University of Technology
Department of Laser & Optoelectronics Engineering
Final Examination 2011/2012

Subject: Engineering Analysis | **Class:** 3^{rd} **year**
Division: Laser & Optoelectronics | **Time: 3 hours**
Examiner: Dr. Refat Taleb Hussein Ali | **Date: 10/6/2012**

Answer FIVE questions only

Q 1: Using Fourier Transform to find the frequency spectrum $Y(j\omega)$ for the differential equation.

$$d^2y(t)/dt^2 + 3dy(t)/dt + 2y(t) = p(t)$$

With $p(t) = \begin{cases} 1 & \text{when } -1/2 \le t \le 1/2 \\ 0 & \text{elsewhere} \end{cases}$ (12 Marks)

Q 2: a) Prove that $(AB)^{-1} = B^{-1}A^{-1}$. (6 Marks)

b) Find $\oint_C \frac{e^s}{(s-1)^2(s^2+4)} ds$ for any contour enclosing the point $s_o = 1$. (6 Marks)

Q 3: Applied the power series method, solve $y'' - y = 0$ (12 Marks)

Q 4: a) Find the Eigen values of the matrix C.

$$C = \begin{bmatrix} \frac{1}{2} & j\sqrt{\frac{3}{4}} \\ j\sqrt{\frac{3}{4}} & \frac{1}{2} \end{bmatrix}$$ (6 Marks)

b) Find the F.T. of the following functions; (6 Marks)

(i) $f(t) = \sin(\omega_o t)$
(ii) $f(t) = \cos(\omega_o t)$

Q 5: Using L.T., solve the initial value problem (12 Marks)

$$y''(t) + 4y'(t) + 3y(t) = 0, \text{ if } y(0) = 3 \text{ and } y'(0) = 1.$$

Q 6: a) Find the roots of $(1+j)^{1/3}$. (6 Marks)

b) Determine whether or not the signal $x(t) = \cos\frac{\pi}{3}t + \sin\frac{\pi}{4}t$ is periodic. If a signal is periodic, determine its fundamental period. (6 Marks)

Q 2: a) Prove that $(AB)^{-1} = B^{-1}A^{-1}$. (6 Marks)

Solution:

If $C = AB \qquad \times A^{-1}$

$A^{-1}C = A^{-1}AB = B \qquad \times B^{-1}$

$B^{-1}A^{-1}C = B^{-1}B = I$

$\therefore C^{-1} = B^{-1}.A^{-1}$

$\therefore (AB)^{-1} = B^{-1}A^{-1}$

$$y'' + y = 0.$$

Solution. By inserting (3) and (4b) into the ODE we have

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^m = 0.$$

To obtain the same general power on both series, we set $m = s + 2$ in the first series and $m = s$ in the second, and then we take the latter to the right side. This gives

$$\sum_{s=0}^{\infty} (s+2)(s+1)a_{s+2}x^s = + \sum_{s=0}^{\infty} a_s x^s.$$

Each power x^s must have the same coefficient on both sides. Hence $(s+2)(s+1)a_{s+2} = -a_s$. This gives the recursion formula

$$a_{s+2} = + \frac{a_s}{(s+2)(s+1)} \qquad (s = 0, 1, \cdots).$$

We thus obtain successively

$$a_2 = + \frac{a_0}{2 \cdot 1} = + \frac{a_0}{2!}, \qquad a_3 = + \frac{a_1}{3 \cdot 2} = + \frac{a_1}{3!}$$

$$a_4 = + \frac{a_2}{4 \cdot 3} = \frac{a_0}{4!}, \qquad a_5 = + \frac{a_3}{5 \cdot 4} = \frac{a_1}{5!}.$$

and so on. a_0 and a_1 remain arbitrary. With these coefficients the series (3) becomes

$$y = a_0 + a_1 x + \frac{a_0}{2!}x^2 + \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 + \cdots.$$

Reordering terms (which is permissible for a power series), we can write this in the form

$$y = a_0\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + + \cdots\right) + a_1\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + + \cdots\right)$$

and we recognize the familiar general solution

b) Determine whether or not the signal $x(t) = \cos\frac{\pi}{3}t + \sin\frac{\pi}{4}t$ is periodic. If a signal is periodic, determine its fundamental period. (6 Marks)

Sol.:

ⓐ $x(t) = \cos\frac{\pi}{3}t + \sin\frac{\pi}{4}t = x_1(t) + x_2(t)$

where

$x_1(t) = \cos\frac{\pi}{3}t = \cos\omega_1 t$ is periodic $\rightarrow T_1 = \frac{2\pi}{\omega_1} = 6$

and $x_2(t) = \sin\frac{\pi}{4}t = \sin\omega_2 t$ is = $\rightarrow T_2 = \frac{2\pi}{\omega_2} = 8$.

