University of Technology
Department of Laser & Opto-electronic Engineering

Final Examination 2011-2012

Subject: Mathematics
Division:
Examiner: Dr. Sabah A.Dhahir

Class: year
Time:3 hours
Date: 10 / 6 /2012

Answer five questions only

Q1: a. Discuses and sketch the following function (conical section): (20m)

$4x^2 - 8x - y^2 + 4y = 4$

b. Find the domain , Range and sketch the following function

$y = a + b\,sin^2x$ **Where a and b are constants.**

Q2: a. Find the volume generated by revolving the area bounded (20m) by the curves $y = x^2$ **and the line** $x + y = 6$ **about x-axis.**

b. Prove that $tanh\,(A + B) = \frac{tanh\,A + tanh\,B}{1 + tanh\,A\;tanh\,B}$

Q3: Answer two of the following: (20m)

a. If $y = e^{ax}sinh\,ax + e^{-ax}cosh\,ax$ **, show that**

1- $y = cosh\,2ax$

2- $(y'')^2 - 4a^2(y')^2 = 16a^4$

b. Obtain three non-vanishing terms of Maclourians expansion of the following function $f(x) = \ln(x + e)$

c. Given $e^x = sin\,(x + 3y)\,, Find\ \frac{dy}{dx}$.

Q4: Solve three of the following integrals (20m)

a. $\int tan^6x\,.\,dx$ b. $\int cos^{-1}4x\,.\,dx$ c. $\frac{dx}{\sqrt{5+4x-x^2}}$

d. $\int \frac{cos\,x}{\sqrt{sin\,x}}\,.\,dx$ e. $\int x\;\; tan^{-1}(x)\,.\,dx$

Q5: a. Find the area bounded by the curves $y^2 = x + 1$ **(20m) and** $y = x - 1$

b. Solve the following integral numerically with (n=4) $\int_0^1 \frac{dx}{\sqrt{1+x^2}}$ **and compare with exact solution.**

Q6: a. Given the three points (-1,2) , (1,-1) and (2,1) . (20m) Find parabola passing through the given points and have its axis parallel to the y-axis

b. Find $\lim_{x\to 0} \frac{\ln(cos\,3x)}{\ln(cos\,2x)}$

Good Louk

Q1. a $4x^2 - 8x - y^2 + 4y = 4$

$4(x^2 - 2x + 1) - (y^2 - 4y + 4) = 4 + \cancel{4} - \cancel{4}$

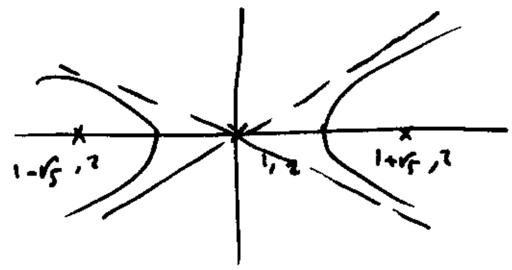
$4(x - 1)^2 - (y - 2)^2 = 4$

$(x - 1)^2 - \frac{(y-2)^2}{4} = 1$

the equ. H. H. P. with center $c(1, 2)$

$a = \pm 1 \quad b = \pm 2 \qquad c = \sqrt{5}$

Asy:-



$(x - 1) \pm \frac{y - 2}{2} = 0$

$2(x - 1) + y - 2 = 0$ 1st.

$2x - 2 + y - 2 = 0$

$y = -2x - 4$ 1st. Asy.

$2(x - 1) - y + 2 = 0$

$2x - 2 - y + 2 = 0$

$y = 2x + 2$ 2nd. As.

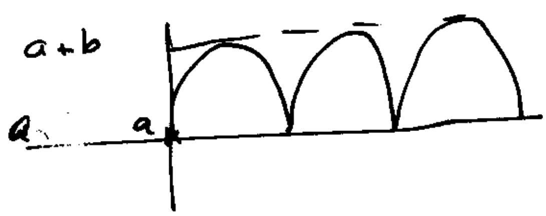
$(\pm 1)^2 \Rightarrow 1$ or 0

b. $y = a + b \sin^2 x$

$a + b$

D: Real No.

