Ministry of Higher Education & Scientific Research
University of Technology
Laser & Optoelectronic Eng. Department
Branch: Laser & Optoelectronics.
Class: Second Year
Subject: Electromagnetic Fields

Final Examinations
Time: 3 Hours
Date: 29 May 2016
Examiner: Asst. L. Ahmed W.
Code: LOPE 2207
Academic Year: 2015-2016

Attempt any FIVE questions

Q1: (A) [6 marks] A field is given as

$$\vec{G} = \frac{25}{(x^2 + y^2)}(x\hat{a}_x + y\hat{a}_y)$$

Find:

i) A unit vector in the direction of $\vec{G}$ at P (3, 4, -2).

ii) The angle between $\vec{G}$ and $\hat{a}_x$ at P.

iii) The value of the following double integral on the plane $y = 7$:

$$\int_0^4 \int_0^2 \vec{G}.\hat{a}_y \, d_z d_x$$

(B) [6 marks] Given the points M (0.1, -0.2, -0.1), N (-0.2, 0.1, 0.3) and P (0.4, 0, 0.1), find the vector $\vec{R}_{MN}$, and the dot product $\vec{R}_{MN}.\vec{R}_{MP}$.

Q2: (A) [6 marks] A 100 nC point charge is located at A (-1, 1, 3) in free space. Find: i) The locus of all points P (x,y,z) at which $E_x = 500\ V/m$. ii) y_1 if P (-2, y_1, 3) lies on that locus.

(B) [6 marks] Uniform line charges of 0.4μC/m and -0.4μC/m are located in the $x = 0$ plane at $y = -0.6$, and $y = 0.6$ mtr respectively. Let $\epsilon = \epsilon_0$, find:

1) The electric field vector $\vec{E}$ at P (x, 0, z).

2) The electric field vector $\vec{E}$ at Q (2, 3, 4).

Q3: [12 marks] Calculate the divergence of $\vec{D}$ at the point specified if:

1) $\vec{D} = (1/z^2)[10xyz\,\hat{a}_x + 5x^2z\,\hat{a}_y + (2z^3 - 5x^2y)\hat{a}_z]$ at P (-2, 3, 5).

2) $\vec{D} = (5z^2\,\hat{a}_\rho + 10\rho z\,\hat{a}_z)$ at P (3, -45°, 5).

3) $\vec{D} = (2r\,sin\theta\,sin\emptyset\,\hat{a}_r + r\cos\theta\,sin\emptyset\,\hat{a}_\emptyset)$ at P (3, 45°, -45°).

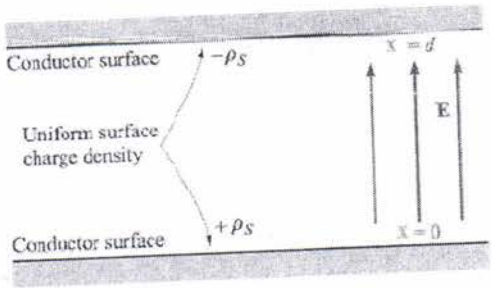
Q4: (A) [5 marks] Let $\vec{E} = 400\hat{a}_x - 300\hat{a}_y + 500\hat{a}_z$ in the neighborhood of point P (6, 2, -3). Find the incremental work done in moving a 4-C charge a distance 1 mm in the direction specified by $\hat{a}_x + \hat{a}_y + \hat{a}_z$.

(B) [7 marks] Given a surface charge density of 8 nC/m^2 on the plane $x = 2$, a line charge density of 30 nC/m on the line $x = 1$, $y = 1$, and a 1μC point charge at P (-1, -1, 2), find V_{AB} for points A (3, 4, 0) and B (4, 0, 1).

Q5: [12 marks] Two perfectly-conducting cylindrical surface are located at $\rho = 3$ and $\rho = 5$ cm. The total current passing radially outward through the medium between cylinders is 3 A dc. Assume the cylinders are both of length 1. Find:

a) The voltage and resistance between the cylinders, and $\vec{E}$ in the region between the cylinders, if a conducting material having $\sigma = 0.005$ S/m is present for $3 < \rho < 5$ cm.
b) Show that integrating the power dissipated per unit volume over the volume gives the total dissipated power.

