Ministry of Higher Education & Scientific Research
University of Technology
Laser & Optoelectronic Eng. Department
Branch: Laser & Optoelectronic Eng.
Class: 2nd Year:
Subject: Mathematics

Final Examination
Time: 3 Hrs
Date: 7/6/ 2016
Examiner: Dr. Razi Al-Azawi
Code: LoPE2201
Academic Year 2015-2016

Attempt only (5) questions

Q1\A) Let $f(x, y, z) = x + \sin(xy) + \cos(xz)$, $x = ts, y = s + t$ and $z = \frac{s}{t}$

Calculate $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$. (6 Marks)

B) If $f(x,y,z) = 2z^3 - 3(x^2 + y^2)z$ prove that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$
(6 Marks)

Q2\ Find $\frac{dw}{dt}$ for the following function (12 Marks)
$w = 2ye^x - \ln(z),\quad x = \ln(t^2 + 1),\quad y = \tan^{-1}(t),\quad z = e^x$ at t=1 by :

1. using chain rule
2. directional method
3. at t=1

Q3\ A) Evaluate the following integral using polar coordinates. (6 Marks)

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \cos\left(x^2 + y^2\right) dx\,dy$$

B) Find the local extreme values of the function (6 Marks)
$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$

Q4\ A)Use undetermined coefficients method to find the general solution of the differential equation (6 Mark)

$$y'' - 2y'' + y = e^x + e^{2x}$$

B) Solve the differential equation: $x\,dy - y = x^2\cos(x)$ (6 Marks)

Q5\ A) Find the general solution of the differential equation (6 Marks)

$$4y'' - 4y' + y = \frac{1}{4}e^{\frac{x}{2}}\sqrt{1 - x^2}$$

B) \ Find the solution of the following differential equation (6 Marks)

$$dx + (xy + y)dy = 0$$

Q6\Find the cross product of $u = 3i - 2j + k$, $v = -3i + 3j + k$ and explain $u \times v$ perpendicular on u and v (12 Marks)

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GOOD LUCK

2015 - 2016 الرياضيات / المرحلة الثانية /

Q1\ A) Let $f(x, y, z) = x + \sin(xy) + \cos(xz)$, $x = ts, y = s + t$ *and* $z = \frac{s}{t}$

Calculate $\frac{\partial f}{\partial s}$ *and* $\frac{\partial f}{\partial t}$

Sol\

1. $$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial f}{\partial x} = 1 + y\cos(xy) - z\sin(xz)\ ,\ \frac{\partial x}{\partial s} = t$$

$$\frac{\partial f}{\partial y} = x\cos(xy)\ ,\ \frac{\partial y}{\partial s} = 1$$

$$\frac{\partial f}{\partial z} = -x\sin(xz)\ ,\ \frac{\partial z}{\partial s} = \frac{1}{t}$$

$$\frac{\partial f}{\partial s} = t\ \ [1 + y\cos(xy) - z\sin(xz)] + x\cos(xy) + \left(\frac{1}{t}\right)(-x\sin(xz))$$

$$\frac{\partial f}{\partial s} = t + ty\cos(xy) - tz\sin(xz) + x\cos(xy) - \frac{x\sin(xz)}{t}$$

2. $$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$\frac{\partial f}{\partial x} = 1 + y\cos(xy) - z\sin(xz)\ ,\ \frac{\partial x}{\partial t} = s$$

$$\frac{\partial f}{\partial y} = x\cos(xy)\ ,\ \frac{\partial y}{\partial t} = 1$$

$$\frac{\partial f}{\partial z} = -x\sin(xz)\ ,\ \frac{\partial z}{\partial t} = \frac{-s}{t^2}$$

$$\frac{\partial f}{\partial t} = s\ \ [1 + y\cos(xy) - z\sin(xz)] + x\cos(xy) + \left(\frac{-s}{t^2}\right)(-x\sin(xz))$$

$$\frac{\partial f}{\partial t} = s + sy\cos(xy) - sz\sin(xz) + x\cos(xy) + \frac{sx\sin(xz)}{t^2}$$

B) B) If $f(x,y,z) = 2z^3 - 3(x^2 + y^2)z$ prove that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$

Sol\ $$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(-6xz) = -6z$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(-6yz) = -6z$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial z}(6z^2 - 3(x^2 + y^2)) = 12z$$

$$\therefore \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = -6z - 6z + 12z = 0$$

Q2/

(1) $$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial t}$$

$$= \frac{2y e^{x}}{t^2+1}(2t) + \frac{2e^{x}}{1+t^2} - \frac{1}{z} e^{t}$$

$$= 2\tan^{-1}(t)\, e^{\ln(t^2+1)} (2t) + \frac{2e^{\ln(t^2+1)}}{1+t^2} - \frac{1}{e^{t}} \cdot e^{t}$$

(2) $$= 2\tan^{-1}(t)\, e^{\ln(t^2+1)} - \ln e^{t}$$

$$= 4\tan^{-1}(t) - \ln e^{t}$$

(3) $$\frac{dw}{dt} = 4\frac{\pi}{4} + 1 = \pi + 1$$

Q3\A) $\int_0^1 \int_0^{\sqrt{1-y^2}} \cos\left(x^2+y^2\right) dxdy$

Sol\

$$0 \le y \le 1$$
$$0 \le x \le \sqrt{1-y^2}$$
$$x = \sqrt{1-y^2}$$

$$0 \le \theta \le \frac{\pi}{2}$$
$$0 \le r \le 1$$

$$dx\,dy = dA = r\,dr\,d\theta$$

and so the integral becomes,

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \cos\left(x^2+y^2\right) dx\,dy = \int_0^{\frac{\pi}{2}} \int_0^1 r\cos\left(r^2\right) dr\,d\theta$$

Note that this is an integral that we can do. So, here is the rest of the work for this integral.