Range $[a, a+b]$



6(a) $(x - h)^2 = +4p(y - k)^2$

x, y
$-1, 2$
$1, -1$
$(2, 1)$

$(-1 - h)^2 = \pm 4p(2 - k)^2$ -- ①

$(1 - h)^2 = \pm 4p(-1 - k)^2$ -- ②

$(2 - h)^2 = \pm 4p(1 - k)^2$ -- ③

$$\left.\begin{aligned} 1 + 2h + h^2 &= 4p(4 - 2k + k^2) \\ 1 - 2h + h^2 &= 4p(1 + 2k + k^2) \\ 4 - 4h + h^2 &= 4p(1 - 2k + k^2) \end{aligned}\right\} \text{Solv for } h, k, p$$

or $Ax^2 + Bx + Cy + F = 0$

$$\begin{aligned} A + B + 2C &= 0 \quad \text{-- ①} \\ A + B + C &= 0 \quad \text{-- ②} \\ 2A + 2B + C &= 0 \quad \text{-- ③} \end{aligned}$$ solv for A, B, C

Q_3. a. $y = e^{ax}\sinh ax + e^{-ax}\cosh ax$

$$y = e^{ax}\frac{e^{ax}-e^{-ax}}{2} + e^{-ax}\frac{e^{ax}+e^{-ax}}{2}$$

$$= \frac{e^{2ax}-1}{2} + \frac{1+e^{-2ax}}{2}$$

$$y = \frac{e^{2ax}+e^{-2ax}}{2} = \cosh 2ax$$

$$y' = \sinh 2ax\,(2a)$$

$$y'' = \cosh 2ax\,(4a^2)$$

$$16a^4(\cosh 2ax)^2 - 4a^2(\sinh^2 2ax\,(4a^2))$$

$$16a^4\cosh^2 2ax - 16a^4\sinh^2 2ax$$

$$16a^4(\cosh^2 2ax - \sinh^2 2ax)$$

$$16a^4 \times 1 = 16a^4$$

Q_3. b $f(x) = \ln(x+e)$

$$f(x) = \ln(x+e) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 - - -$$

$$f(0) = \ln e = 1$$

$$f'(x) = \frac{1}{x+e}$$

$$f'(0) = \frac{1}{e}$$

$$f''(x) = -(x+e)^{-2}$$

$$f'(0) = \frac{-1}{e^2}$$

$$f'''(x) = 2(x+e)^{-3}$$

$$f'''(0) = \frac{2}{e^3}$$

$$\ln(x+e) = 1 + \frac{x}{e} - \frac{x^2}{2e} + \frac{2x^3}{6e^3} + - - -$$

Q_3 c:- $e^x = \sin(x+3y)$

$$e^x = \cos(x+3y)\left(1+3\frac{dy}{dx}\right)$$

$$e^x = \cos(x+3y) + 3\cos(x+3y)\frac{dy}{dx}$$

$$3\cos(x+3y)\frac{dy}{dx} = e^x - \cos(x+3y)$$

$$\frac{dy}{dx} = \frac{e^x - \cos(x+3y)}{3\cos(x+3y)}$$

Q5 :-

a. $\int \tan^6 x \cdot dx = \int \tan^4 x \cdot \tan^2 x \cdot dx =$

$\int \tan^4 x \cdot (\sec^2 x - 1) \cdot dx$

$\int (\tan^4 x \cdot \sec^2 x - \tan^4 x) dx$

$\int \tan^4 x \cdot \sec^2 x - \int \tan^4 x \cdot dx$

$\tan^5 x - \int \tan^2 x \cdot (\sec^2 x - 1) \cdot dx$

$\tan^5 x - \int (\tan^7 x \cdot \sec^2 x - \tan^7 x) \cdot dx$

$\tan^5 x - \tan^3 x - \int (\sec^2 - 1) \cdot dx$

$\tan^5 x - \tan^5 x + \tan - x + c$

b. $\int \cos^{-1} x \cdot dx$

let $\cos^{-1} x = u$ $\qquad dx = dv$

$\frac{-dx}{\sqrt{1-x^2}} = du$ $\qquad x = v$

$\int u \cdot dv = u \cdot v - \int v \cdot du$

$= x \cos^{-1} x \cdot - \int \frac{-x \cdot dx}{\sqrt{1-x^2}}$

$= x \cos^{-1} x + \int \frac{x \cdot dx}{\sqrt{1-x^2}}$

$= x \cos^{-1} x + \frac{1}{2} \int 2x \cdot dx \, (1-x^2)^{-1/2}$

$= x \cos^{-1} x + \frac{2}{2} (1-x^2)^{1/2}$

$= x \cos^{-1} x + \frac{1}{\sqrt{1-x^2}}$

c. $\int \frac{dx}{\sqrt{5+4x-x^2}} = \int \frac{dx}{\sqrt{-(x^2-4x+5)}} = \int \frac{dx}{\sqrt{-1(x^2+4x+4+1)}}$