Q6: [12 marks] Using the solution of Laplace's equation and referring to the figure given below, find the capacitance of a parallel-plate capacitor. Considering the potential is function of only x-direction.



Physical constants you may need:

Permittivity of free space $\epsilon_0 = 8.854\times10^{-12}$ *F/m.*
Permeability of free space $\mu_0 = 4\pi10^{-7}$ *H/m.*
Electron charge $e = 1.602\times10^{-19}$ *C.*

Good luck

Q1: (A)

A field is given as

$$\mathbf{G} = \frac{25}{(x^2+y^2)}(x\mathbf{a}_x + y\mathbf{a}_y)$$

Find:

a) a unit vector in the direction of $\mathbf{G}$ at $P(3,4,-2)$: Have $\mathbf{G}_p = 25/(9+16) \times (3,4,0) = 3\mathbf{a}_x + 4\mathbf{a}_y$, and $|\mathbf{G}_p| = 5$. Thus $\mathbf{a}_G = \underline{(0.6, 0.8, 0)}$.

b) the angle between $\mathbf{G}$ and $\mathbf{a}_x$ at P: The angle is found through $\mathbf{a}_G \cdot \mathbf{a}_x = \cos\theta$. So $\cos\theta = (0.6, 0.8, 0) \cdot (1, 0, 0) = 0.6$. Thus $\underline{\theta = 53^\circ}$.

c) the value of the following double integral on the plane $y = 7$:

$$\int_0^4 \int_0^2 \mathbf{G} \cdot \mathbf{a}_y dz dx$$

$$\int_0^4 \int_0^2 \frac{25}{x^2+y^2}(x\mathbf{a}_x + y\mathbf{a}_y) \cdot \mathbf{a}_y dz dx = \int_0^4 \int_0^2 \frac{25}{x^2+49} \times 7\, dz dx = \int_0^4 \frac{350}{x^2+49} dx$$

$$= 350 \times \frac{1}{7}\left[\tan^{-1}\left(\frac{4}{7}\right) - 0\right] = \underline{26}$$

Q1: (B)

Given the points $M(0.1, -0.2, -0.1)$, $N(-0.2, 0.1, 0.3)$, and $P(0.4, 0, 0.1)$, find:

a) the vector $\mathbf{R}_{MN}$: $\mathbf{R}_{MN} = (-0.2, 0.1, 0.3) - (0.1, -0.2, -0.1) = \underline{(-0.3, 0.3, 0.4)}$.

b) the dot product $\mathbf{R}_{MN} \cdot \mathbf{R}_{MP}$: $\mathbf{R}_{MP} = (0.4, 0, 0.1) - (0.1, -0.2, -0.1) = (0.3, 0.2, 0.2)$. $\mathbf{R}_{MN}$ $\mathbf{R}_{MP} = (-0.3, 0.3, 0.4) \cdot (0.3, 0.2, 0.2) = -0.09 + 0.06 + 0.08 = \underline{0.05}$.

c) the scalar projection of $\mathbf{R}_{MN}$ on $\mathbf{R}_{MP}$:

$$\mathbf{R}_{MN} \cdot \mathbf{a}_{RMP} = (-0.3, 0.3, 0.4) \cdot \frac{(0.3, 0.2, 0.2)}{\sqrt{0.09 + 0.04 + 0.04}} = \frac{0.05}{\sqrt{0.17}} = \underline{0.12}$$

d) the angle between $\mathbf{R}_{MN}$ and $\mathbf{R}_{MP}$:

$$\theta_M = \cos^{-1}\left(\frac{\mathbf{R}_{MN} \cdot \mathbf{R}_{MP}}{|\mathbf{R}_{MN}||\mathbf{R}_{MP}|}\right) = \cos^{-1}\left(\frac{0.05}{\sqrt{0.34}\sqrt{0.17}}\right) = \underline{78^\circ}$$

