Ordinary differential equations "ODE"

Ordinary D.E is an equation of one independent vareable ~~and~~ one dependent vareable, with it's derivative to order of derivatives (integer).

ex's

$\frac{dy}{dx} + 2y = 0$ 1^{st} order

$y'' - y = e^x$ 2^{nd} order

$y''' - 3y'' + y = \sin x$ 3^{rd} order

Methods of solve first order ODE

1- separable of vareable

2- reduciable to separable of vareable (Homogenuous)

3- Non-Homo. + (special case)

4- Exact D.equation and Non exact (Integral factor)

5- Linear ODE

6- Non-Linear ODE (Bernoully ODE)

Methods of solveing hiegher order ODE

There are several methods for example

1- undertemint coefficients

2. Viration of parameters

3- D- opereators

in oure course we will use "undetermint coefficients"
and some other techinques such That Laplace Trans.
and power series, That will be explaned Leate.

Now: we will explan The sixth methods and in next lecturs
undetermint coffficients and Laplace Transformation

1 separable of vareiable

$$y' = \frac{f(x)}{g(y)}$$

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \Longrightarrow g(y)\,dy = f(x)\,dx$$

Integral Both side } $\xrightarrow{I.B.S}$ $\int g(y)\,dy = \int f(x)\,dx$

$$\Longrightarrow y = h(x,y) \text{ or } h(x)$$

ex1

$$y' - y = 0$$

$$\frac{dy}{dx} = y$$

$$\frac{dy}{y} = dx \qquad I.B.S$$

$$\ln y = x + c$$

$$y = e^{x} \cdot e^{c} \longrightarrow y = K e^{x}$$

ex2 solve $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

$$y(1-ay) = \frac{dy}{dx}(a+x)$$

$$\frac{y(1-ay)}{dy} = \frac{a+x}{dx}$$

$$\frac{dy}{y(1-ay)} = \frac{dx}{a+x}$$

$$\left(\frac{a}{ay-1} - \frac{1}{y}\right)dy = \frac{dx}{a+x} \qquad I.B.S$$

$$\ln(ay-1) - \ln y = \ln(a+x) + c$$

$$\ln\frac{ay-1}{y} = \ln(a+x) + c$$

$$a - \frac{1}{y} = (a+x) \cdot K$$

$$\frac{1}{y} = a - (a+x) \cdot K$$

$$y = \frac{1}{a - (a+x) \cdot K}$$

(2) Reduciable to separable of var.

Homogenuos eq.

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad \text{or} \quad \frac{dy}{dx} = f\left(\frac{x}{y}\right)$$

to solve This Type let $v = \frac{y}{x} \Rightarrow y = xv$

$$\Rightarrow \frac{dy}{dx} = \frac{dx}{dx} \cdot v + x\frac{dv}{dx}$$

$$\Rightarrow y' = v + xv'$$

$$\Rightarrow f(v) = v + x\frac{dv}{dx}$$

$$\Rightarrow f(v) - v = x\frac{dv}{dx}$$

$$\Rightarrow \frac{dv}{f(v) - v} = \frac{dx}{x} \qquad \text{I.B.S.}$$

$$v = h(x) \quad \text{or} \quad h(x, v)$$

$$\frac{y}{x} = h(x) \quad \text{or} \quad h\left(x, \frac{y}{x}\right)$$

$$y = x\,h(x) \quad \text{or} \quad x\,h\left(x, \frac{y}{x}\right)$$

ex1 solve $(x^2 + y^2)\,dx = 2xy\,dy$

sol.

$$\frac{x^2+y^2}{2xy} = \frac{dy}{dx} \quad * \quad \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$\Rightarrow \frac{1+\frac{y^2}{x^2}}{2\frac{y}{x}} = \frac{dy}{dx} \qquad \text{let} \quad v = \frac{y}{x} \Rightarrow \frac{dy}{dx} = v + \frac{dv}{dx}$$

$$\frac{1+v^2}{2v} = \left(v + \frac{dv}{dx}\right)$$

$$\Rightarrow \frac{dv}{\frac{1+v^2}{2v} - v} = \frac{dx}{x}$$

$$\frac{2v\,dv}{1-v^2} = \frac{dx}{x} \qquad \text{I.B.S}$$

$$-\ln(1-v^2) = \ln(x) + c$$

$$\ln(1-v^2)^{-1} = \ln(x) + c$$

$$\frac{1}{1-v^2} = k \cdot x$$

$$x(1-v^2) = \frac{1}{k} = k_1$$

$$x\left(1 - \frac{y^2}{x^2}\right) = K_1$$

$$y^2 = x^2 - xK_1$$

$$y = \sqrt{x^2 - xK_1}$$

H.W solve $x\frac{dy}{dx} = y(\ln y - \ln x + 1)$

(3) Non Homog. equations

$$\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$$

(I) let $\left.\begin{array}{l} x = x_1 + h \\ y = y_1 + k \end{array}\right\} \Rightarrow \frac{dy}{dx} = \frac{dy_1}{dx_1} + 0, \quad \frac{dx_1}{dy_1} = \frac{dx}{dy}$