$$\int \frac{dx}{\sqrt{-[(x+2)^2+1]}} = \int \frac{dx}{\sqrt{1-(x-2)^2}}$$ let $x-2=u$, $dx = du$

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u = \sin^{-1}(x-2) + c$$

d. $$\int \frac{\cos x}{\sqrt{\sin x}} \cdot dx$$ let $\sin x = u$, $\cos x \cdot dx = du$

$$\int \frac{du}{\sqrt{u}} = \int u^{-1/2} \cdot du = 2u^{1/2} + c = 2 \frac{1}{\sqrt{\sin x}} + c$$

e. $$\int x^3 \tan^{-1} x$$

$$\tanh(A+B) = \frac{\tanh A + \tanh B}{1 + \tanh A \tanh B}$$

$$\tanh(A+B) = \frac{e^{A+B} - e^{-A-B}}{e^{A+B} + e^{-A-B}}$$

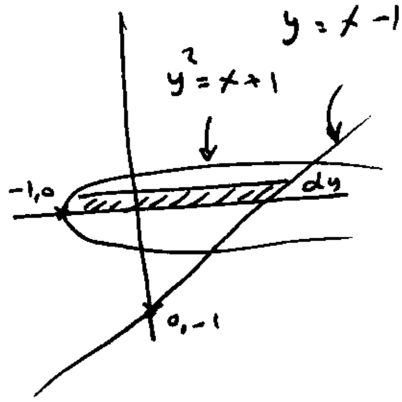
$$\frac{\tanh A + \tanh B}{1 + \tanh A \tanh B} = \frac{\frac{e^{A} - e^{-A}}{e^{A}+e^{-A}} + \frac{e^{B} - e^{-B}}{e^{B}+e^{-B}}}{1 + \frac{e^{A}-e^{-A}}{e^{A}+e^{-A}} \cdot \frac{e^{B}-e^{-B}}{e^{B}+e^{-B}}}$$

$$= \frac{\frac{(e^{A}-e^{-A})(e^{B}+e^{-B}) + (e^{A}+e^{-A})(e^{B}-e^{-B})}{\cancel{(e^{A}+e^{-A})(e^{B}+e^{-B})}}}{\frac{(e^{A}+e^{-A})(e^{B}+e^{-B}) + (e^{A}-e^{-A})(e^{B}-e^{-B})}{\cancel{(e^{A}+e^{-A})(e^{B}+e^{-B})}}}$$

$$= \frac{e^{A+B} + e^{A-B} - e^{-A+B} - e^{-A-B} + e^{A+B} + e^{A-B} - e^{-A+B} - e^{-A-B}}{\cancel{e^{A+B}}\; e^{A}e^{B} + \cancel{e^{A}e^{-B}} + \cancel{e^{-A}e^{B}} + e^{-A-B} + e^{A}e^{B} - \cancel{e^{A}e^{-B}} + \cancel{e^{-A}e^{B}} + e^{-A-B}}$$

$$\frac{\cancel{2}(e^{A} - e^{-A})(e^{B} + e^{-B})}{\cancel{2}e^{A+B} + \cancel{2}e^{-A-B}} = \frac{e^{A+B} - \cancel{e^{-A+B}} + \cancel{e^{A-B}} + e^{-A-B}}{e^{A+B} + e^{-A-B}}$$

Q5:- a. $\int_{y_1}^{y_2} (x_1 - x_2) \cdot dy$



$\int_{y_1}^{y} [(\cancel{x}+1) - (y^2-1)]\, dy$

$\int_{-1}^{2} [(y+1) - (y^2-1)] \cdot dy$

$x = y+1$
$x = y^2-1$

$y+1 = y^2-1$
$y^2 - y - 2 = 0$
$(y+1)(y-2) = 0$
$y = -1, y = 2$

$\int_{-1}^{2} (y - y^2 + 2) \cdot dy$

$\left[\frac{y^2}{2} - \frac{y^3}{2} + 2y \right]_{-1}^{2}$

$\left(\frac{4}{2} - \frac{8}{2} + 4\right) - \left(\frac{1}{2} - \frac{1}{3} + 2\right)$