Q2 (A)

a) Find the locus of all points $P(x, y, z)$ at which $E_x = 500$ V/m: The total field at P will be:

$$\mathbf{E}_P = \frac{100 \times 10^{-9}}{4\pi\epsilon_0} \frac{\mathbf{R}_{AP}}{|\mathbf{R}_{AP}|^3}$$

where $\mathbf{R}_{AP} = (x+1)\mathbf{a}_x + (y-1)\mathbf{a}_y + (z-3)\mathbf{a}_z$, and where $|\mathbf{R}_{AP}| = [(x+1)^2 + (y-1)^2 + (z-3)^2]^{1/2}$. The x component of the field will be

$$E_x = \frac{100 \times 10^{-9}}{4\pi\epsilon_0} \left[\frac{(x+1)}{[(x+1)^2 + (y-1)^2 + (z-3)^2]^{1.5}} \right] = 500 \text{ V/m}$$

And so our condition becomes:

$$\underline{(x+1) = 0.56\,[(x+1)^2 + (y-1)^2 + (z-3)^2]^{1.5}}$$

b) Find y_1 if $P(-2, y_1, 3)$ lies on that locus: At point P, the condition of part a becomes

$$3.19 = \left[1 + (y_1 - 1)^2\right]^3$$

from which $(y_1 - 1)^2 = 0.47$, or $y_1 = \underline{1.69 \text{ or } 0.31}$

Q2: (B)

a) Find $\mathbf{E}$ at $P(x, 0, z)$: In general, we have

$$\mathbf{E}_P = \frac{\rho_l}{2\pi\epsilon_0} \left[\frac{\mathbf{R}_{+P}}{|\mathbf{R}_{+P}|} - \frac{\mathbf{R}_{-P}}{|\mathbf{R}_{-P}|} \right]$$

where $\mathbf{R}_{+P}$ and $\mathbf{R}_{-P}$ are, respectively, the vectors directed from the positive and negative line charges to the point P, and these are normal to the z axis. We thus have $\mathbf{R}_{+P} = (x, 0, z) - (0, -.6, z) = (x, .6, 0)$, and $\mathbf{R}_{-P} = (x, 0, z) - (0, .6, z) = (x, -.6, 0)$. So

$$\mathbf{E}_P = \frac{\rho_l}{2\pi\epsilon_0} \left[\frac{x\mathbf{a}_x + 0.6\mathbf{a}_y}{x^2 + (0.6)^2} - \frac{x\mathbf{a}_x - 0.6\mathbf{a}_y}{x^2 + (0.6)^2} \right] = \frac{0.4 \times 10^{-6}}{2\pi\epsilon_0} \left[\frac{1.2\mathbf{a}_y}{x^2 + 0.36} \right] = \underline{\frac{8.63\mathbf{a}_y}{x^2 + 0.36} \text{ kV/m}}$$

b) Find $\mathbf{E}$ at $Q(2, 3, 4)$: This field will in general be:

$$\mathbf{E}_Q = \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{\mathbf{R}_{+Q}}{|\mathbf{R}_{+Q}|} - \frac{\mathbf{R}_{-Q}}{|\mathbf{R}_{-Q}|}\right]$$

where $\mathbf{R}_{+Q} = (2, 3, 4) - (0, -.6, 4) = (2, 3.6, 0)$, and $\mathbf{R}_{-Q} = (2, 3, 4) - (0, .6, 4) = (2, 2.4, 0)$. Thus

$$\mathbf{E}_Q = \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{2\mathbf{a}_x + 3.6\mathbf{a}_y}{2^2 + (3.6)^2} - \frac{2\mathbf{a}_x + 2.4\mathbf{a}_y}{2^2 + (2.4)^2}\right] = \underline{-625.8\mathbf{a}_x - 241.6\mathbf{a}_y}\ \mathrm{V/m}$$

Q3:

a) $\mathbf{D} = (1/z^2)\left[10xyz\,\mathbf{a}_x + 5x^2z\,\mathbf{a}_y + (2z^3 - 5x^2y)\,\mathbf{a}_z\right]$ at $P(-2, 3, 5)$: We find

$$\nabla \cdot \mathbf{D} = \left[\frac{10y}{z} + 0 + 2 + \frac{10x^2y}{z^3}\right]_{(-2,3,5)} = \underline{8.96}$$

b) $\mathbf{D} = 5z^2\mathbf{a}_\rho + 10\rho z\,\mathbf{a}_z$ at $P(3, -45°, 5)$: In cylindrical coordinates, we have