(II) let The constants $= 0 \Rightarrow$ $\begin{array}{l} a_1 h + b_1 h + c_1 = 0 \\ a_2 k + b_2 k + c_2 = 0 \end{array}$

(III) find h & k ; This Transfer eq. to homog.

(IV) solve by The priviaus method.

ex1 solve $\frac{dy}{dx} = \frac{y - x + 1}{y + x + 5}$

let $x = x_1 + h$
$y = y_1 + k$

$$\frac{dy_1}{dx_1} = \frac{\cancel{y_1} + k - x_1 - \cancel{h} + 1}{y_1 + k + x_1 + \cancel{h} + 5}$$

$$\left.\begin{array}{l} k - h + 1 = 0 \\ k + h + 5 = 0 \end{array}\right\} \rightarrow \begin{array}{l} h = -2 \\ k = -3 \end{array}$$

$$\Rightarrow \frac{dy_1}{dx_1} = \frac{y_1 - x_1}{y_1 + x_1} = \frac{\frac{y_1}{x} - 1}{\frac{y_1}{x_1} + 1}$$

let $v = \frac{y_1}{x_1}$

$$v + x_1\frac{dv}{dx_1} = \frac{v - 1}{v + 1}$$

$$\frac{dv}{\frac{v-1}{v+1} - v} = \frac{dx_1}{x_1}$$

$$-\frac{v + 1}{1 + v^2}\,dv = \frac{dx_1}{x_1}$$

$$\left(\frac{-v}{1+v^2} - \frac{1}{1+v^2}\right)dv = \frac{dx_1}{x_1}$$

$$-\ln(1+v^2) - \tan^{-1} v = \ln x_1 + c$$

⋮

H.W.

special case

if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{1}{\alpha}$

$\Rightarrow a_2 = \alpha a_1 \quad \Longrightarrow \quad V = \alpha(a_1 x + b_1 y)$

$b_2 = \alpha b_1$

$\cancel{c_2 = \alpha}$

$$\frac{dV}{dx} = \alpha a_1 + b_1 \frac{dy}{dx}$$

ex

$$\frac{dy}{dx} = \frac{3y+2x+4}{6y+4x+5} = \frac{3y+2x+4}{2(3y+2x)+5}$$

let $V = 3y + 2x \Rightarrow \frac{dV}{dx} = \frac{3\,dy}{dx} + 2$

$$\frac{1}{3}\frac{dV}{dx} - \frac{2}{3} = \frac{V+4}{2V+5}$$

$$\frac{dV}{dx} = \frac{21V+66}{6V+15}$$

$$\frac{6V+15}{21V+66}\,dV = dx \Longrightarrow$$

Long division

$$\frac{6}{21} + \frac{15-\frac{132}{7}}{21V+66}$$

$21V+66 \,\big)\, 6V + 15$

$6V + \frac{132}{7}$

$\left(15 - \frac{132}{7}\right)$

$\left(15 - \frac{132}{7}\right)$

00

$$\frac{6}{21} + \frac{15-\frac{132}{7}}{21V+66}\,dV = dx$$

$$\frac{2}{7} + \frac{-\frac{27}{7}}{21V+66}\,dV = dx \quad \text{I.B.S.}$$

$$\frac{2}{7}V - \frac{9}{7}\cdot\frac{1}{7}\ln(21V+66) = x + c$$

to be continued

H.W. solve

$(y-x+5)\,dy =$
$= (y-x+1)\,dx$

$3x-2y+1\,dy = 6x-4y+3\,dx$

(4) Exact Method

General Form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

it called exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

ex1

$$y\,dx + x\,dy = 0 \quad (*) \Rightarrow \quad M = y \Rightarrow \frac{\partial M}{\partial y} = 1$$
$$N = x \rightarrow \frac{\partial N}{\partial x} = 1$$

$\therefore$ (*) is exact.

