University of Technology

Department of Laser & Optoelectronic Engineering

Final Examination 2011-2012

Subject: Quantum Mechanics — *Class: 3^{rd} year*

Division: Laser — *Time: 3 hours*

Examiner: Dr. Mohammed Abdul-Redha — *Date:29/5 /2012*

Answer five questions only

(12 Marks for every question)

Q (1): a) A photon with wavelength ($5x10^{-12}$ m) hit an electron due to Compton scattering, after the collision, the photon wavelength become ($7\ x10^{-12}$ m). Find the angle of the scattering photon.

b) A photon has energy (15 eV) hit a material causing eject an electron with velocity ($2x10^6$ m/sec). Find the work function of the material.

Q (2): a) The stationary state for a particle in a box of length (1 nm) is:

$$\psi = \sqrt{x}\ e^{-\frac{x^2}{2}}$$ **, Find the probability of finding this particle in the range (0 to 0.5 nm)**

b) For an operator $\widehat{P} = -i\hbar\frac{\partial}{\partial x}$, find the Eigen value of that operator for the wave function: $\psi = A\ e^{i(kx-wt)}$

Q (3): a) Find the commutation relation of $[L_y, z]$

b) Find the reflectance of a particle hit a barrier surface, where the wave function of the particle is: $\psi = 6\ e^{2ikx} + 3\ e^{-2ikx}$

Q (4): Find the complete wave function $\psi_{210}\ (r,\theta,\emptyset)$ of Hydrogen atom.

Q (5): a) Find the normalized solution and energy of particle move in oscillating field $\psi_2(x)$.

b) Prove that the quantum number of hydrogen atom (m) must be less than the quantum number (l).

Q (6): Derive the complete solution in one dimension of particle moving in a potential box, and the state energy:

$$V = \begin{cases} \infty & for\ x < 0 \\ V_0 & for\ a > x > 0 \\ \infty & for\ x > a \end{cases} \quad \text{where } E > V_0$$

Note (h=$6.6x10^{-34}$ J.sec, m_e=$9.1x10^{-31}$ kg, c=$3x10^8$ m/sec, e=$16x10^{-20}$ col)

ا.د.محمد عبد الرضا

الحلول النموذجية / أسئلة الميكانيك الكمي

Q1) a): $\lambda' - \lambda = \frac{h}{m_e c}(1-\cos\theta)$

$$7\times10^{-12} - 5\times10^{-12} = \frac{6.6\times10^{-34}}{9.1\times10^{-31}\times 3\times10^{8}}(1-\cos\theta)$$

$$2\times10^{-12} = 2.43\times10^{-12}\,(1-\cos\theta)$$

$$\therefore 1-\cos\theta = \frac{2\times10^{-12}}{2.43\times10^{-12}} =$$

$$\therefore \cos\theta = 1-0.823 \quad ; \cos\theta = 0.1769$$

$$\therefore \theta = 79.81^{\circ}$$

b) $E = T + W$

$$hf = \frac{1}{2}mv^2 + W$$

$$\therefore 15\times16\times10^{-20} = \frac{1}{2}\times9.1\times10^{-31}(2\times10^{6})^2 + W$$

$$2.4\times10^{-18} = 1.82\times10^{-18} + W$$

$$\therefore W = 0.58\times10^{-18}\ \text{J}$$

$$= \frac{0.58\times10^{-18}}{16\times10^{-20}}\ \text{eV} = 3.625\ \text{eV}$$

Q2) a)

$$P = \int_0^{0.5} \psi \psi^* dx = \int_0^{0.5} \sqrt{x}\, e^{-\frac{x^2}{2}} . \sqrt{x}\, e^{-\frac{x^2}{2}} dx .$$

$$= \int x e^{-x^2} dx = -\tfrac{1}{2} e^{-x^2} \Big|_0^{0.5} = -\tfrac{1}{2}\left(e^{-0.5^2} - 1\right)$$

$$= -\tfrac{1}{2}(0.77 - 1) = 0.11 = 11\ \%$$

b) $\hat{p}\psi = a\psi$ $\qquad a = $ eigen value.