$\frac{4-8+8}{2} - \frac{3-2+6}{6}$

$2 - \frac{7}{6} = \frac{5}{6}$ unit^2

b. $\int_0^1 \frac{dx}{\sqrt{1+x^2}}$

$x_0 = 0$ $x_1 = \frac{1}{4}$ $x_2 = \frac{1}{2}$ $x_3 = \frac{3}{4}$ $x_4 = 1$

$y_0 = \frac{1}{1}$ $y_1 \frac{1}{\sqrt{1+\frac{1}{16}}}$ $y_2 = \frac{1}{\sqrt{1+\frac{1}{4}}}$ $y_3 = \frac{1}{\sqrt{1+\frac{9}{16}}}$

$h = \frac{1-0}{4} = \frac{1}{4}$

$y_4 = \frac{1}{\sqrt{2}}$

$T = \frac{h}{2} (y_0 + 2y_1 + 2y_2 + 2y_3 + y_4)$

$T = \frac{0.25}{2} \left(1 + \frac{2}{\sqrt{1+\frac{1}{6}}} + \frac{2}{\sqrt{1+\frac{1}{4}}} + \frac{2}{\sqrt{1+\frac{3}{16}}} + \frac{2}{\sqrt{2}} \right)$ unit^2.

Exact solution $= \sinh^{-1} x \Big]_0^1 = \sinh^{-1} 1 - \sinh^{-1} 0$

$= ?$

Q6:- a. $\lim_{x \to 0} \dfrac{\ln \cos 3x}{\ln \cos 2x} = \dfrac{\ln 1}{\ln 1} = \dfrac{0}{0}$

we have L.H.R case $\frac{0}{0}$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

let $f(x) = \ln \cos 3x \Rightarrow f'(x) = -\dfrac{3}{\cos 3x} \cdot \sin 3x = -3 \tan 3x$

$g(x) = \ln \cos 2x \Rightarrow g'(x) = \dfrac{-2 \sin 2x}{\cos 2x} = -2 \tan 2x$

$$\lim_{x \to 0} \frac{3 \tan 3x}{2 \tan 2x} = \frac{3 \tan 0}{2 \tan 0} = \frac{0}{0}$$

$f''(x) = -3 \sec^2 3x \,(3) = -9 \sec^2 3x$

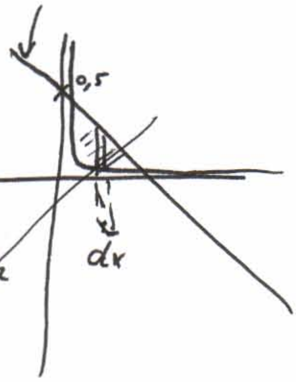
$g''(x) = -2 \sec^2 2x \,(2) = -4 \sec^2 2x$

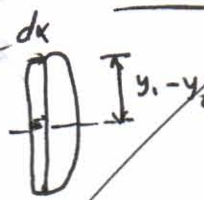
$$\lim_{x \to 0} \frac{9}{4} \frac{\sec^2 3x}{\sec^2 2x} = \frac{9}{4}$$

b. $xy = 4$ $\quad\quad x + y = 5$

$y = \dfrac{4}{x}$ $\quad\quad y = -x + 5$

$y = \frac{4}{x} - x + 5$





$$\boxed{dV = \int_{x_1}^{x_2} \pi (y_1 - y_2)^2 \cdot dx}$$

or $\pi \int_{x_1}^{x_2} (y_1^2 - y_2^2) \cdot dx$

$$\pi \int_{x_1}^{x_2} \left[\left(\frac{4}{x}\right)^2 - (-x+5)^2 \right] \cdot dx$$

$\dfrac{4}{x} = -x + 5$

$4 = -x^2 + 5x$

$x^2 - 5x + 4 = 0$

$(x - 1)(x - 4) = 0 \Rightarrow x = 1, \; x = 4$

$$\int_1^4 \left(\frac{4}{x}\right)^2 - (-x+5)^2 \cdot dx$$