Advance

Calculuse

the first course

BioMedical Engineering Dept.

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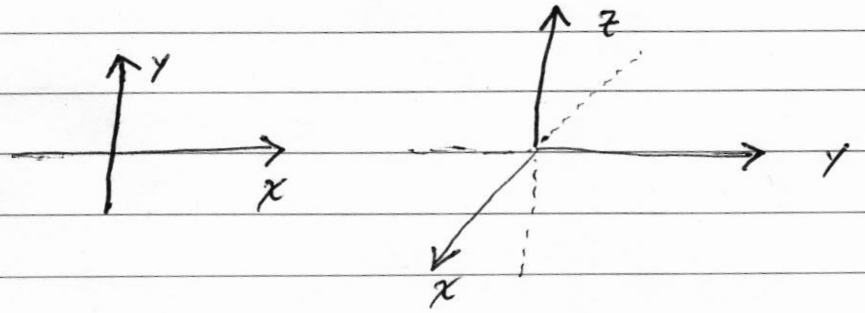
2016 - 10 - 3

2016 - 10 - 5

Vectors



The vector $\vec{V} = iV_1 + jV_2 = \langle V_1, V_2 \rangle$

$\vec{V} = iV_1 + jV_2 + kV_3 = \langle V_1, V_2, V_3 \rangle$

where V_1, V_2 are V components in R^2 space

and V_1, V_2, V_3 " " " " R^3 " .

~~where~~ The unit vector $\vec{u} = \langle 0, 1 \rangle$ or

$\vec{u} = \langle 1, 0 \rangle$ or

$\vec{u} = \langle \cos\theta, \sin\theta \rangle$

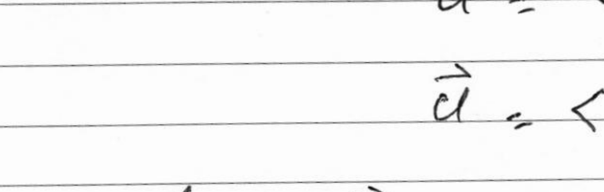
The length of vector $\|\vec{V}\|$ (norm of V)

$\|\vec{V}\| = \sqrt{V_1^2 + V_2^2}$ or $\|\vec{V}\| = \sqrt{V_1^2 + V_2^2 + V_3^2}$

ex

$\vec{V} = \langle 3, 2 \rangle = 3i + 2j$

$\vec{u} = \langle -2, 1 \rangle = -2i + j$



<u>note</u>

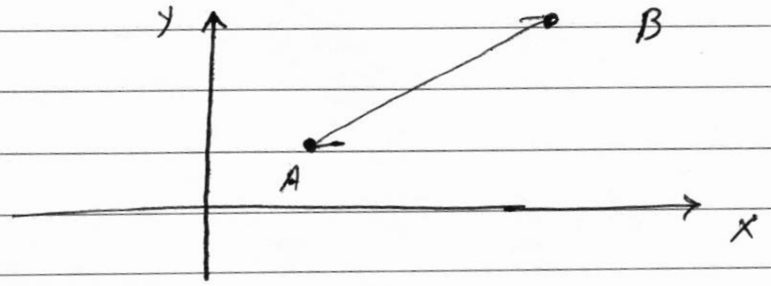
The vector $\vec{v}$ between any two point $A = (a_1, a_2)$

$B = (b_1, b_2)$

$$\vec{V} = \vec{AB} = \langle b_1 - a_1 , b_2 - a_2 \rangle$$

$A = (1, 1)$

$B = (3, 4)$



$$\vec{V} = \overline{AB} = \langle 3-1 , 4-1 \rangle = \langle 2, 3 \rangle$$

$$\vec{u} = \overrightarrow{BA} = \langle 1-3 , 1-4 \rangle = \langle -2, -3 \rangle$$

clear That

$$\overrightarrow{u} \neq \overrightarrow{V}$$

but

$$\| u \| = \| V \| \quad \text{since}$$

$$\| u \| = \sqrt{-2^2 + -3^2} = \sqrt{13}$$

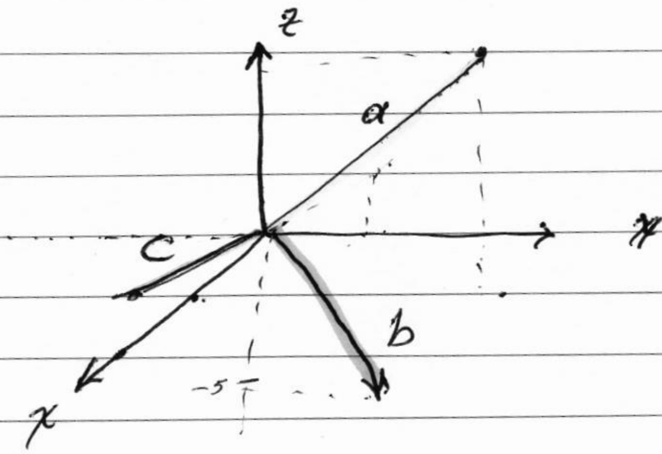
$$\| V \| = \sqrt{2^2 + 3^2} = \sqrt{13}$$

ex in ($\mathbb{R}^3$)

let $\vec{a} = \langle 2, 6, 7 \rangle \Longrightarrow \|a\| = \sqrt{89}$

$\vec{b} = \langle -2, 4, -5 \rangle \Longrightarrow \|b\| = \sqrt{45}$

$\vec{c} = \langle 4, 0, 1 \rangle \Longrightarrow \|c\| = \sqrt{17}$

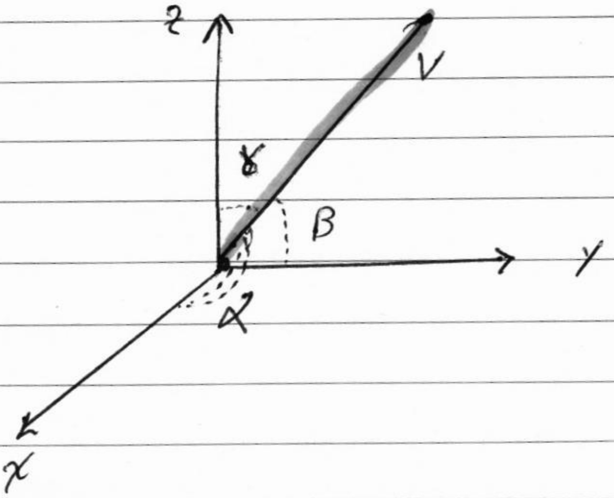


The angle in $\mathbb{R}^3$ is compute as follows :-

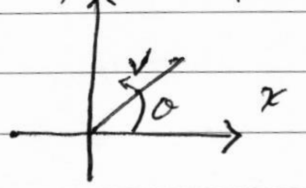
$$\cos\alpha = \frac{V_1}{\|V\|}$$

$$\cos\beta = \frac{V_2}{\|V\|}$$

$$\cos\gamma = \frac{V_3}{\|V\|}$$



While in R^2



$0 < \theta < \pi$

ante clocwise

Operations on vectors

① Addition

let $\bar{A} = \langle a_1, a_2 \rangle$ or $\langle a_1, a_2, a_3 \rangle$

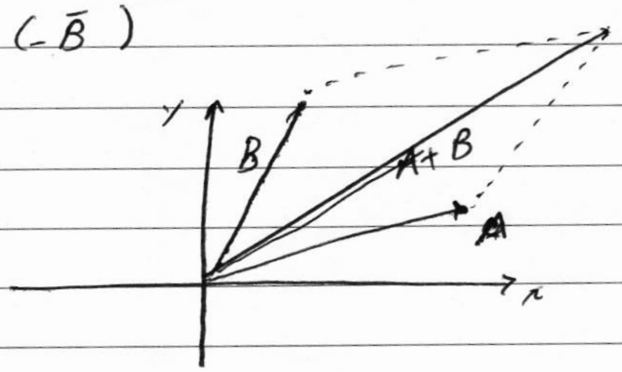
$\bar{B} = \langle b_1, b_2 \rangle$ or $\langle b_1, b_2, b_3 \rangle$

if They have The same intial point Then :-

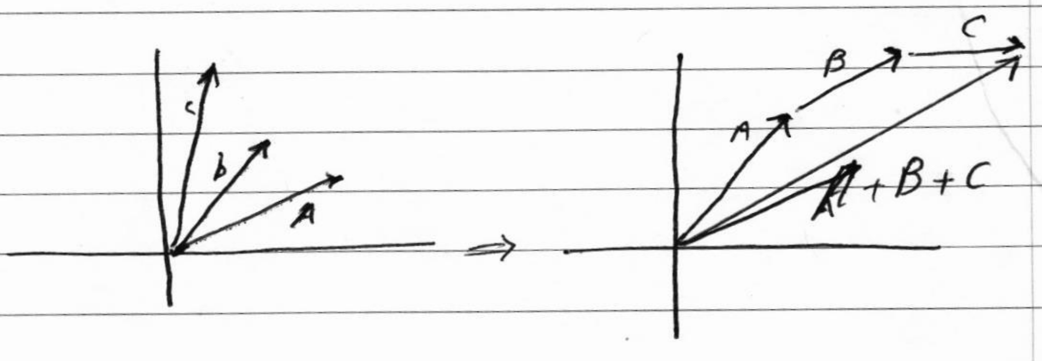
define $\bar{A} + \bar{B} = \langle a_1 + b_1, a_2 + b_2 \rangle$ or $\langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