$$\therefore\ -i\hbar \frac{\partial \psi}{\partial x} = -i\hbar \frac{\partial}{\partial x}\left[A e^{i(kx-wt)}\right]$$

$$= -i\hbar A e^{i(kx-wt)} . ik = \hbar k A e^{i(kx-wt)}$$

$$= \hbar k \psi$$

$$\therefore\ a = k\hbar$$

Q3) a)

$$[\hat{L}_y, \hat{z}] = [\hat{L}_y \hat{z} - \hat{z}\hat{L}_y]$$

since: $\hat{L}_y = \hat{z}\hat{P}_x - \hat{x}\hat{P}_z$

$$\therefore [\hat{L}_y, \hat{z}] = (\hat{z}\hat{P}_x - \hat{x}\hat{P}_z)\hat{z} - \hat{z}(\hat{z}\hat{P}_x - \hat{x}\hat{P}_z)$$

$$= \hat{z}\hat{P}_x\hat{z} - \hat{x}\hat{P}_z\hat{z} - \hat{z}\hat{z}\hat{P}_x + \hat{z}\hat{x}\hat{P}_z$$

$$\therefore [\hat{L}_y, \hat{z}]\psi = \hat{z}\hat{P}_x\hat{z}\psi - \hat{x}\hat{P}_z\hat{z}\psi - \hat{z}\hat{z}\hat{P}_x\psi + \hat{z}\hat{x}\hat{P}_z\psi$$

$$= \hat{z}\hat{P}_x(z\psi) - \hat{x}\hat{P}_z(z\psi) - \hat{z}\hat{z}\left(-i\hbar\frac{\partial\psi}{\partial x}\right) + \hat{z}\hat{x}\left(-i\hbar\frac{\partial\psi}{\partial z}\right)$$

$$= \hat{z}(-i\hbar)\frac{\partial z\psi}{\partial x} - \hat{x}(-i\hbar)\frac{\partial z\psi}{\partial z} + i\hbar z^2\frac{\partial\psi}{\partial x} - i\hbar zx\frac{\partial\psi}{\partial z}$$

$$= -i\hbar z^2\frac{\partial\psi}{\partial x} + i\hbar x\left(z\frac{\partial\psi}{\partial z} + \psi\right) + i\hbar z^2\frac{\partial\psi}{\partial x} - i\hbar zx\frac{\partial\psi}{\partial z}$$

$$= i\hbar x\psi$$

$$\therefore [\hat{L}_y, \hat{z}] = i\hbar x$$

b) ~~~~ If $\psi = Ae^{iakx} + Be^{-iakx}$

$$\therefore \text{Reflectance} = R = \frac{|B|^2}{|A|^2} =$$

$$= \frac{|3|^2}{|6|^2} = \frac{9}{36} = \frac{1}{4}$$

$$= 0.25 = 25\%$$

Q4) $\psi_{210}(r,\theta,\phi) = ?$

$n=2,\ l=1,\ m=0$

$\psi_{nlm} = R_{nl} \cdot A_{ml} \cdot B_m$

$\therefore \psi_{210} = R_{21} \cdot A_{01} \cdot B_0$

since: $R_{nl} = \frac{1}{r} \rho^{l+1} e^{-\rho} v(\rho)$

and $v(\rho) = \sum_{j=0}^{j_{max}} a_j \rho^j$, $j_{max} = n-l-1$

$\therefore j_{max} = 2-1-1 = 0$

$\therefore v(\rho) = \sum_0^0 a_j \rho^j = a_0$

$\therefore R_{2\,l} = \frac{1}{r} \rho^2 e^{-\rho} a_0$, $\rho = \frac{r}{na} = \frac{r}{2a}$

$\therefore R_{21} = \frac{1}{r}\left(\frac{r^2}{4a^2}\right) e^{-\frac{r}{2a}} a_0$

$\therefore R_{21} = \left(\frac{a_0}{4a^2}\right) r e^{-\frac{r}{2a}}$

Then find A_{01}:

