constants.

   Given $V=300$ volts, $C=0.12\times 10^{-6}$ F and $R=4\times 10^6\ \Omega$ find (a) the initial rate of change of voltage, and (b) the rate of change of voltage after 0.5 s.

   [(a) -625 V/s (b) -220.5 V/s]

4. The pressure p of the atmosphere at height h above ground level is given by $p=p_0\mathrm{e}^{-h/c}$, where p_0 is the pressure at ground level and c is a constant. Determine the rate of change of pressure with height when $p_0=1.013\times 10^5$ pascals and $c=6.05\times 10^4$ at 1450 metres. [-1.635 Pa/m]

tangents and normals

For the curves in problems 1 to 5, at the points given, find (a) the equation of the tangent, and (b) the equation of the normal.

1. $y=2x^2$ at the point (1, 2) $\left[\begin{array}{l}\text{(a) } y=4x-2\\ \text{(b) } 4y+x=9\end{array}\right]$

2. $y=3x^2-2x$ at the point (2, 8)

   $\left[\begin{array}{l}\text{(a) } y=10x-12\\ \text{(b) } 10y+x=82\end{array}\right]$

3. $y=\dfrac{x^3}{2}$ at the point $\left(-1,-\dfrac{1}{2}\right)$

   $\left[\begin{array}{l}\text{(a) } y=\frac{3}{2}x+1\\ \text{(b) } 6y+4x+7=0\end{array}\right]$

4. $y=1+x-x^2$ at the point $(-2,-5)$

   $\left[\begin{array}{l}\text{(a) } y=5x+5\\ \text{(b) } 5y+x+27=0\end{array}\right]$

5. $\theta=\dfrac{1}{t}$ at the point $\left(3,\dfrac{1}{3}\right)$

   $\left[\begin{array}{l}\text{(a) } 9\theta+t=6\\ \text{(b) } \theta=9t-26\frac{2}{3} \text{ or } 3\theta=27t-80\end{array}\right]$

maximum and minimum

1. $y = 3x^2 - 4x + 2$ $\left[\text{Minimum at } \left(\frac{2}{3}, \frac{2}{3}\right)\right]$
2. $x = \theta(6 - \theta)$ [Maximum at (3, 9)]
3. $y = 4x^3 + 3x^2 - 60x - 12$

$\left[\begin{array}{l}\text{Minimum } (2, -88); \\ \text{Maximum}(-2.5, 94.25)\end{array}\right]$

4. $y = 5x - 2\ln x$

[Minimum at (0.4000, 3.8326)]

5. $y = 2x - e^x$

[Maximum at (0.6931, −0.6136)]

6. $y = t^3 - \dfrac{t^2}{2} - 2t + 4$

$\left[\begin{array}{l}\text{Minimum at } (1, 2.5); \\ \text{Maximum at } \left(-\dfrac{2}{3}, 4\dfrac{22}{27}\right)\end{array}\right]$

7. $x = 8t + \dfrac{1}{2t^2}$ [Minimum at (0.5, 6)]
8. Determine the maximum and minimum values on the graph $y = 12\cos\theta - 5\sin\theta$ in the range $\theta = 0$ to $\theta = 360°$. Sketch the graph over one cycle showing relevant points.

$\left[\begin{array}{l}\text{Maximum of 13 at } 337°23', \\ \text{Minimum of } -13 \text{ at } 157°23'\end{array}\right]$

9. Show that the curve $y = \frac{2}{3}(t-1)^3 + 2t(t-2)$ has a maximum value of $\frac{2}{3}$ and a minimum value of -2.

maximum and minimum

1. The speed, v, of a car (in m/s) is related to time t s by the equation $v = 3 + 12t - 3t^2$. Determine the maximum speed of the car in km/h. [54 km/h]

2. Determine the maximum area of a rectangular piece of land that can be enclosed by 1200 m of fencing. [90000 m^2]

3. A shell is fired vertically upwards and its vertical height, x metres, is given by $x = 24t - 3t^2$, where t is the time in seconds. Determine the maximum height reached. [48 m]

4. A lidless box with square ends is to be made from a thin sheet of metal. Determine the least area of the metal for which the volume of the box is 3.5 m^3. [11.42 m^2]

5. A closed cylindrical container has a surface area of 400 cm^2. Determine the dimensions for maximum volume.

$\left[\begin{array}{l}\text{radius} = 4.607 \text{ cm}; \\ \text{height} = 9.212 \text{ cm}\end{array}\right]$

6. Calculate the height of a cylinder of maximum volume which can be cut from a cone of height 20 cm and base radius 80 cm. [6.67 cm]

7. The power developed in a resistor R by a battery of emf E and internal resistance r is given by $P = \dfrac{E^2 R}{(R+r)^2}$. Differentiate P with respect to R and show that the power is a maximum when $R = r$.

8. Find the height and radius of a closed cylinder of volume 125 cm^3 which has the least surface area.

$\left[\begin{array}{l}\text{height} = 5.42 \text{ cm}; \\ \text{radius} = 2.71 \text{ cm}\end{array}\right]$

9. Resistance to motion, F, of a moving vehicle, is given by $F = \frac{5}{x} + 100x$. Determine the minimum value of resistance. [44.72]

10. An electrical voltage E is given by $E = (15\sin 50\pi t + 40\cos 50\pi t)$ volts, where t is the time in seconds. Determine the maximum value of voltage. [42.72 volts]

11. The fuel economy E of a car, in miles per gallon, is given by:

$$E = 21 + 2.10 \times 10^{-2} v^2 - 3.80 \times 10^{-6} v^4$$

where v is the speed of the car in miles per hour.
Determine, correct to 3 significant figures, the most economical fuel consumption, and the speed at which it is achieved.

[50.0 miles/gallon, 52.6 miles/hour]