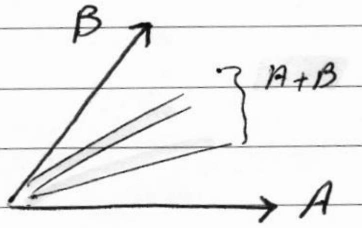
and $\bar{A} - \bar{B} = \bar{A} + (-\bar{B})$

Geomatrically

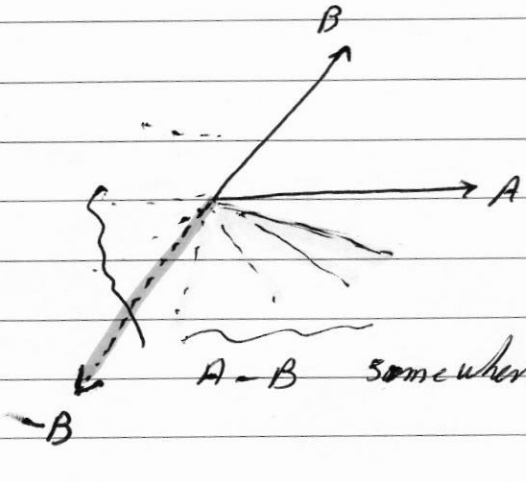


or





some where
between
Them



A - B somewhere between A and -B

2) Product

a) dot product "."

$\bar{A} = \langle a_1, a_2 \rangle$
$\bar{B} = \langle b_1, b_2 \rangle$

$\Rightarrow \bar{A} \cdot \bar{B} = a_1 \cdot b_1 + a_2 \cdot b_2$

$\bar{A} \cdot \bar{B} = \|A\| \cdot \|B\| \cos\theta$

ex

let $\bar{A} = \langle 2, -3 \rangle$, $\bar{B} = \langle 1, 2 \rangle$. find a) $\bar{A} \cdot \bar{B}$ b) angle betwen Them

a)

$$\bar{A} \cdot \bar{B} = (2 \cdot 1) + (-3 \cdot 2) = -4$$

b)

$$-4 = \sqrt{13} \cdot \sqrt{5} \cos\theta$$

$$\Rightarrow \cos\theta = \frac{-4}{\sqrt{65}} \Longrightarrow \theta = \cos^{-1}\left(-\frac{4}{\sqrt{65}}\right)$$

ex2 prove That: $\bar{A} \perp \bar{B}$ if and only if $\bar{A} \cdot \bar{B} = 0$

sol:

let $\bar{A} \perp \bar{B} \Rightarrow \theta = 90^\circ \Rightarrow \bar{A} \cdot \bar{B} = \|A\| \cdot \|B\| \cdot \cos 90$
$= 0$

Now let $\bar{A} \cdot \bar{B} = 0$

$\Rightarrow \|A\| \cdot \|B\| \cos\theta = 0$

$\Rightarrow \cos\theta = 0 \Rightarrow \theta = \cos^{-1} 0 = 90^\circ \Rightarrow \bar{A} \perp \bar{B}$

(b) Croos product "$\times$"

$\bar{A} = \langle a_1, a_2, a_3 \rangle$

$\bar{B} = \langle b_1, b_2, b_3 \rangle$ Then

$$\bar{A} \times \bar{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

ex let $\bar{u} = \langle 1, 2, -2 \rangle$

$\bar{V} = \langle 3, 0, 1 \rangle$

$$\bar{u} \times \bar{V} = \begin{vmatrix} i & j & k \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix}$$

$$= i \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} - j \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + k \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}$$

$$= 2i - 7j - 6k$$

properties of croos product

1. $\bar{A} \times \bar{B} \perp \bar{A}$ and $\bar{A} \times \bar{B} \perp \bar{B}$
2. $\bar{A} \times \bar{B} = 0 \iff \bar{A} \parallel \bar{B}$ $(\bar{A} \times \bar{A} = 0)$
3. $\bar{A} \times \bar{B} = -(\bar{B} \times \bar{A})$ i.e $\bar{A} \times \bar{B} \neq \bar{B} \times \bar{A}$
4. $\bar{u} \times (\bar{A} + \bar{B}) = \bar{u} \times \bar{A} + \bar{u} \times \bar{B}$
5. $(\bar{A} \times \bar{B}) \cdot 5 = 5\bar{A} \times \bar{B} = \bar{A} \times 5\bar{B}$

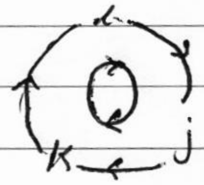
notes for croos prodects

(1)

$i \times j = K$, $j \times i = -K$

$j \times K = i$, $K \times j = -i$

$K \times i = j$, $i \times K = -j$



(2) $\|\bar{A} \times \bar{B}\|$ = Area of ~~Trapezium~~ equiletral Generated by $\bar{A}$ and $\bar{B}$.

(3)

$$\|\bar{A} \times \bar{B}\|^2 = \|A\|^2 \|B\|^2 - (\bar{A} \cdot \bar{B})^2$$

~~$A \times B$ = Area of Trapezium between $\bar{A}$ and $\bar{B}$~~

Lagrangs identity

properties of dot product

(1) $i \cdot i = j \cdot j = K \cdot K = 1$ & $i \cdot j = i \cdot K = j \cdot K = 0$

(2) $\bar{u} \cdot \bar{V} = \bar{V} \cdot \bar{u}$

(3) $\bar{u} \cdot (\bar{A} + \bar{B}) = \bar{u} \cdot \bar{A} + \bar{u} \cdot \bar{B}$

(4) $s \cdot (\bar{A} \cdot \bar{B}) = s\bar{A} \cdot \bar{B} = \bar{A} \cdot s\bar{B}$

(5) $\bar{A} \cdot \bar{A} = \|A\|^2$

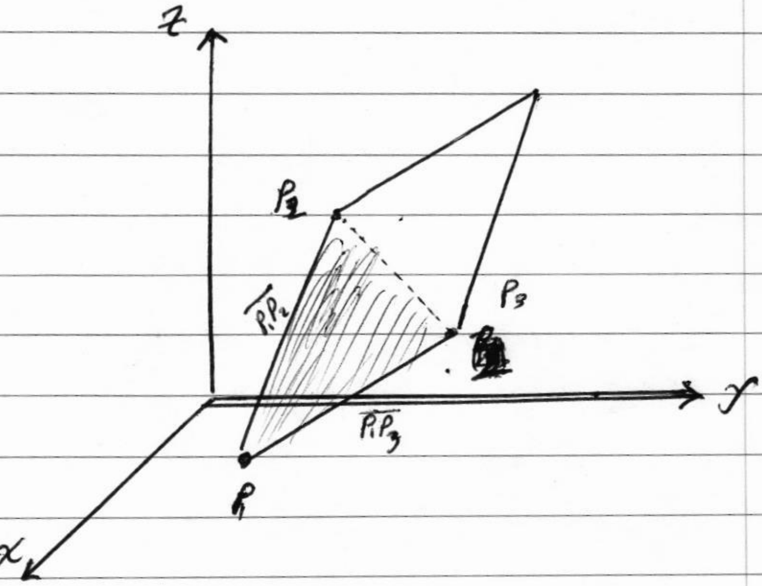
ex 8

Find The area of Traiangle it's heads The points

$P_1 = (2, 2, 0)$

$P_2 = (-1, 0, 2)$

$P_3 = (0, 4, 3)$



$\overline{P_1P_2} = \langle -3, -2, 2 \rangle$

$\overline{P_1P_3} = \langle -2, 2, 3 \rangle$

$$\overline{P_1P_2} \times \overline{P_1P_3} = \begin{vmatrix} i & j & k \\ -3 & -2 & 2 \\ -2 & 2 & 3 \end{vmatrix} = -10i + 5j - 10k$$

$$\| \overline{P_1P_2} \times \overline{P_1P_3} \| = \sqrt{100 + 25 + 100} = 15$$

$$\text{area of } P_1P_2P_3 \, \Delta = \frac{1}{2} \cdot 15 = 7.5$$

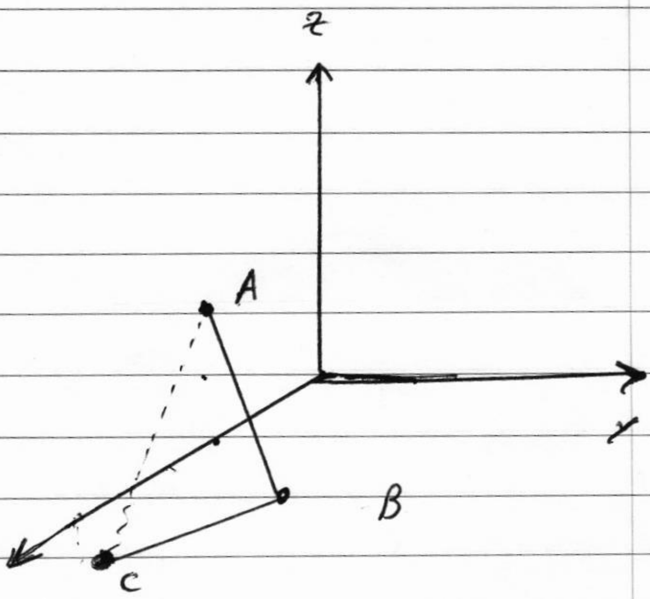
H.w.