H.w. (2)

$$(x^4 - 2xy^2 + y^4)\,dx - (2x^2y - 4xy^3 + \sin y)\,dy = 0$$

show it is exact. (......)

$$\frac{\partial M}{\partial y} = -4xy + 4y^3 = \frac{\partial N}{\partial x}$$

To sol. Exact eq

I) let $u = \int M\,dx + g(y)$ or

$\boxed{u = \int N\,dy + h(x)}$ ——— (1) ← constant of integration

II) find $g(y)$ or $\boxed{h(x)}$ by using

$\frac{\partial u}{\partial y} = N$ or $\boxed{\frac{\partial u}{\partial x} = M}$ ——— (2) [الطريقة الثانية]

III) subsitut (2) in (1)

u is The requiered solution ...

ex solve $y\,dx + x\,dy = 0$ ~~by ex1~~ it is

$\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial y} \Rightarrow$ it is exact

method (1)	method (2)
$u = \int y\,dx + g(y)$	$u = \int x\,dy + h(x)$
$u = yx + g(y)$ ——— (1)	$u = xy + h(x)$ ——— (1)
$\frac{\partial u}{\partial y} = N \Rightarrow x + g' = x$	$\frac{\partial u}{\partial x} = M \Rightarrow y + h' = y$
$\Rightarrow g' = 0 \Rightarrow g = C$ — (2) constant	$\Rightarrow h' = 0$ $\Rightarrow h = C$ ——— (2)
$\therefore u = yx + C$	$\therefore u = xy + C$

ex solve

$$y^2 e^{xy^2} + 4x^3 \; dx + (2xy e^{xy^2} - 3y^2)\, dy = 0$$

$$\frac{\partial M}{\partial y} = 2y e^{xy^2} + y^2 e^{xy^2} \cdot 2xy$$

$$\frac{\partial N}{\partial x} = 2y \cdot e^{xy^2} + e^{xy^2} y^2 \cdot 2xy$$

$\Rightarrow$ it is exact.

let $u = \int y^2 e^{xy^2} + 4x^3 \; dx + g(y)$

$$u = e^{xy^2} + x^4 + g(y) \quad \text{———} \; 1$$

$$\frac{\partial u}{\partial y} = N \quad \Rightarrow 2yx\, e^{xy^2} + g' = 2xy\, e^{xy^2} - 3y^2$$

$$\Rightarrow g' = -3y^2$$

$$\Rightarrow g = -\frac{\cancel{3}}{\cancel{3}} y^3 + c = -y^3 + c \quad \text{———} \; 2$$

$$\therefore u = e^{xy^2} + x^4 - y^3 + c$$

H.w. solve ex2 page 66

Non-Exact eq.

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

it is not exact eq. ; but we can made it exact by multiplicat it by Integral factor (I.F.)

There are four (methodes / rules) for find (I.F.)

rule 1

$$I.F. = \frac{1}{xM + yN}$$

ex solve $(x^2y - 2xy^2)\,dx + (-x^3 + 3x^2y)\,dy = 0$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= x^2 - 4xy \\ \frac{\partial N}{\partial x} &= -3x^2 + 6xy \end{aligned}\right\} \longrightarrow \text{not exact}$$

$$I.F = \frac{1}{xM + yN} = \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2}$$

$$= \frac{1}{x^2y^2}$$

$I.F.\,M\,dx + I.F\,N\,dy = 0$

$M_1\,dx + N_1\,dy = 0$

$\left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(\frac{-x}{y^2} + \frac{3}{y}\right)dy = 0$

$$\left\{\begin{aligned} \frac{\partial M_1}{\partial y} &= \frac{-1}{y^2} \\ \frac{\partial N_1}{\partial x} &= \frac{-1}{y^2} \end{aligned}\right.$$

$\therefore$ it is exact

complete The sol. H.w.

Rule 2 $\quad I.F. = \dfrac{1}{xM - yN}$

ex1

sol. $\quad y\,(xy + 2x^2y^2)\,dx + x\,(xy - x^2y^2)\,dy = 0 \quad \longrightarrow (*)$

check That:

(a) $(*)$ it is not exact H.w.

(b) find I.F. by Rule (2) H.w.

(c) find M_1 & N_1 H.w.

(d) Show $M_1\,dx + N_1\,dy = 0$ is exact H.w.