$$\nabla \cdot \mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} = \left[\frac{5z^2}{\rho} + 10\rho\right]_{(3,-45°,5)} = \underline{71.67}$$

c) $\mathbf{D} = 2r\sin\theta\sin\phi\,\mathbf{a}_r + r\cos\theta\sin\phi\,\mathbf{a}_\theta + r\cos\phi\,\mathbf{a}_\phi$ at $P(3, 45°, -45°)$: In spherical coordinates, we have

$$\nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

$$= \left[6\sin\theta\sin\phi + \frac{\cos 2\theta\sin\phi}{\sin\theta} - \frac{\sin\phi}{\sin\theta}\right]_{(3,45°,-45°)} = \underline{-2}$$

Q4: (A)

Let $\mathbf{E} = 400\mathbf{a}_x - 300\mathbf{a}_y + 500\mathbf{a}_z$ in the neighborhood of point $P(6, 2, -3)$. Find the incremental work done in moving a 4-C charge a distance of 1 mm in the direction specified by:

a) $\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$: We write

$$dW = -q\mathbf{E} \cdot d\mathbf{L} = -4(400\mathbf{a}_x - 300\mathbf{a}_y + 500\mathbf{a}_z) \cdot \frac{(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)}{\sqrt{3}}(10^{-3})$$

$$= -\frac{(4 \times 10^{-3})}{\sqrt{3}}(400 - 300 + 500) = \underline{-1.39\ \text{J}}$$

Q4: (B)

know to be

$$V_p(r) = \frac{Q}{4\pi\epsilon_0 r}$$

Potential functions for the sheet and line charges can be found by taking indefinite integrals of the electric fields for those distributions. For the line charge, we have

$$V_l(\rho) = -\int \frac{\rho_l}{2\pi\epsilon_0\rho} d\rho + C_1 = -\frac{\rho_l}{2\pi\epsilon_0}\ln(\rho) + C_1$$

For the sheet charge, we have

$$V_s(x) = -\int \frac{\rho_s}{2\epsilon_0} dx + C_2 = -\frac{\rho_s}{2\epsilon_0}x + C_2$$

$$V = \frac{Q}{4\pi\epsilon_0 r} - \frac{\rho_l}{2\pi\epsilon_0}\ln(\rho) - \frac{\rho_s}{2\epsilon_0}x + C$$

The terms in this expression are not referenced to a common origin, since the charges are at different positions. The parameters r, ρ, and x are *scalar distances* from the charges, and will be treated as such here. For point A we have $r_A = \sqrt{(3-(-1))^2+(4-(-1))^2+(-2)^2} = \sqrt{45}$, $\rho_A = \sqrt{(3-1)^2+(4-2)^2} = \sqrt{8}$, and its distance from the sheet charge is $x_A = 3-2 = 1$. The potential at A is then

$$V_A = \frac{10^{-6}}{4\pi\epsilon_0\sqrt{45}} - \frac{30\times10^{-9}}{2\pi\epsilon_0}\ln\sqrt{8} - \frac{8\times10^{-9}}{2\epsilon_0}(1) + C$$

At point B, $r_B = \sqrt{(4-(-1))^2+(0-(-1))^2+(1-2)^2} = \sqrt{27}$,
$\rho_B = \sqrt{(4-1)^2+(0-2)^2} = \sqrt{13}$, and the distance from the sheet charge is $x_B = 4-2 = 2$.
The potential at A is then

$$V_B = \frac{10^{-6}}{4\pi\epsilon_0\sqrt{27}} - \frac{30\times10^{-9}}{2\pi\epsilon_0}\ln\sqrt{13} - \frac{8\times10^{-9}}{2\epsilon_0}(2) + C$$

Then

$$V_A - V_B = \frac{10^{-6}}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{45}} - \frac{1}{\sqrt{27}}\right] - \frac{30\times10^{-9}}{2\pi\epsilon_0}\ln\left(\sqrt{\frac{8}{13}}\right) - \frac{8\times10^{-9}}{2\epsilon_0}(1-2) = \underline{193\ \mathrm{V}}$$