prove That $\bar{A} \times \bar{B} = 0 \overset{iff}{\Longleftrightarrow} \bar{A} \parallel \bar{B}$

ex

prove That : The points $\begin{cases} A = (2, -1, 1) \\ B = (3, 2, -1) \\ C = (7, 0, -2) \end{cases}$ construct a right Triangle from right side.

sol.



$\overline{AB} = \langle 1, 3, -2 \rangle$

$\overline{BC} = \langle 4, -2, -1 \rangle$

$\overline{AB} \cdot \overline{BC} = 4 - 6 + 2 = 0$

$\therefore \overline{AB} \perp \overline{BC}$

or one can use pythagors Theorem

$\overline{AC} = \langle 5, 1, -3 \rangle$

$\Longrightarrow \quad \| \overline{AC} \|^2 = 35 \quad \& \quad \| \overline{AB} \|^2 = 14 \quad \& \quad \| \overline{BC} \|^2 = 21$

$\therefore \quad \| \overline{AC} \|^2 = \| \overline{AB} \|^2 + \| \overline{BC} \|^2$

$\therefore \quad AB \perp BC \qquad \square$

Triple scalar product (T.S.P.)

$$\bar{A}\cdot(\bar{B}\times\bar{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

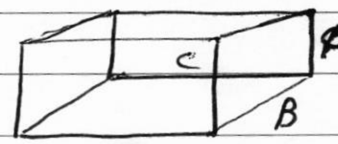
clear That

$$\bar{A}\cdot(\bar{B}\times\bar{C}) = \bar{B}\cdot(\bar{C}\times\bar{A}) = \bar{C}\cdot(\bar{A}\times\bar{B})$$

T.S.P evalute The volume of cubic $\bar{A},\bar{B},\bar{C}$
parallelogram

ex

let The parallelogram its sides



$\bar{A} = \langle 3,2,1\rangle$, $\bar{B} = \langle 1,1,2\rangle$, $\bar{C} = \langle 1,3,2\rangle$

comput: (a) Volum, (b) area of Triangle between A & c
(c) find The angle between $\bar{A}$ and The plan CB

sol.

(a) $\bar{A}\cdot(\bar{B}\times\bar{C}) = \begin{vmatrix} 3 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 3 & 2 \end{vmatrix} = -10 \Rightarrow V = 10$ lower line

(b) $\bar{A}\times\bar{C} = \begin{vmatrix} i & j & k \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{vmatrix} = i - 5j + 7k \Rightarrow \|\bar{A}\times\bar{C}\| = \sqrt{75}$

$\Rightarrow$ area of $\Delta = \sqrt{75}/2$

(c) $\underset{\text{plan}}{\bar{C}\times\bar{B}} = \begin{vmatrix} i & j & k \\ 1 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} = 4i - 2k \Rightarrow \|\bar{C}\times\bar{B}\| = \sqrt{20}$

$$A\cdot(\bar{C}\times\bar{B}) = \|A\|\cdot\|\bar{C}\times\bar{B}\|\cos\theta$$

$$10 = \sqrt{14}\cdot\sqrt{20}\cos\theta$$

$$\sqrt{\frac{5}{14}} = \cos\theta \Longrightarrow \theta = \cos^{-1}\sqrt{\frac{5}{14}}$$

$\Box$

2016 - 10 - 16

Parametric equation

for Lines in R^2 and R^3

Def.

The line "L" in R^2 (R^3) That passes Through the point $P_0 = (x_0, y_0)$ ($P_0 = (x_0, y_0, z_0)$) and parallel to The non-zero vector $\vec{V} = \langle a, b \rangle$ ($\vec{V} = \langle a, b, c \rangle$)

has The following parametric equation :-

L:

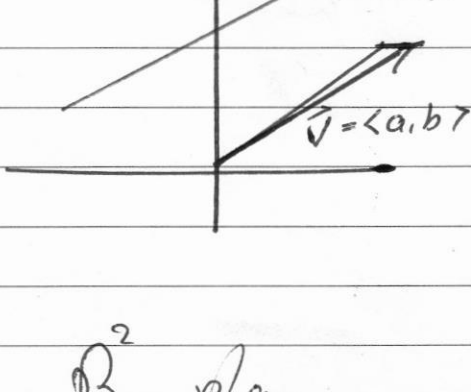
$$x = x_0 + at$$
$$y = y_0 + bt$$

or in R^3

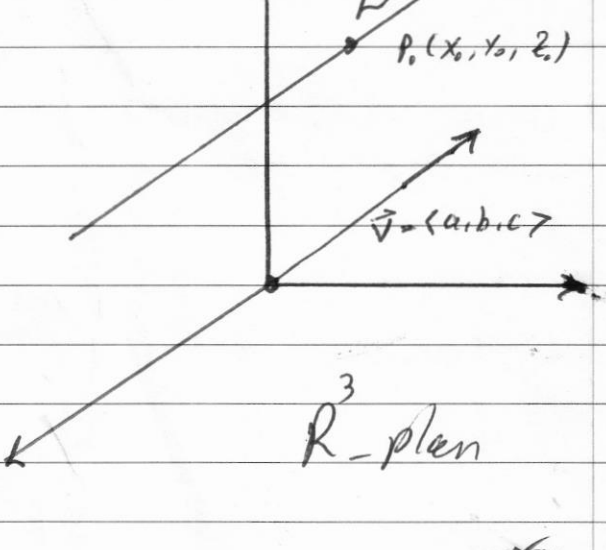
L:

$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

$t \in (-\infty, \infty)$



R^2 _ plan



R^3 _ plan

ex1 find The parametric eq. of The lines:

1) passing through $(4,2)$ and parallel to $\vec{V}=\langle -1,5\rangle$

2) " " $(1,2,-3)$ " " " $\vec{V}=\langle 4,5,-7\rangle$

3) " " $(0,0,0)$ " " " $\vec{V}=\langle 1,1,1\rangle$

sol.

1) $L:$ $x=4-t$, $y=2+5t$

2) $L:$ $x=1+4t$, $y=2+5t$, $z=-3-7t$

3) $L:$ $x=t$, $y=t$, $z=t$

ex2 (a) find The parametric eq. of The line L which passing through the points $p=(2,4,-1)$ and $q=(5,0,7)$

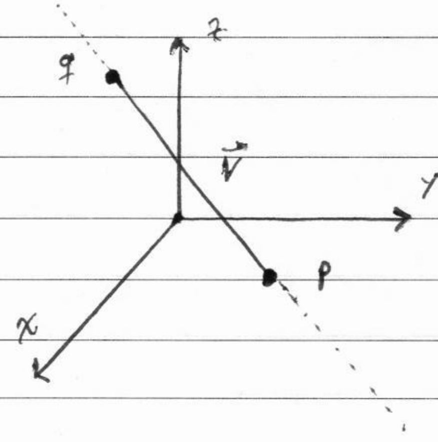
(b) When dose The line intersect The xy-plan (R^2)

c) the parametric eq. of the line segment from p to q.

sol.

a) $\vec{V}=\overrightarrow{pq}=\langle 5-2,\ 0-4,\ 7+1\rangle=\langle 3,-4,8\rangle$

$$\Rightarrow L:\left\{\begin{array}{l} x=2+3t \\ y=4-4t \\ z=-1+8t \end{array}\right.$$



(b) intersect R^2 in $z=0$

$\Rightarrow$

$0=-1+8t \Rightarrow t=\frac{1}{8}$

$$\Rightarrow \left.\begin{array}{l} x=2+\frac{3}{8} \\ y=4-\frac{4}{8} \end{array}\right\} \Rightarrow \text{intersect at } p=\left(\frac{19}{8},\frac{28}{8}\right)$$

c)

The Line L_1 = $\begin{cases} x = 2 + 3t \\ y = 4 - 4t \\ z = -1 + 8t \end{cases}$ pass through p

at $t = 0$ and through q at $t = 1$

by use any point coordinates x or y or z.

so; The eq. becom for The line segement $\overline{pq}$

$$L_2: \begin{aligned} x &= 2 + 3t \\ y &= 4 - 4t \\ z &= -1 + 8t \end{aligned} \qquad 0 \leq t \leq 1$$

ملاحظة

1) لايجاد معادلة مستقيم يجب ان يكون لدينا نقطة على المستقيم ومتجه . يوازي المستقيم .