Since $T_1/T_2 = 6/8 = \frac{3}{4}$ is a rational number,

$\therefore$ $x(t)$ is periodic with fundamental period

$T_0 = 4T_1 = 3T_2 = 24$.

b) Find the F.T. of the following functions; (6 Marks)

(i) $f(t) = \sin(\omega_o t)$

(ii) $f(t) = \cos(\omega_o t)$

Sol.: (i) $f(t) = \sin(\omega_o t) = \frac{1}{2j}\left(e^{j\omega_o t} - e^{-j\omega_o t}\right)$

$$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$\mathcal{F}\{\sin(\omega_o t)\} = \frac{1}{2j}\int_{-\infty}^{\infty}\left(e^{j\omega_o t} - e^{-j\omega_o t}\right) e^{-j\omega t}\, dt$$

Note that: shift in frequency

$e^{\mp j\omega_o t} f(t) = F(\omega \pm \omega_o)$

Since;

$$\mathcal{F}\{e^{j\omega_o t}\cdot 1\} = \delta(f - f_o)$$

$$\mathcal{F}\{e^{-j\omega_o t}\cdot 1\} = \delta(f + f_o)$$

However;

$$\mathcal{F}[\sin(\omega_o t)] = \frac{1}{2j}\{\delta(f - f_o) - (f + f_o)\}$$

Ex: Find the I.F.T. of the functions

(1) $F(f) = 0.5\{\delta(f-f_o) + \delta(f+f_o)\}$

Sol.

$$\mathcal{F}^{-1}\{\delta(f-f_o)\} = e^{j\omega_o t}$$

$$\mathcal{F}^{-1}\{\delta(f+f_o)\} = e^{-j\omega_o t}$$

$$\therefore \mathcal{F}^{-1}\{0.5[\delta(f-f_o) + \delta(f+f_o)]\} = \tfrac{1}{2}(e^{j\omega_o t} + e^{-j\omega_o t})$$

$$= \cos(\omega_o t)$$

Ex: Find the ~~values~~ ~~of~~ eigenvalues of the matrix

$$C = \begin{bmatrix} \frac{1}{2} & j\sqrt{\frac{3}{4}} \\ j\sqrt{\frac{3}{4}} & \frac{1}{2} \end{bmatrix}$$

Sol:

$$|C - \lambda I| = \begin{vmatrix} \frac{1}{2}-\lambda & j\sqrt{\frac{3}{4}} \\ j\sqrt{\frac{3}{4}} & \frac{1}{2}-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - \lambda + 1 = 0$$

$$\lambda = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda = \frac{1 \mp \sqrt{1-4}}{2} = \frac{1 \mp j\sqrt{3}}{2}$$

$$\therefore \lambda_1 = \frac{1}{2} + j\frac{\sqrt{3}}{2}, \quad \lambda_2 = \frac{1}{2} - j\frac{\sqrt{3}}{2}$$

Ex: Solve the initial value problem

$$y'' + 4y' + 3y = 0, \quad y(0) = 3, \quad y'(0) = 1.$$

Sol: Let $Y(s) = \mathcal{L}\{y\}$ be the L.T. of the (unknown) solution $y(t)$. Then

$$\mathcal{L}(y') = sY(s) - y(0) = sY(s) - 3$$

$$\mathcal{L}(y'') = s^2 Y(s) - s\,y(0) - y'(0) = s^2 Y(s) - 3s - 1$$

we substitute this into the laplace transform of the given differential equation, finding

$$s^2 Y(s) + 4sY(s) + 3Y(s) = 3s + 1 + 4*3$$

or

$$(s+3)(s+1)Y(s) = 3s + 13$$

Solving algebraically for $Y(s)$ and using partial fractions, we obtain

$$Y(s) = \frac{3s+13}{(s+3)(s+1)} = \frac{-2}{s+3} + \frac{5}{s+1}$$

Now, the I.L.T: we found that

$$\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

Using the linearity, we see the solution of this problem is

$$y(t) = -2e^{-3t} + 5e^{-t}$$

Q: Find the roots of $(1+j)^{1/3}$.

