University of Technology
Department of Laser & Optoelectronic Engineering
Final Examination 2011-2012

Subject: Mathematics
Division: Laser & Optoelectronic
Examiner: Dr. Mohammed Abdul-Redha

Class: 2nd year
Time: 3 hours
Date: 7 / 6 /2012

Answer five questions only

(Six marks for every branch)

Q (1): a) Find the area bounded by $(r = 0)$ **to** $(r = \sqrt{2(1 + \cos\theta)})$ **for** (θ) **from 0 to** 2π

b) Find the area of the region bounded by the parabola $y = 2x^2$ **and the line** $y = 2$

Q (2): a) If $w = x^2 + xyz + z^2$, **and**

$$x = e^{2rs} + s^2, \quad y = r^2 - \tan(s), \quad z = e^{rs}, \quad \text{Find } \frac{\partial w}{\partial r}$$

(b): Find the local extreme values of the function:

$$f(x, y) = x^2 - 2y - 2xy$$

Q (3): a) Evaluate the integral $\int_{-1}^{1}\int_{0}^{y} y^2 e^{xy}\,dx\,dy$

b) Find in power series for the third order the function: $f(x) = \sqrt{x+1}$

Q (4): a) Find the equation of the plane that has the points $P_1\,(1, 2, 3)$, $P_2\,(2, 0, 1)$ **and** $P_3\,(1, 1, 0)$.

b) Find $\nabla . \vec{A}$ **for the vector**

$$\vec{A} = \vec{\imath}\, xyz + \vec{\jmath}\, ze^x + \vec{k}(x + y + z)$$

Q (5): a) Solve using power series to third order the differential equation:

$$y'' + xy = 0$$

b) Determine the convergence or divergence of: $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

Q (6): a) Solve the differential equation $x\frac{dy}{dx} + y = \sin(x)$

b) Find Fourier series for the following function in sine form (odd function):

$f(x) = 2 \quad for \quad 0 < x < 1$

حلول الاسئلة (2)

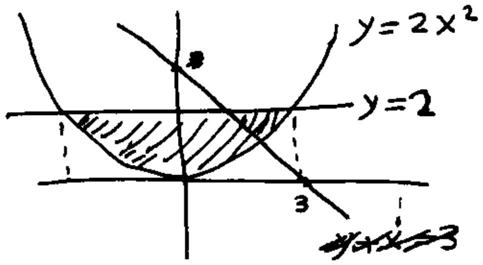
Q1)a):

$$r=\sqrt{2(1+\cos\theta)}$$

$$A=\int_0^{2\pi}\int_0^{r} r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{2}r^2\Big)_0^{\sqrt{2(1+\cos\theta)}} d\theta$$

$$=\frac{1}{2}\int_0^{2\pi} 2(1+\cos\theta)\,d\theta = \int_0^{2\pi}(1+\cos\theta)\,d\theta = \theta+\sin\theta\Big)_0^{2\pi}$$

$$=(2\pi+\sin 2\pi)-(0+\sin 0) = 2\pi$$



b) $$A=\int_{-1}^{1}\int_{2x^2}^{2} dy\,dx$$

The intersection points .

$2x^2=2$; $x=\mp 1$

$$\therefore A=\int_{-1}^{1} y\Big)_{2x^2}^{2} dx = \int_{-1}^{1}(2-2x^2)\,dx$$

$$=2\int_{-1}^{1}(1-x^2)\,dx = 2\left[x-\frac{x^3}{3}\right]_{-1}^{1}$$

$$=2\left[\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right]=2\left(1-\frac{1}{3}+1-\frac{1}{3}\right)=2\left(2-\frac{2}{3}\right)=2\,\frac{6-2}{3}$$

$$=\frac{8}{3}=2.666$$

Q2)a) $w=x^2+xyz+z^2$

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\therefore \frac{\partial w}{\partial x}=2x+yz \quad ; \quad \frac{\partial x}{\partial r}=2s\,e^{2rs}$$

$$\frac{\partial w}{\partial y}=xz \quad ; \quad \frac{\partial y}{\partial r}=2r$$

$$\frac{\partial w}{\partial z}=xy+2z \quad ; \quad \frac{\partial z}{\partial r}=s\,e^{rs}$$

$$\therefore \frac{\partial w}{\partial r}=2s(2x+yz)e^{2rs}+2xzr+s\,e^{rs}(xy+2z)$$

Q2) b)

$f = x^2 - 2y - 2xy$

$f_x = 2x - 2y \Rightarrow y = x$

$f_y = -2 - 2x \Rightarrow x = -1, \; y = -1$ $\quad \therefore$ The point $(-1, -1)$

$f_{xx} = 2, \quad f_{yy} = 0, \quad f_{xy} = -2$

$f_{xx} f_{yy} - f_{xy}^2 = 2(0) - (-2)^2 = -4$

$\therefore$ since $f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 < 0$

$\therefore$ The point $(-1, -1)$ is ~~maximum~~ minimum

Q3) a)

$$\int_{-1}^{1} \int_{0}^{y} y \, e^{xy} \, dx \, dy = \int_0^1 y \frac{e^{xy}}{y} \Big|_0^{y} dy$$

$$\cancel{= \int y^2 (e^{y^3} - 1) dy = \int_0^1 (y^2 e^{y^3} - y^2) dy}$$

$$\cancel{= \frac{1}{3} e^{y^3} - \left(\frac{y^3}{3}\right)\Big|_0^1 = \left(\frac{1}{3} e^{27} - \frac{1}{3}\right) - \left(\frac{1}{3}\right)}$$

$$\cancel{= \frac{1}{3} e^{27} - \frac{2}{3}} = \int_0^1 y e^{xy} \Big|_0^{y} dy = \int_0^1 (y e^{y^2} - y) \, dy$$

$$\cancel{= (e^y - y)\Big|_0^1 = (e - 1) - (1 - 0) = e - 2}$$

$$= \int_{-1}^{1} (y e^{y^2} - y) \, dy = \frac{1}{2} e^{y^2} - \frac{y^2}{2} \Big|_{-1}^{1}$$

$$= \left(\frac{1}{2} e^1 - \frac{1}{2}\right) - \left(\frac{1}{2} e^1 - \frac{1}{2}\right) = 0$$