2) في حالة وجود نقطتين فيمكن ايجاد المتجه الذي يحويهما ثم ناخذ احدهما مع المتجه لايجاد المعادلة . واذا طلب تحديد المعلمة t لقطعة المستقيم نحددها باستخدام احد مساقط النقاط كما فعلنا في المثال السابق .

ex 3

Let L_1 & L_2 be two lines with

$$L_1: \begin{cases} x = 1 + 4t \\ y = 5 - 4t \\ z = -1 + 5t \end{cases} \qquad L_2: \begin{cases} x = 2 + 8t \\ y = 4 - 3t \\ z = 5 + t \end{cases}$$

a) are The lines parallel ?
b) dose " " intersect ?

Sol.

a) $\vec{V}_1 = \langle 4, -4, 5 \rangle$
$\vec{V}_2 = \langle 8, -3, 1 \rangle$

$$\Rightarrow \quad \vec{V}_1 \times \vec{V}_2 = \begin{vmatrix} i & j & k \\ 4 & -4 & 5 \\ 8 & -3 & 1 \end{vmatrix} = \langle 11, 36, 20 \rangle \neq 0$$

$$\Rightarrow \vec{V}_1 \nparallel \vec{V}_2 \Rightarrow L_1 \nparallel L_2$$

b) if the lines intersect Then There is a point

$P(x, y, z)$ satisfies both of L_1 & L_2 so that :-

$x_1 = x_2$, $y_1 = y_2$, $z_1 = z_2$

$$\left.\begin{aligned} 1 + 4t_1 &= 2 + 8t_2 \quad — 1 \\ 5 - 4t_1 &= 4 - 3t_2 \quad — 2 \\ -1 + 5t_1 &= 5 + t_2 \quad — 3 \end{aligned}\right\} \xrightarrow{1+2} 6 = 6 + 5t_2 \Rightarrow t_2 = 0 \quad — 4$$

$$4 \text{ in } 1 \Rightarrow t_1 = \frac{1}{4} \quad — 5 \quad \xrightarrow{5 \text{ in } 3} -1 + \frac{5}{4} = 5 \Rightarrow 5 = 24$$
$$\Rightarrow C!$$

$\therefore L_1$ not intersect L_2 $\square$

H.W. شعبة حافظ احمد idea Plus

def

The distance between line $L_1 = \; x = x_0 + at$ — ①

$y = y_0 + bt$

$z = z_0 + ct$

and The point $p = (\alpha, \beta, \gamma)$ can be computed

as follows :-

① find The eq. $D = \Sigma($ ~

$$D(t) = (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 \quad \text{— ②}$$

② find t by set $\frac{dD}{dt} = 0$

③ find coordinate of L_1 using t (say (Q))

④ comput $\sqrt{D}$ again with out "t" $\equiv d(P, Q)$

ex find The distance between $p = (1,1,5)$ and L_1

$x = 1+t \;,\; y = 3-t \;,\; z = 2t$

sol.

step ① $D(t) = (1+t-1)^2 + (3-t-1)^2 + (2t-5)^2$

$= 6t^2 - 24t + 29$

step ② $\frac{dD}{dt} = 0 = 12t - 24 \implies t = 2$

step 3 $L_1 = (1+2,\; 3-2,\; 2*2) = (3, 1, 4)$

step 4

$D = (3-1)^2 + (1-1)^2 + (4-5)^2 = 4 + 0 + 1$

$\sqrt{D} = \sqrt{5}$

method (2) to find the distance between the points S and the line L (passing through P and parallel to vector V ~~and the points~~.

$$d = \frac{\| \overrightarrow{PS} \times V \|}{\| V \|}$$

ex recall the above example $P = (1,1,5)$

$L : \begin{cases} x = 1+t \\ y = 3-t \\ z = 2t \end{cases}$

sol.

from L $S = (1,3,0)$, $V = \langle 1,-1,2 \rangle$

$\Rightarrow \overrightarrow{PS} = (0,-2,5)$

$\Rightarrow \overrightarrow{PS} \times V = \begin{vmatrix} i & j & k \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \langle 1,5,2 \rangle$

$\Rightarrow \| \overrightarrow{PS} \times V \| = \sqrt{1+25+4} = \sqrt{30}$

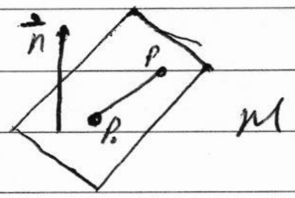
$\& \ \| V \| = \sqrt{1+1+4} = \sqrt{6}$

$\Rightarrow$

$$d = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{\frac{30}{6}} = \sqrt{5}$$

Equation of plane in space

A plane M in space can be determined by knowing a point on plan (P_0) and a vector $\vec{n}$ that perpendicular to M, since $\forall P \in M$



$\overrightarrow{P_0P} \cdot \vec{n} = 0$ ——— (1) where $\vec{n} = \langle A, B, C \rangle$, $P = (x, y, z)$

$\Longrightarrow$

$(Ai + Bj + Ck) \cdot (x - x_0\, i + y - y_0\, j + z - z_0\, k) = 0$

$\Rightarrow A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ ——— (2)

$\Rightarrow Ax + By + Cz = d$ ——— (3)

(1) is called vector eqution.

(2) " " component eqution.

(3) " " simplified " " .

ملاحظة

لإيجاد معادلة المستوى نحتاج الى نقطة في المستوى ومتجه عمودي على المستوى كما يمكن ايجاد المتجه العمودي اذا علمت ثلاث نقاط ليست على استقامة واحدة وكذلك اذا علم متجهين ونقطة .

<u>ex1</u> find plane eq. That perpendicular to vector

$\vec{n} = \langle 5, 2, -1 \rangle$ and pass Through point $(-3, 0, 7)$

<u>sol.</u>

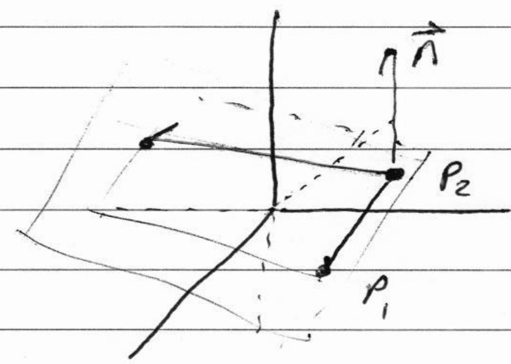
$$5(x+3) + 2(y-0) - 1(z-7) = 0$$

$$5x + 2y - z = -15 - 7 = -22$$

ليس على استقامة واحدة

<u>ex2</u> find an eq. for The plane Through Tn colinear

$P_1(1, 2, -1)$, $P_2(2, 3, 1)$, $P_3(3, -1, 2)$



$\overrightarrow{P_1P_2} = \langle 1, 1, 2 \rangle$
$\overrightarrow{P_2P_3} = \langle 1, -4, 1 \rangle$

$$\vec{n} = \overline{P_1P_2} \times \overline{P_2P_3} = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 1 & -4 & 1 \end{vmatrix}$$

$$= \langle 9, 1, -5 \rangle$$

لا يهم اي نقطة

now use one of the point (it does not matter which)

$$9(x-2) + (y-3) - 5(z-1) = 0$$

$$9x + y - 5z = 16$$

<u>H.w.</u> same ex2 but $P_1(0,0,1)$, $P_2(2,0,0)$, $P_3(0,3,0)$

ex.

Is the Line $L_1 : \begin{cases} x = 3 + 8t \\ y = 4 + 5t \\ z = -3 - t \end{cases}$ parallel to plane M

$x - 3y + 5z = 12$? if not find points of intersection.

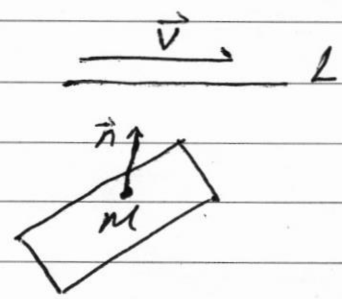
sol.

$\vec{V}$ parallel to L_1 is $\vec{V} = \langle 8, 5, -1 \rangle$

$\vec{n}$ perpendicular to M is $\vec{n} = \langle 1, -3, 5 \rangle$

$\vec{V} \cdot \vec{n} = 8 - 15 - 5 = -12 \neq 0$

$\therefore$ M not parallel to L_1



Let $P_0(x_0, y_0, z_0)$ is the point of inter.