Q5:

a) Find the voltage and resistance between the cylinders, and $\mathbf{E}$ in the region between the cylinders, if a conducting material having $\sigma = 0.05$ S/m is present for $3 < \rho < 5$ cm: Given the current, and knowing that it is radially-directed, we find the current density by dividing it by the area of a cylinder of radius ρ and length l:

$$\mathbf{J} = \frac{3}{2\pi\rho l}\,\mathbf{a}_\rho\ \mathrm{A/m^2}$$

Then the electric field is found by dividing this result by σ:

$$\mathbf{E} = \frac{3}{2\pi\sigma\rho l}\,\mathbf{a}_\rho = \underline{\frac{9.55}{\rho l}\mathbf{a}_\rho\ \mathrm{V/m}}$$

The voltage between cylinders is now:

$$V = -\int_5^3 \mathbf{E}\cdot d\mathbf{L} = \int_3^5 \frac{9.55}{\rho l}\mathbf{a}_\rho\cdot\mathbf{a}_\rho d\rho = \frac{9.55}{l}\ln\left(\frac{5}{3}\right) = \underline{\frac{4.88}{l}}\ \mathrm{V}$$

Now, the resistance will be

$$R = \frac{V}{I} = \frac{4.88}{3l} = \underline{\frac{1.63}{l}}\ \Omega$$

b) Show that integrating the power dissipated per unit volume over the volume gives the total dissipated power: We calculate

$$P = \int_v \mathbf{E} \cdot \mathbf{J}\, dv = \int_0^l \int_0^{2\pi} \int_{.03}^{.05} \frac{3^2}{(2\pi)^2 \rho^2 (.05) l^2}\, \rho\, d\rho\, d\phi\, dz = \frac{3^2}{2\pi(.05) l} \ln\left(\frac{5}{3}\right) = \frac{14.64}{l}\ \mathrm{W}$$

We also find the power by taking the product of voltage and current:

$$P = VI = \frac{4.88}{l}(3) = \frac{14.64}{l}\ \mathrm{W}$$

which is in agreement with the power density integration.

Q6:

$$\frac{\partial^2 V}{\partial x^2} = 0$$

$$\frac{d^2 V}{dx^2} = 0$$

$$\frac{dV}{dx} = A$$

$$V = Ax + B$$

To be very general, let $V = V_1$ at $x = x_1$ and $V = V_2$ at $x = x_2$. These values are then substituted into (12), giving

$$V_1 = Ax_1 + B \qquad V_2 = Ax_2 + B$$

$$A = \frac{V_1 - V_2}{x_1 - x_2} \qquad B = \frac{V_2 x_1 - V_1 x_2}{x_1 - x_2}$$

and

$$V = \frac{V_1(x - x_2) - V_2(x - x_1)}{x_1 - x_2}$$

A simpler answer would have been obtained by choosing simpler boundary conditions. If we had fixed $V = 0$ at $x = 0$ and $V = V_0$ at $x = d$, then

$$A = \frac{V_0}{d} \quad B = 0$$

$$\boxed{V = \frac{V_0 x}{d}}$$

1. Given V, use $\mathbf{E} = -\nabla V$ to find $\mathbf{E}$.
2. Use $\mathbf{D} = \epsilon\mathbf{E}$ to find $\mathbf{D}$.
3. Evaluate $\mathbf{D}$ at either capacitor plate, $\mathbf{D} = \mathbf{D}_S = D_N \mathbf{a}_N$.
4. Recognize that $\rho_S = D_N$.
5. Find Q by a surface integration over the capacitor plate, $Q = \int_S \rho_S dS$.

Here we have

$$V = V_0 \frac{x}{d}$$

$$\mathbf{E} = -\frac{V_0}{d}\mathbf{a}_x$$

$$\mathbf{D} = -\epsilon\frac{V_0}{d}\mathbf{a}_x$$

$$\mathbf{D}_S = \mathbf{D}\Big|_{x=0} = -\epsilon\frac{V_0}{d}\mathbf{a}_x$$

$$\mathbf{a}_N = \mathbf{a}_x$$

$$D_N = -\epsilon\frac{V_0}{d} = \rho_S$$

$$Q = \int_S \frac{-\epsilon V_0}{d} dS = -\epsilon\frac{V_0 S}{d}$$

and the capacitance is

$$\boxed{C = \frac{|Q|}{V_0} = \frac{\epsilon S}{d}}$$