$A_{ml} = C P_l^m(\cos\theta)$, $P_l^m(x) = (1-x^2)^{\frac{|m|}{2}} \frac{d^m}{dx}(P_l(x))$

and $P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx}(x^2-1)^l$

$\therefore P_1(x) = \frac{1}{2^1 1!} \frac{d}{dx}(x^2-1) = x$

$\therefore P_1^0(x) = (1-x^2)^0 \frac{d^0}{dx}(x) = x$

$\therefore A_{ml} = C \cdot \cos\theta$

Then find: B_m

$B_m = e^{-im\phi} \Rightarrow B_0 = e^0 = 1$

$\therefore \psi_{210} = R_{21} \cdot A_{01} \cdot B_0$

$= \frac{a_0}{4a^2} r e^{-\frac{r}{2a}} \cdot C \cdot \cos\theta \cdot 1$

$= \left(\frac{Ca_0}{4a^2}\right) r\cos\theta\, e^{-\frac{r}{2a}}$

Q5) a) $\psi_2(x) = ?$

$$\psi_n(x) = C_n H_n(y) \cdot e^{-\frac{y^2}{2}} \quad , y = x\sqrt{\frac{m\omega}{\hbar}}$$

$$C_n = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \quad , H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy}\left(e^{-y^2}\right)$$

since: $n = 2$

$$\therefore H_2(y) = e^{y^2} \frac{d^2}{dy^2}\left(e^{-y^2}\right) = e^{y^2} \frac{d}{dy}\left[-2y\, e^{-y^2}\right]$$

$$= -2\, e^{y^2}\left[-2y^2 e^{-y^2} + e^{-y^2}\right] = 4y^2 - 2$$

and: $C_2 = \frac{1}{\sqrt{4 \cdot 2}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} = \frac{1}{2\sqrt{2}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}$

$$\therefore \psi_2(y) = \frac{1}{2\sqrt{2}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} (4y^2 - 2) \cdot e^{-\frac{y^2}{2}}$$

change y to x:

$$\psi_2(x) = \frac{1}{\sqrt{2}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \left(2\frac{m\omega}{\hbar}x^2 - 1\right) e^{-\frac{x^2 m\omega}{2\hbar}}$$

$$\text{Energy} = \left(2n + \frac{1}{2}\right)\hbar\omega = \left(2 + \frac{1}{2}\right)\hbar\omega = \frac{5}{2}\hbar\omega$$

b) $\underline{m < \ell}$

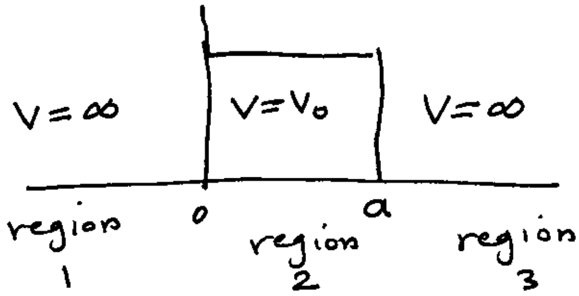
since legendre's polynomial $P_\ell(x)$ have the degree of (ℓ). Then there is no derivative ~~when~~ for the degree greater than ℓ, which means:

$$\frac{d^m}{dx}(P_\ell) = 0 \quad \text{when } m > \ell$$

That will let the wave function to be zero.

Then $m < \ell$ is the only possible solution for finding ψ.

Q6)



$E > V_0$

There are no solutions in regions 1 and 3

Then only in region 2:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + (V-E)\psi = 0$$

or $$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - (E-V_0)\psi = 0$$

or $$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E-V_0)\psi = 0$$

let $\frac{2m}{\hbar^2}(E-V_0) = K^2$

$\therefore \frac{d^2\psi}{dx^2} + K^2\psi = 0$ which give: $\psi = Ae^{iKx} + Be^{-iKx}$

The boundary conditions:

at $x=0$, $\psi = 0$

$\therefore 0 = A + B \Rightarrow B = -A$; $\therefore \psi = A(e^{iKx} - e^{-iKx})$

or $\psi = 2iA\sin Kx$

The boundary condition:

at $x = a$, $\psi = 0 \Rightarrow 0 = 2iA\sin Ka$

$\therefore Ka = \pi n \quad n = 0, 1, 2, \ldots$

$\therefore \psi = 2iA\sin\left(\frac{\pi n x}{a}\right)$

and has energy:

$Ka = \pi n \Rightarrow K^2a^2 = \pi^2 n^2 \Rightarrow K^2 = \frac{\pi^2 n^2}{a^2}$

$$\frac{2m(E-V_0)}{\hbar^2} = \frac{\pi^2 n^2}{a^2} \Rightarrow E - V_0 = \left(\frac{\pi^2\hbar^2}{2ma^2}\right)n^2$$

$$\therefore E = V_0 + \left(\frac{\pi^2\hbar^2}{2ma^2}\right)n^2$$