$\Longrightarrow$

$x_0 - 3y_0 + 5z_0 = 12$ —— ①

$x_0 = 3 + 8t \quad , \quad y_0 = 4 + 5t \quad , \quad z_0 = -3 - t$ —— ②

② in ① $\Rightarrow (3 + 8t) - (4 + 5t) + 5(-3 - t) = 12$

$\Rightarrow -12t - 24 = 12 \Rightarrow t = -3$ —— ③

③ in ② $\Rightarrow \left.\begin{matrix} x_0 = (-21) \\ y_0 = (-11) \\ z_0 = 0 \end{matrix}\right\} \Rightarrow P_0(-21, -11, 0)$

Line intersection between two planes

ex

Find The parametric eq. for the line L in which The planes

$m_1 : 3x - 6y - 2z = 15$

$m_2 : 2x + y - 2z = 5$ are intersection.

sol.

$\vec{n_1} = \langle 3, -6, -2 \rangle$

$\vec{n_2} = \langle 2, 1, -2 \rangle$ $\Rightarrow$

$$\vec{n_1} \times \vec{n_2} = \begin{vmatrix} i & j & k \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = \langle 14, 2, 15 \rangle = \vec{V}$$

clear That $\vec{V}$ parallel to a line of intersection (L)

since $\vec{V}$ perpendicular to $\vec{n}_1$ and $\vec{n}_2$

Now we need a point p to find eq. of L

Let $z = 0$ in m_1 & m_2

$$\Rightarrow \begin{aligned} 3x - 6y &= 15 \\ 2x + y &= 5 \end{aligned}$$

$15x = 45$

$\Rightarrow x = 3 \Rightarrow y = -1$

$\Rightarrow p = (3, -1, 0)$

$$\therefore L : \begin{cases} x = 3 + 14t \\ y = -1 + 2t \\ z = 15t \end{cases}$$

ملاحظة

اختيار $z=0$ ممكن :
ان يتغير وعليه تتغير
معادلة L فمن الممكن
ان اجعل $z=1$ او
x مكان ال z

ex Find a vector $\vec{V}$ That parallel to line of intersection of planes

$m1 \quad 3x - 2z = 3$

$m2 \quad y - z = 5$

sol.

$\vec{n}_1 = \langle 3, 0, -2 \rangle$

$\vec{n}_2 = \langle 0, 1, -1 \rangle$

$$\Rightarrow \vec{V} = \vec{n}_1 \times \vec{n}_2 = \left| \begin{matrix} i & j & k \\ 3 & 0 & -2 \\ 0 & 1 & -1 \end{matrix} \right| = \langle 2, 3, 3 \rangle$$

Distance between point S and plane m

for any $P \in m$ and $\vec{n} \perp m$, The distance from any point S to The plane m is The ~~distance~~ length of The vector projection $\overrightarrow{PS}$ onto $\vec{n}$, i.e.,

$$D = \left\| \overrightarrow{PS} \cdot \frac{\vec{n}}{\|n\|} \right\|$$

ex Find The distance between $S(1,1,3)$ to m:

$3x + 2y + 6z = 0$

sol.

$\vec{n} = \langle 3, 2, 6 \rangle \Rightarrow \|\vec{n}\| = \sqrt{9 + 4 + 36} = \sqrt{49} = 7$

let $P = \langle 0, y, 0 \rangle$ in $m1$ $\quad 2y = 6 \Rightarrow y = 3 \Rightarrow P(0,3,0)$

$$\therefore D = \left\| \frac{\langle -1, 2, -3 \rangle \cdot \langle 3, 2, 6 \rangle}{7} \right\| = \left| -\frac{3}{7} + \frac{4}{7} - \frac{18}{7} \right| = \frac{17}{7}$$

Other way to find D from $S(x_0, y_0, z)$ to to $m: ax+by+cz=d$

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\|n\|} = \frac{|ax_0 + by_0 + cz + d|}{\sqrt{a^2+b^2+c^2}} \quad (*)$$

Ex repeat privious ex H.W.

Distance between two planes

case 1 : if The plane m1 intersect plan m2

Then $D = 0$ and The angle between Them

is The angle between $\vec{n}_1$ and $\vec{n}_2$ That

can found by $\vec{n}_1 \cdot \vec{n}_2 = \|n_1\| \cdot \|n_2\| \cos\theta$

case 2 : if The plane m_1 parllel to plane m2

Then D can be found by using (*) for

a point $S \in m_1$ and $\vec{n}_2$ or

$S \in m_2$ and $\vec{n}_1$

here

$S = (0,0,z)$ or $(0,y,0)$ or $(x,0,0)$

ex1 for case 1) find The angle between

$m1 : 3x - 6y - 2z = 15$

$m2 : 2x + y - 2z = 5$

sol. $\because \vec{n}_1 \times \vec{n}_2 \neq 0$ (by ex in page 20)

Then $m_1 \not\parallel m_2$

so;

$\vec{n}_1 = \langle 3, -6, -2 \rangle$

$\vec{n}_2 = \langle 2, 1, -2 \rangle$

$\Rightarrow \bar{n}_1 \cdot \bar{n}_2 = 6 - 6 + 4 = 4$

$\|n_1\| = \sqrt{49} = 7$

$\|n_2\| = \sqrt{9} = 3$

$\Rightarrow$

$$4 = (7 \times 3) \cos \theta$$

$$\theta = \cos^{-1}\left(\frac{4}{21}\right) \cong 1.38 \qquad (79^\circ)$$

ex2 find The distance between $m_1 = x + 2y - 2z = 3$

$m_2 = 2x + 4y - 4z = 7$

sol. $\vec{n}_1 = \langle 1, 2, -2 \rangle$

$\vec{n}_2 = \langle 2, 4, -4 \rangle$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & -2 \\ 2 & 4 & -4 \end{vmatrix} = 0 \Rightarrow m_1 \parallel m_2$$

let $P \in m_1$ s.t $P(x,0,0) \Rightarrow x + 0 + 0 = 3 \Rightarrow x = 3$

$\therefore P(3,0,0)$

by (*)

$$D = \frac{|2 * 3 + 4 * 0 + -4 * 0 - 7|}{\sqrt{4 + 16 + 16}} = \frac{1}{\sqrt{36}} = \frac{1}{6}$$

24

The points on the plane easiest to find from the plane's equation are the intercepts. If we take P to be the y-intercept $(0, 3, 0)$, then

$$\begin{aligned} \overrightarrow{PS} &= (1 - 0)\mathbf{i} + (1 - 3)\mathbf{j} + (3 - 0)\mathbf{k} \\ &= \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}, \\ |\mathbf{n}| &= \sqrt{(3)^2 + (2)^2 + (6)^2} = \sqrt{49} = 7. \end{aligned}$$

The distance from S to the plane is

$$\begin{aligned} d &= \left| \overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| \qquad \text{length of } \operatorname{proj}_{\mathbf{n}} \overrightarrow{PS} \\ &= \left| (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot \left(\frac{3}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k} \right) \right| \\ &= \left| \frac{3}{7} - \frac{4}{7} + \frac{18}{7} \right| = \frac{17}{7}. \end{aligned}$$

■

Angles Between Planes

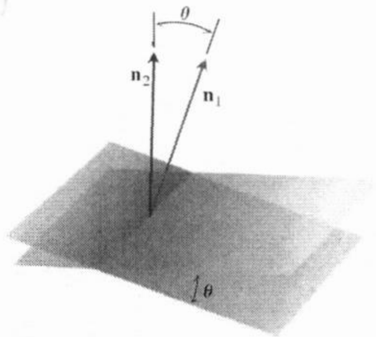


FIGURE 12.42 The angle between two planes is obtained from the angle between their normals.

The angle between two intersecting planes is defined to be the acute angle between their normal vectors (Figure 12.42).

EXAMPLE 12 Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Solution The vectors

$$\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}, \qquad \mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

are normals to the planes. The angle between them is

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} \right) \\ &= \cos^{-1}\left(\frac{4}{21} \right) \\ &\approx 1.38 \text{ radians.} \qquad \text{About 79 deg} \end{aligned}$$

■

Exercises 12.5

Lines and Line Segments

Find parametric equations for the lines in Exercises 1–12.

1. The line through the point $P(3, -4, -1)$ parallel to the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$
2. The line through $P(1, 2, -1)$ and $Q(-1, 0, 1)$
3. The line through $P(-2, 0, 3)$ and $Q(3, 5, -2)$
4. The line through $P(1, 2, 0)$ and $Q(1, 1, -1)$
5. The line through the origin parallel to the vector $2\mathbf{j} + \mathbf{k}$
6. The line through the point $(3, -2, 1)$ parallel to the line $x = 1 + 2t, y = 2 - t, z = 3t$
7. The line through $(1, 1, 1)$ parallel to the z-axis
8. The line through $(2, 4, 5)$ perpendicular to the plane $3x + 7y - 5z = 21$
9. The line through $(0, -7, 0)$ perpendicular to the plane $x + 2y + 2z = 13$
10. The line through $(2, 3, 0)$ perpendicular to the vectors $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$
11. The x-axis
12. The z-axis

Find parametrizations for the line segments joining the points in Exercises 13–20. Draw coordinate axes and sketch each segment, indicating the direction of increasing t for your parametrization.