Sol:

$$G(s) = \sqrt[n]{s}$$

$$G(s) = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + j\sin\frac{\theta + 2k\pi}{n}\right)$$

where $\theta \neq 0$ and $k = 0, 1, 2, \ldots, n-1$

$$G(s) = (1+j)^{1/3}$$

$$r = \sqrt{2} = (2)^{1/2}$$

$$\theta = \tan^{-1}(1) = 45^\circ = \frac{\pi}{4}\ \text{rad}$$

and $n = 3$, $k = 0, 1, 2$

$$\therefore G(\sqrt[3]{1+j}) = \sqrt[6]{2}\left(\cos\frac{\frac{\pi}{4} + 2k\pi}{3} + j\sin\frac{\frac{\pi}{4} + 2k\pi}{3}\right)$$

$$= \sqrt[6]{2}\left[\cos\left(\frac{\pi + 8k\pi}{12}\right) + j\sin\left(\frac{\pi + 8k\pi}{12}\right)\right]$$

$$G((1+j)^{1/3}) = \sqrt[6]{2}\left[\cos\left(\frac{\pi}{12}\right) + j\sin\left(\frac{\pi}{12}\right)\right]$$

$$= \sqrt[6]{2}\left[\cos\left(\frac{9\pi}{12}\right) + j\sin\left(\frac{9\pi}{12}\right)\right]$$

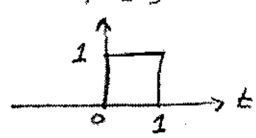
$$= \sqrt[6]{2}\left[\cos\left(\frac{17\pi}{12}\right) + j\sin\left(\frac{17\pi}{12}\right)\right]$$

Ex; Consider the differential equation

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = P(t) \qquad \text{--} (1)$$

with $p(t)$ is the pulse

$$P(t) = \begin{cases} 1 & , \quad 0 \leq t \leq 1 \\ 0 & , \quad \text{elsewhere} \end{cases}$$



Sol.;

$$\mathcal{F}\left[\frac{dy(t)}{dt}\right] = j\omega Y(j\omega)$$

and

$$\mathcal{F}\left[\frac{d^2y(t)}{dt^2}\right] = -\omega^2 Y(j\omega)$$

Using F.T. of eq.(1) term by term, then

$$-\omega^2 Y(j\omega) + j3\omega Y(j\omega) + 2Y(j\omega) = P(j\omega)$$

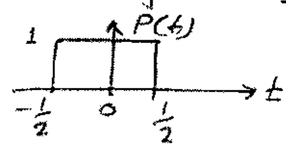
$$\therefore \quad Y(j\omega) = \frac{1}{2-\omega^2+j3\omega} P(j\omega)$$

$$|Y(j\omega)| = \frac{1}{\sqrt{(2-\omega^2)^2+(3\omega)^2}} |P(j\omega)|$$

$$|Y(j\omega)| = \frac{1}{\sqrt{\omega^4+5\omega^2+4}} |P(j\omega)|$$

The F.T. of the pulse at the origon is even function.

$$\mathcal{F}[P(t)] = \int_{-\infty}^{\infty} P(t)\, e^{-j\omega t}\, dt$$



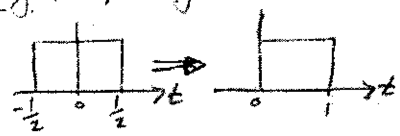
$$\mathcal{F}[P(t)] = \int_{-1/2}^{1/2} e^{-j\omega t}\,dt = -\frac{e^{-j\omega t}}{j\omega}\Bigg|_{-1/2}^{1/2} = \frac{2}{\omega}\sin\left(\frac{\omega}{2}\right)$$

or

$$P(j\omega) = \frac{\sin\left(\frac{\omega}{2}\right)}{\left(\frac{\omega}{2}\right)}$$

By using the time shifting property

$$\mathcal{F}[P(t-t_0)] = e^{-j\omega t_0} P(j\omega)$$



However

$$P(j\omega) = \frac{\sin\left(\frac{\omega}{2}\right)}{\left(\frac{\omega}{2}\right)}\, e^{-j\frac{\omega}{2}}$$

The result for the magnitude of the spectrum of the pulse is

$$|P(j\omega)| = \left|\frac{\sin\left(\frac{\omega}{2}\right)}{\left(\frac{\omega}{2}\right)}\right|$$

However the output signal has a frequency Spectrum

$$|Y(j\omega)| = \frac{1}{\sqrt{\omega^4 + 5\omega^2 + 4}}\left|\frac{\sin\left(\frac{\omega}{2}\right)}{\left(\frac{\omega}{2}\right)}\right|$$

b) Find $\oint_C \frac{e^s}{(s-1)^2(s^2+4)} ds$ for any contour enclosing the point $s_o = 1$. (6 Marks)

$$\oint_C \frac{e^s}{(s-1)^2(s^2+4)} ds = 2\pi j \left(\frac{e^s}{s^2+4}\right)' \Big|_{s=1}$$

$$= 2\pi j \; \frac{e^s(s^2+4) - e^s \cdot 2s}{(s^2+4)^2} \Big|_{s=1}$$

$$= \frac{6\pi e}{25} j \approx 2.050 j$$