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \cos\left(x^2+y^2\right) dx\,dy = \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin\left(r^2\right)\Big|_0^1 \,d\theta$$
$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin(1)\,d\theta$$
$$= \frac{\pi}{4}\sin(1)$$

B) Find the local extreme values of the function

$$f\ (x,y)\ = xy - x^2 - y^2 - 2x - 2y + 4$$

$$f_x = y - 2x - 2 = 0, \qquad f_y = x - 2y - 2 = 0,$$

or $$x = y = -2.$$

Therefore, the point $(-2,-2)$ is the only point where f may take on an extreme value.

$$f_{xx} = -2, \qquad f_{yy} = -2, \qquad f_{xy} = 1.$$

The discriminant of f at $(a,b) = (-2,-2)$ is

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3$$

The combination $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ tells us that f has a local maximum at $(-2,-2)$. The value of f at this point is $f(-2,-2) = 8$.

Q5\A) $y = y_h + y_p$

$4y'' - 4y' + y = 0$

$4D^2 - 4D + 1 = 0 \Rightarrow (2D-1)^2 = 0 \Rightarrow r_1 = r_2 = \frac{1}{2}$

$\Rightarrow y_h = C_1 e^{\frac{1}{2}x} + C_2 x e^{\frac{1}{2}x}$

$\therefore u_1 = e^{x/2} \quad , u_2 = x e^{x/2}$

$$v_1' = \frac{\begin{vmatrix} 0 & x e^{x/2} \\ 16 e^{x/2}\sqrt{1-x^2} & \frac{1}{2} x e^{x/2} + e^{x/2} \end{vmatrix}}{\begin{vmatrix} e^{x/2} & x e^{x/2} \\ \frac{1}{2} e^{x/2} & \frac{1}{2} x e^{x/2} + e^{x/2} \end{vmatrix}} = -\frac{1}{16} \frac{x e^{x}\sqrt{1-x^2}}{e^{x}} = -\frac{1}{16} x\sqrt{1-x^2}$$

$$v_2' = \frac{\begin{vmatrix} e^{x/2} & 0 \\ \frac{1}{2} e^{x/2} & 16 e^{x/2}\sqrt{1-x^2} \end{vmatrix}}{\begin{vmatrix} e^{x/2} & x e^{x/2} \\ \frac{1}{2} e^{x/2} & \frac{1}{2} x e^{x/2} + e^{x/2} \end{vmatrix}} = \frac{1}{16}\sqrt{1-x^2}$$

$v_1 = \int v_1' \, dv = \frac{1}{48}(1-x^2)^{3/2} \quad , v_2 = \int v_2' \, du = \frac{1}{32}\left[x\sqrt{1-x^2} + \sin^{-1} x\right]$

$\Rightarrow y_p = \frac{1}{48} e^{x/2} (1-x^2)^{3/2} + \frac{x e^{x/2}}{32}\left[x\sqrt{1-x^2} + \sin^{-1} x\right]$

B) \ Find the solution of the following differential equation (6 Marks)

$$dx + (xy + y)dy = 0$$

Sol\

$$dx + (xy + y)dy = 0 \Rightarrow dx + (x+1)ydy = 0 \Rightarrow ydy = -\frac{dx}{(x+1)} \Rightarrow \frac{y^2}{2} = -\ln|x+1| + c$$

$$y^2 = -2\ln|x+1| + 2c \Rightarrow y = \sqrt{-2\ln|x+1| + 2c}$$

Q4\A) Use undetermined coefficients method to find the general solution of the differential equation .

$y'' - 2y'' + y = e^{x} + e^{2x}$

$y'' - 2y' + y = 0 \Rightarrow D^2 - 2D + 1 = 0 \Rightarrow (D-1)^2 = 0$

$r_1 = r_2 = 1 \Rightarrow y_h = (C_1 + C_2 x)\, e^{x}$

$y_p = A x^2 e^{x} + B e^{2x}$

$y_p' = 2Axe^{x} + Ax^2e^{x} + 2Be^{2x}$

$y_p'' = 2Ae^{x} + 2Axe^{x} + Ax^2e^{x} + 2Axe^{x} + 4Be^{2x}$

Then

$= 2Ae^{x} + 2Axe^{x} + Ax^2e^{x} + 2Axe^{x} + 4Be^{2x} - 2Ax^2e^{x}$
$-4Be^{2x} + Ax^2e^{x} + Be^{2x}$

$\Rightarrow 2Ae^{x} + Be^{2x} = e^{x} + e^{2x} \Rightarrow A = \frac{1}{2}, \; B = 1$

$\therefore y_p = \frac{1}{2}x^2e^{x} + e^{2x}$

$\therefore y = (C_1 + C_2x)e^{x} + \frac{1}{2}x^2e^{x} + e^{2x}$

B) $x\dfrac{dy}{dx} - y = x^2 \cos x$

$\dfrac{dy}{dx} - \dfrac{1}{x}y = x\cos x$

$\therefore \dfrac{dy}{dx} + P(x)\,y = Q(x)$

$P(x) = -\dfrac{1}{x}, \quad Q(x) = x\cos x$

$\rho(x) = e^{\int P(x)dx} = e^{\int -\frac{dx}{x}} = e^{-\ln x} = \dfrac{1}{x}$

$\therefore y = \dfrac{1}{\rho(x)}\int \rho(x)Q(x)\,dx$

$y = x\int \cos x\,dx = \sin x + C$

Q6\

$$u \times v = \begin{vmatrix} i & j & k \\ 3 & -2 & 1 \\ -3 & 3 & 1 \end{vmatrix}$$

$$= i(-2-3) - j(3+3) + k(9-6)$$

$$= -5i - 6j + 3k$$