13. $(0, 0, 0)$, $(1, 1, 3/2)$
14. $(0, 0, 0)$, $(1, 0, 0)$
15. $(1, 0, 0)$, $(1, 1, 0)$
16. $(1, 1, 0)$, $(1, 1, 1)$
17. $(0, 1, 1)$, $(0, -1, 1)$
18. $(0, 2, 0)$, $(3, 0, 0)$
19. $(2, 0, 2)$, $(0, 2, 0)$
20. $(1, 0, -1)$, $(0, 3, 0)$

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Planes

Find equations for the planes in Exercises 21–26.

21. The plane through $P_0(0, 2, -1)$ normal to $\mathbf{n} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$
22. The plane through $(1, -1, 3)$ parallel to the plane
$$3x + y + z = 7$$
23. The plane through $(1, 1, -1)$, $(2, 0, 2)$, and $(0, -2, 1)$
24. The plane through $(2, 4, 5)$, $(1, 5, 7)$, and $(-1, 6, 8)$
25. The plane through $P_0(2, 4, 5)$ perpendicular to the line
$$x = 5 + t, \quad y = 1 + 3t, \quad z = 4t$$
26. The plane through $A(1, -2, 1)$ perpendicular to the vector from the origin to A
27. Find the point of intersection of the lines $x = 2t + 1$, $y = 3t + 2$, $z = 4t + 3$, and $x = s + 2$, $y = 2s + 4$, $z = -4s - 1$, and then find the plane determined by these lines.
28. Find the point of intersection of the lines $x = t$, $y = -t + 2$, $z = t + 1$, and $x = 2s + 2$, $y = s + 3$, $z = 5s + 6$, and then find the plane determined by these lines.

In Exercises 29 and 30, find the plane determined by the intersecting lines.

29. $L1: x = -1 + t, \quad y = 2 + t, \quad z = 1 - t; \quad -\infty < t < \infty$
$L2: x = 1 - 4s, \quad y = 1 + 2s, \quad z = 2 - 2s; \quad -\infty < s < \infty$
30. $L1: x = t, \quad y = 3 - 3t, \quad z = -2 - t; \quad -\infty < t < \infty$
$L2: x = 1 + s, \quad y = 4 + s, \quad z = -1 + s; \quad -\infty < s < \infty$
31. Find a plane through $P_0(2, 1, -1)$ and perpendicular to the line of intersection of the planes $2x + y - z = 3$, $x + 2y + z = 2$.
32. Find a plane through the points $P_1(1, 2, 3)$, $P_2(3, 2, 1)$ and perpendicular to the plane $4x - y + 2z = 7$.

Distances

In Exercises 33–38, find the distance from the point to the line.

33. $(0, 0, 12)$; $\quad x = 4t, \quad y = -2t, \quad z = 2t$
34. $(0, 0, 0)$; $\quad x = 5 + 3t, \quad y = 5 + 4t, \quad z = -3 - 5t$
35. $(2, 1, 3)$; $\quad x = 2 + 2t, \quad y = 1 + 6t, \quad z = 3$
36. $(2, 1, -1)$; $\quad x = 2t, \quad y = 1 + 2t, \quad z = 2t$
37. $(3, -1, 4)$; $\quad x = 4 - t, \quad y = 3 + 2t, \quad z = -5 + 3t$
38. $(-1, 4, 3)$; $\quad x = 10 + 4t, \quad y = -3, \quad z = 4t$

In Exercises 39–44, find the distance from the point to the plane.

39. $(2, -3, 4)$, $\quad x + 2y + 2z = 13$
40. $(0, 0, 0)$, $\quad 3x + 2y + 6z = 6$
41. $(0, 1, 1)$, $\quad 4y + 3z = -12$
42. $(2, 2, 3)$, $\quad 2x + y + 2z = 4$
43. $(0, -1, 0)$, $\quad 2x + y + 2z = 4$
44. $(1, 0, -1)$, $\quad -4x + y + z = 4$
45. Find the distance from the plane $x + 2y + 6z = 1$ to the plane $x + 2y + 6z = 10$.
46. Find the distance from the line $x = 2 + t$, $y = 1 + t$, $z = -(1/2) - (1/2)t$ to the plane $x + 2y + 6z = 10$.

Angles

Find the angles between the planes in Exercises 47 and 48.

47. $x + y = 1, \quad 2x + y - 2z = 2$
48. $5x + y - z = 10, \quad x - 2y + 3z = -1$

T Use a calculator to find the acute angles between the planes in Exercises 49–52 to the nearest hundredth of a radian.

49. $2x + 2y + 2z = 3, \quad 2x - 2y - z = 5$
50. $x + y + z = 1, \quad z = 0$ (the xy-plane)
51. $2x + 2y - z = 3, \quad x + 2y + z = 2$
52. $4y + 3z = -12, \quad 3x + 2y + 6z = 6$

Intersecting Lines and Planes

In Exercises 53–56, find the point in which the line meets the plane.

53. $x = 1 - t, \quad y = 3t, \quad z = 1 + t; \quad 2x - y + 3z = 6$
54. $x = 2, \quad y = 3 + 2t, \quad z = -2 - 2t; \quad 6x + 3y - 4z = -12$
55. $x = 1 + 2t, \quad y = 1 + 5t, \quad z = 3t; \quad x + y + z = 2$
56. $x = -1 + 3t, \quad y = -2, \quad z = 5t; \quad 2x - 3z = 7$

Find parametrizations for the lines in which the planes in Exercises 57–60 intersect.

57. $x + y + z = 1, \quad x + y = 2$
58. $3x - 6y - 2z = 3, \quad 2x + y - 2z = 2$
59. $x - 2y + 4z = 2, \quad x + y - 2z = 5$
60. $5x - 2y = 11, \quad 4y - 5z = -17$

Given two lines in space, either they are parallel, or they intersect, or they are skew (imagine, for example, the flight paths of two planes in the sky). Exercises 61 and 62 each give three lines. In each exercise, determine whether the lines, taken two at a time, are parallel, intersect, or are skew. If they intersect, find the point of intersection.

61. $L1: x = 3 + 2t, \quad y = -1 + 4t, \quad z = 2 - t; \quad -\infty < t < \infty$
$L2: x = 1 + 4s, y = 1 + 2s, z = -3 + 4s; \quad -\infty < s < \infty$
$L3: x = 3 + 2r, \ y = 2 + r, \ z = -2 + 2r; \quad -\infty < r < \infty$
62. $L1: x = 1 + 2t, \quad y = -1 - t, \quad z = 3t; \quad -\infty < t < \infty$
$L2: x = 2 - s, \quad y = 3s, \quad z = 1 + s; \quad -\infty < s < \infty$
$L3: x = 5 + 2r, \quad y = 1 - r, \quad z = 8 + 3r; \quad -\infty < r < \infty$

Theory and Examples

63. Use Equations (3) to generate a parametrization of the line through $P(2, -4, 7)$ parallel to $\mathbf{v}_1 = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Then generate another parametrization of the line using the point $P_2(-2, -2, 1)$ and the vector $\mathbf{v}_2 = -\mathbf{i} + (1/2)\mathbf{j} - (3/2)\mathbf{k}$.
64. Use the component form to generate an equation for the plane through $P_1(4, 1, 5)$ normal to $\mathbf{n}_1 = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Then generate another equation for the same plane using the point $P_2(3, -2, 0)$ and the normal vector $\mathbf{n}_2 = -\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} - \sqrt{2}\mathbf{k}$.
65. Find the points in which the line $x = 1 + 2t$, $y = -1 - t$, $z = 3t$ meets the coordinate planes. Describe the reasoning behind your answer.
66. Find equations for the line in the plane $z = 3$ that makes an angle of $\pi/6$ rad with $\mathbf{i}$ and an angle of $\pi/3$ rad with $\mathbf{j}$. Describe the reasoning behind your answer.
67. Is the line $x = 1 - 2t$, $y = 2 + 5t$, $z = -3t$ parallel to the plane $2x + y - z = 8$? Give reasons for your answer.