Q3) b)

$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \cdots$

$f(x) = (x+1)^{\frac{1}{2}} \quad ; \quad f(0) = 1$

$f'(x) = \frac{1}{2}(x+1)^{-\frac{1}{2}} \quad ; \quad f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(x+1)^{-\frac{3}{2}} \quad ; \quad f''(0) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8}(x+1)^{-\frac{5}{2}} \quad ; \quad f'''(0) = \frac{3}{8}$

$$\therefore f(x) = 1 + \frac{1}{2} \frac{x}{1!} - \frac{1}{4} \frac{x^2}{2!} + \frac{3}{8} \frac{x^3}{3!} + \cdots$$

Q4)a)

$\overrightarrow{P_1P_2} = (1-2)i + (2-0)j + (3-1)K = -1+2j+2K$

$\overrightarrow{P_2P_3} = (2-1)i + (0-1)j + (1-0)K = i - j + K$

The normal vector is:

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_2P_3} = \begin{vmatrix} i & j & K \\ -1 & 2 & 2 \\ 1 & -1 & 1 \end{vmatrix} = 4i+3j-K$$

$\therefore a=4,\ b=3,\ c=-1$

$x_0=1,\ y_0=2,\ z_0=3$

$\therefore$ The plane is:

$ax+by+cz = ax_0+by_0+cz_0$

$4x+3y-z = 4+6-3 = 7$

Q4)b)

$\vec{A} = ixyz + jze^{x} + K(x+y+z)$

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial x} = (yz)+(0)+(1)$$

$= yz+1$

Q5)a) $y'' + y\cancel{x} = 0$

let $y = \sum a_n x^n \Rightarrow \sum_{n=0} n(n-1)a_n x^{n-2} + x\sum a_n x^n = 0$

$\therefore \sum_{n=0} n(n-1)a_n x^{n-2} + \sum_{0} a_n x^{n+1} = 0$

$\sum_{-2} (n+1)(n+2)a_{n+2} x^n + \sum_{1} a_{n-1} x^n = 0$

$\sum_{-2} \left[(n+1)(n+2)a_{n+2} + a_{n-1}\right] x^n = 0$

$0+0+2a_2 + \sum_{1} [(n+1)(n+2)a_{n+2} + a_{n-1}]$

$\therefore a_2 = 0$

$n=1$: $2\cdot3\, a_3 + a_0 = 0 \Rightarrow a_3 = -\frac{a_0}{2\cdot3}$

$n=2$: $3\cdot4\, a_4 + a_1 = 0 \Rightarrow a_4 = -\frac{a_1}{3\cdot4}$

$n=3$: $4\cdot5\, a_5 + a_2 = 0 \Rightarrow a_5 = 0$

$\therefore y = a_0 + a_1 x - \frac{a_0}{2\cdot3}x^3 - \frac{a_1}{3\cdot4}x^4 + \cdots$

Q5) b) $f(x) = (x+1)^3$

$f(x) = (x+1)^3$; $f(0) = 1$

$f'(x) = 3(x+1)^2$; $f'(0) = 3$

$f''(x) = 6(x+1)$; $f''(0) = 6$

$f'''(x) = 6$; $f'''(0) = 6$

$\therefore f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$

$= 1 + \frac{x}{1}(3) + \frac{x^2}{2}(6) + \frac{x^3}{6}(6) + 0$

$= 1 + 3x + 3x^2 + x^3$

Q5) b): $a_n = \frac{x^n}{n!}$, $a_{n+1} = \frac{x^{n+1}}{(n+1)!}$

$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}\, n!}{(n+1)!\, x^n} = \frac{x}{(n+1)}$

$\therefore \rho = \lim_{n\to\infty} \left(\frac{x}{n+1}\right) = 0$ $\therefore$ the series is convergence

since $\rho < 1$

Q6) a): $x\frac{dy}{dx} + y = \frac{\sin x}{x}$

$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = \left(\frac{\sin x}{x^2}\right)$

$\therefore P = \frac{1}{x}$; $Q = \frac{\sin x}{x^2}$

$\therefore v = e^{\int p\,dx} = e^{\int \frac{1}{x}dx} = x$

$y = \int vQ\,dx$; $xy = \int x \frac{\sin x}{x} dx = \int \sin x\, dx = -\cos x + C$

$\therefore y = -\frac{\cos x}{x} + \frac{C}{x}$

Q6) b)

since the function is odd, then:

$a_0 = 0$, $a_n = 0$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{\pi n x}{L}\,dx$$

$2L = 4$, $L = 2$

$$\therefore b_n = \frac{2}{2}\int_0^2 2\cdot\sin\frac{\pi n x}{2}\,dx = 4\int_0^2 \sin\left(\frac{\pi n x}{2}\right)dx = -4\cos\left(\frac{\pi n x}{2}\right)\Big)_0^2$$

$$= -4\left[\cos\pi n - 1\right] = 4(1-\cos\pi n)$$

$$\therefore f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{\pi n x}{L} + b_n\sin\frac{\pi n x}{L}$$

$$= \sum_{1}^{\infty} 4(1-\cos\pi n)\sin\left(\frac{\pi n x}{2}\right)$$