Partial Derivative

2016-10-26

chapter 14 calculus 12

functions of more Than one variables

In several functions and applications The function $y = f(x)$ has one independent variable "x" and called some time "input" with one dependent variable "y" or "output".

in This chapter we will used The function with more Than one independent variable; for example

$V = f(r, h) = \pi r^2 h$

which calculates The volume of cylinder from it's reduce "r" and hight "h". or more examples

$z = f(x, y) = x^2 + y^2$ $\qquad f: R^2 \longrightarrow R$

$z = \frac{1}{xy} + y$ $\qquad f: R^2 \longrightarrow R$

$w = f(x, y, z) = \frac{2z}{3xy^2}$ $\qquad f: R^3 \longrightarrow R$

and so on ....

partial derivative

as we know if $y = f(x)$. Then the derivative for x is $\frac{dy}{dx} = y'$

But if we have $z = f(x, y)$ or

$w = f(x, y, z)$ or

$\alpha = f(x_1, x_2, \ldots x_n)$

Then we need to interduce the concept of partial Derivative which are what we get when hold all but one of the variables of function as a constant and differentiate ~~the~~ with respect to that one variable; i.e.

Concider all variables constant except one which we need to find the derivative resp. it.

This operation denoted by "∂" read partial or f_x when we need derive f respect to x.

ex find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ for

① $z = x^2 + 2xy + y^2$

② $z = wx + 3x^3 + 1 - w + y$

③ $\frac{1}{z} = 2x + xy + 1$

sol.

① $z_x = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2 + 2xy + y^2)$ اعتبرنا ان y ثابت والاشتقاق لـ x فقط

$= 2x + 2y$

$z_y = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2 + 2xy + y^2)$ اعتبرنا ان x ثابت والاشتقاق لـ y

$= 2x + 2y$

② $z_x = w + 9x^2$

$z_y = 1$

③

$\frac{\partial}{\partial x}\left(\frac{1}{z}\right) = \frac{\partial}{\partial x}(2x + xy + 1)$

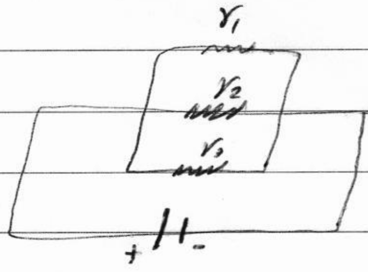
$\frac{-\frac{\partial z}{\partial x}}{z^2} = 2 + y \Rightarrow +\frac{\partial z}{\partial x} = -z^2(2+y)$

$\frac{\partial}{\partial y}\left(\frac{1}{z}\right) = \frac{\partial}{\partial y}(2x + xy + 1)$

$\frac{-\frac{\partial z}{\partial y}}{z^2} = x \Rightarrow \frac{\partial z}{\partial y} = -z^2 x$

ex if The resistors r_1, r_2 r_3 ohms are connected ~~th~~ in parallel to make an R-ohms resister, The value of R can be fund from eq.

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$



find $\frac{\partial R}{\partial r_2}$ when $[r_1, r_2, r_3] = [30, 45, 90]$

sol.

$$\frac{\partial}{\partial r_2}\left(\frac{1}{R}\right) = \frac{\partial}{\partial r_2}\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right)$$

$$\frac{-\frac{\partial R}{\partial r_2}}{R^2} = -\frac{1}{r_2^2}$$

$$\frac{\partial R}{\partial r_2} = \frac{R^2}{r_2^2} = \left(\frac{R}{r_2}\right)^2$$

at $[30, 45, 90] \Rightarrow \frac{\partial R}{\partial r_2} = \left(\frac{\frac{1}{\frac{1}{30} + \frac{1}{45} + \frac{1}{90}}}{45}\right)^2$

$$= \left(\frac{15}{45}\right)^2 = \frac{1}{9}$$

Higher Order Derivative

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = z_{xx}$$

$$\frac{\partial^n z}{\partial x^n} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\underbrace{\ldots\ z\ )))\ \ldots}_{n\text{-times}}\right)\right)$$

mixed derivative

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} z\right) = z_{yx}$$

$\xleftarrow{\text{R. to L}}$ $\xrightarrow{}$ L to R

note if f and it partial derivative are continuous function. Then

$$f_{xy} = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

ex find z_{xyyx} when $z = x^3y^2 + xy + y^2$

$$z_{xyyx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}(x^3y^2 + xy + y^2)\right)\right)\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}(2x^2y^2 + y)\right)\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}(4x^2y + 1)\right)$$

$$= \frac{\partial}{\partial x}(4x^2) = 8x$$

ex $w = e^{y} \cos(x)$

find (a) w_{xyy} (b) w_{yxx}

a = $(e^{y} \cos x)_{xyy}$

$= (-e^{y} \sin x)_{yy}$

$= (-e^{y} \sin x)_{y}$

$= -e^{y} \sin x$

b ≡ $(e^{y} \cos x)_{yxx}$

$= (e^{y} \cos x)_{xx}$

$= (-e^{\cancel{x}} \sin x)_{x}$

$= -e^{y} \cos x$

ex $f(x,y) = y^{3} e^{-5x}$ find f_{yyxx} at $(0,1)$

sol.

$f_y = 3y^{2} e^{-5x}$

$f_{yy} = 6y\, e^{-5x}$

$f_{yyx} = -30\, y\, e^{-5x}$

$f_{yyxx} = 150\, y\, e^{-5x}$

$f_{yyxx}(0,1) = 150\,(1) \cdot e^{0} = 150$

$\square$

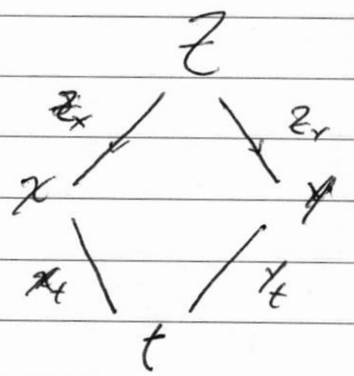
Chain Rule

Chain Rule for partial derivative have ~~two~~ Three form

(A)

$z = f(x, y)$ and $x = x(t)$ & $y = y(t)$

find $z_t = \frac{\partial z}{\partial t}$

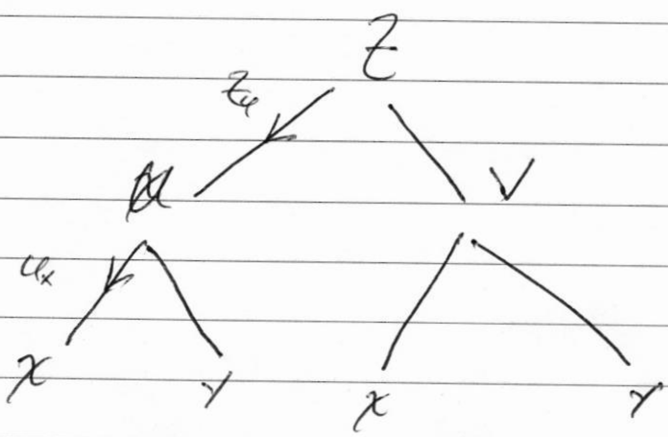


$$z_t = \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

(B) if $z = f(u, v)$ and $u = h(x, y)$ $v = g(x, y)$

$$\frac{\partial z}{\partial x} = z_u u_x + z_v v_x$$

$$\frac{\partial z}{\partial y} = z_u u_y + z_v v_y$$



$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

(c) $z = f(u) \quad , \quad u = g(x, y)$

$$\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y}$$

$$\begin{tikzcd} & z \arrow[d] & \\ & u \arrow[dl] \arrow[dr] & \\ x & & y \end{tikzcd}$$

ex for (c)

Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ & show that $x z_y + y z_x = 0$

sol. when $z = f(x^2 - y^2)$

let $u = x^2 - y^2 \Rightarrow z = f(u)$

$$\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du} \cdot (2x) = z' \, 2x$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = \frac{dz}{du} (-2y) = -z' \, 2y$$

$$\Rightarrow 2xy z' + (-2xy z') = 0$$

H.W. $R = \sqrt{x^2 + y^2}$ ~~show that~~ find $\frac{\partial R}{\partial x}$, $\frac{\partial^2 R}{\partial y^2}$

$\frac{x}{R}$ "

" $\frac{x^2}{R^3}$

ex (B)

Let $z = e^{uv}$, $u = 2x + y$
$v = \frac{x}{y}$

Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$

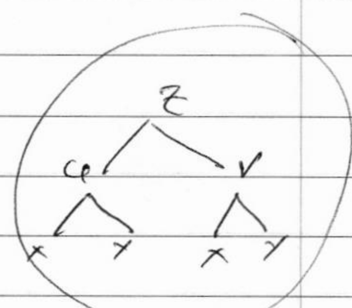
sol.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= v e^{uv} \cdot 2 + u e^{uv} \cdot \frac{1}{y}$$

$\frac{\partial z}{\partial y}$ = H.w. class work

ex2 , $z = f([x-y], [-x+y])$



Show That $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$ $(z_x + z_y = 0)$

sol. let $u = x - y$ & $v = x + y$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= z_u \cdot 1 + z_v \cdot (-1) = z_u - z_v \quad \text{①}$$

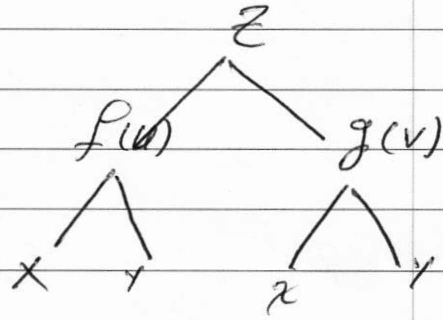
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= z_u \cdot (-1) + z_v \cdot (1) = -z_u + z_v \quad \text{②}$$

$\therefore z_x + z_y = 0$

ex $\quad z = f(y + cx) + g(y - cx) \qquad c \neq 0$

Show That
$$\frac{\partial^2 z}{\partial x^2} = c^2 \frac{\partial^2 z}{\partial y^2}$$



Sol.

Let $u = y + cx$

$\quad\; v = y - cx$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= f'(u) \cdot (c) + g'(v) \cdot (-c)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = c\,\frac{\partial}{\partial x}\left(f'(u) - g'(v)\right)$$

$$= c\left[\frac{\partial f'(u)}{\partial u} \cdot \frac{\partial u}{\partial x} - \frac{\partial g'(v)}{\partial v} \cdot \frac{\partial v}{\partial x}\right]$$

$$= c\left[f''(u) \cdot c - g''(v) \cdot (-c)\right]$$

$$= c^2\left[f''(u) + g''(v)\right] \quad \text{——— } (1)$$

$$\frac{\partial z}{\partial y} = f'(u) \cdot 1 + g'(v) \cdot 1 = f' + g'$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left[f'(u) + g'(v)\right] = f'' \cdot 1 + g'' \cdot 1$$

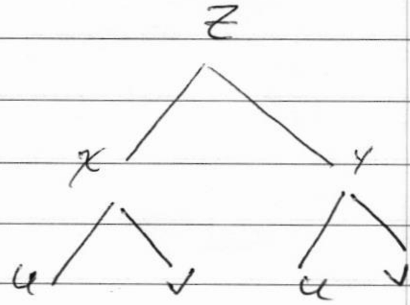
$$= f''(u) + g''(v)$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = c^2 \frac{\partial^2 z}{\partial y^2} \qquad \text{Q.E.D.} \quad \square$$

ex $\quad z = \sin xy, \quad x = u + v, \quad y = u \cdot v$

Show That :-

$$\frac{\partial^2 z}{\partial u \partial x} = \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial x}{\partial u} + \frac{\partial^2 z}{\partial y \partial x} \cdot \frac{\partial y}{\partial u}$$



sol. L.H.S $\quad \frac{\partial z}{\partial x} = y \cos xy$

$$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(y \cos xy\right)$$

$$\circledast \quad = \frac{\partial (y \cos xy)}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial \ (y \cos xy)}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= -y^2 \sin xy \cdot (1) + [\cos xy - yx \sin xy]\, v \quad (1)$$

R.H.S.

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(y \cos xy) = -y^2 \sin xy$$

$$\frac{\partial x}{\partial u} = 1$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(y \cos xy\right) = [\cos xy - yx \sin xy]$$

$$\frac{\partial y}{\partial u} = v$$

$$\Rightarrow (1) = (2) \Rightarrow \text{Q.E.D.} \ \square$$

note from $\circledast$ one can see if we replace ($y \cos xy$ by $\frac{\partial z}{\partial x}$) we get the sol. without long detailes ....! so let $\left| \begin{array}{l} z = f(x,y) \\ x = g(u,v) \\ y = h(u,v) \end{array} \right.$

resol. above ex ....

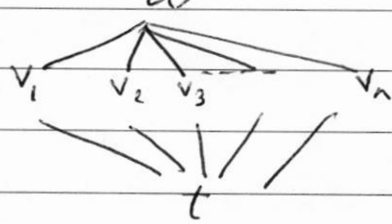
Total Differential (extend of Chain Rule)

if $w = f(v_1, v_2, ---, v_n)$ Then
$w' =$

$$\frac{\partial w}{\partial v_1} \cdot dv_1 + \frac{\partial w}{\partial v_2} dv_2 + ----- + \frac{\partial w}{\partial v_n} \cdot dv_n$$

and if $v_1, \_\_, v_n$ is a functions of t Then

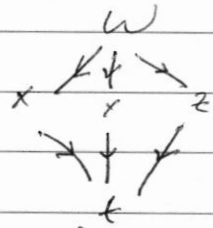
$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial v_1} \cdot \frac{dv_1}{dt} + \frac{\partial w}{\partial v_2} \cdot \frac{dv_2}{dt} + --- + \frac{\partial w}{\partial v_n} \cdot \frac{dv_n}{dt}$$



ex! let $w = xy + yz$

$x = t^2$, $y = \sin t$, $z = e^t$,

find $\frac{\partial w}{\partial t}$?



$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

$$= y \cdot 2t + (x+z) \cos t + y \cdot e^t$$

$$= 2t \sin t + (t^2 + e^t) \cos t + e^t \sin t$$

note

$$\frac{dw}{dt} = \left\langle \frac{\partial w}{\partial v_1}, \frac{\partial w}{\partial v_2}, ---, \frac{\partial w}{\partial v_n} \right\rangle \cdot \left\langle \frac{dv_1}{dt}, ---, \frac{dv_n}{dt} \right\rangle$$

and

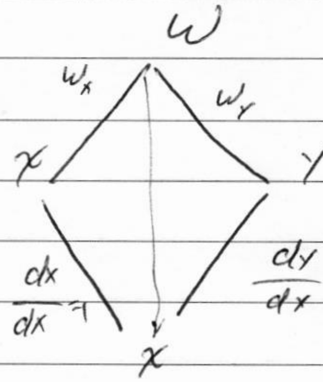
$$\frac{\partial w}{\partial r} = \left\langle \frac{\partial w}{\partial v_1}, --- \frac{\partial w}{\partial v_n} \right\rangle \cdot \left\langle \frac{\partial v_1}{\partial r}, -----, \frac{\partial v_n}{\partial r} \right\rangle$$

$$\frac{\partial w}{\partial s} = \left\langle \frac{\partial w}{\partial v_1}, --- \frac{\partial w}{\partial v_n} \right\rangle \cdot \left\langle \frac{\partial v_1}{\partial s}, -----, \frac{\partial v_n}{\partial s} \right\rangle$$

Implicit Differentiation (partial)

~~x~~Case 1

$W = F(x, y) = 0$



and $y = h(x)$

$$\frac{dW}{dx} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dx}$$

$$0 = \frac{\partial w}{\partial x} \cdot 1 + \frac{\partial w}{\partial y} \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\partial w/\partial x}{\partial w/\partial y} = \frac{-F_x}{F_y}$$

ex1 find $\frac{dy}{dx}$ if $x^2 + \sin y - 2y = 0$

sol.

$w = F(x, y) = x^2 + \sin y - 2y = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-2x}{\cos y - 2}$$

Case 2 when $W = F(x, y) \neq 0$ or in General

$w = F(x, y, z, \dots) \neq 0$

$w = F(v_1, v_2, \dots, v_n) \neq 0$

~~then~~ we can find $\frac{dw}{dv_j}$ when all of $v_1, v_2, v_3, \dots$

can be write as a function of v_j

for ex
~~in R^3~~

$$\frac{dw}{dx} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dx}$$

$$\frac{dw}{dy} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dy} \text{ (same way) } + \frac{\partial w}{\partial y} \cdot \frac{dy}{dy} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dy}$$

ex 2

if $w = \sqrt{x^2 + y^2 + z^2}$ $\quad x = \cos 2y$, $\quad z = \sqrt{y}$

find (a) $\frac{dw}{dy}$, $\frac{dw}{dx}$, $\frac{dw}{dz}$

sol.

(a)

$$\frac{dw}{dy} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dy} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dy} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dy}$$

$$= -2x\, w^{-1} \cdot \sin 2y + y w^{-1}\cdot 1 + \cancel{z} w^{-1} \cdot \frac{1}{2}\cancel{y}^{-\frac{1}{2}}$$

$$= w^{-1}\left[-2x \sin 2y + y + \frac{1}{2}\right]$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}}\left[-2x \sin 2y + y + \frac{1}{2}\right]$$

(b) $\cancel{H} = w.$

(c) $y = z^2 \Rightarrow x = \cos 2z^2$

$$\Rightarrow \frac{dw}{dz} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dz} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dz} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dz}$$

$$= -4zx\, w^{-1} \sin 2z^2 + 2zy\, w^{-1} + z w^{-1}\cdot 1$$

$$= z w^{-1}\left(-4x \sin 2z^2 + 2y + 1\right)$$

$$= \frac{z}{\sqrt{x^2+y^2+z^2}}\left(-4x \sin 2z^2 + 2y + 1\right)$$

∎