(e) solve The equation in (d)

Rule 3

$$I.F. = e^{\int f(x)\,dx}$$

when $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = f(x)$ function of x alon

ex1

$$(x^2 + y^2 + x)\,dx + (xy)\,dy = 0 \quad \text{—— } (*)$$

by using The procedure in ex1 in page 70

ⓐ $\frac{\partial M}{\partial y} = 2y \quad , \quad \frac{\partial N}{\partial x} = y \Rightarrow (*)$ not exact

ⓑ $I.F. = e^{\int f(x)\,dx}$

$$f(x) = \frac{1}{xy} \cdot [2y - y] = \frac{1}{x}$$

$$I.F. = e^{\int \frac{1}{x}} = e^{\ln x} = x$$

ⓒ $M_1 = x^3 + xy^2 + x^2$

$N_1 = x^2 y$

ⓓ $\frac{\partial M_1}{\partial y} = 2yx \quad , \quad \frac{\partial N_1}{\partial x} = 2xy \Rightarrow$ it exact eq.

ⓔ Shave $M_1\,dx + N_1\,dx = 0$ H.W.

$$\left\{ u = \frac{x^4}{4} + \frac{x^2 y^2}{2} + \frac{x^3}{3} + c \right\}$$

Rule (4)

$$I.F. = e^{\int g(y)\,dy}$$

when $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = g(y)$ function of y alon

ex1 solve

$$(2xy^4 e^y + 2xy^3 + y)\,dx + (x^2 y^4 e^y - x^2 y^2 - 3x)\,dy = 0$$

do The same of privious example .....

H.W.

$$U = x^2 e^y + \frac{x^2}{y} + \frac{x}{y^3} + c$$

(5) Linear ODE

General form

$$y' + p(x)\, y = q(x) \quad \text{———} \ (*)$$

to solve this eq.

multiplicate $(*)$ by $e^{\int p(x)\,dx}$

$$y' e^{\int p(x)} + p(x)\, y\, e^{\int p(x)} = q(x)\, e^{\int p(x)}$$

$$\left[y\, e^{\int p(x)\,dx} \right]' = q(x)\, e^{\int p(x)} \qquad \text{I.B.S.}$$

$$y\, e^{\int p(x)\,dx} = \left[\int q(x) \cdot e^{\int p(x)} \right] + C$$

or

$$y = e^{-\int p(x)\,dx} \left[\left(\int q(x)\, e^{\int p(x)\,dx} \right) + C \right]$$

ex1

$$y' + y x^2 = x^2$$

$$\Rightarrow p(x) = x^2 \quad \& \quad q(x) = x^2$$

$$\therefore y \cdot e^{\int x^2} = \left[\int x^2 e^{\int x^2} + c \right]$$

$$y \cdot e^{\frac{x^3}{3}} = \left[\int x^2 e^{\frac{1}{3}x^3} dx \right] + c$$

$$y \cdot e^{\frac{x^3}{3}} = e^{\frac{x^3}{3}} + c$$

$$y = e^{\frac{-x^3}{3}} \cdot e^{\frac{x^3}{3}} + c \cdot e^{\frac{-x^3}{3}}$$

$$y = 1 + c \cdot e^{-x^3/3}$$

ex2 $\quad x \ln x \, y' + y = 2 \ln x \qquad \div x \ln x$

$$y' + \frac{1}{x \ln x} y = \frac{2}{x} \Rightarrow p = \frac{1}{x \ln x}, \; q = \frac{2}{x}$$

$$y = e^{-\int \frac{1}{x \ln x} dx} \left[\left(\int \frac{2}{x} \cdot e^{\int \frac{1}{x \ln x} dx} dx \right) + c \right]$$

$$= e^{-\ln(\ln x)} \left[\left(\int \frac{2}{x} e^{\ln(\ln x)} dx \right) + c \right]$$

$$= \frac{1}{\ln x} \left[\left((\ln x)^2 \right) + c \right] = \frac{c}{\ln x} + \ln x$$

(6) Non-~~Exact~~ Linear equation

Bernolli's equation

Theres no. fixed Rule to solve This Type of eq. , for example

$$\sin y\; y' = \cos x\;(2\cos y - \sin^2 x)$$

$$\sin y\; y' = 2\cos x \cos y - \cos x \sin^2 x$$

$$-z' = 2\cos x\; z - \cos x \sin^2 x \qquad \Leftarrow \left\{\begin{array}{l} \text{let } z = \cos y \\ z' = -\sin y\; y' \end{array}\right.$$

$$z' + z\; 2\cos x = \cos x \sin^2 x$$

, it becom Linear

$$z = e^{-2\int \cos x}\left[\left(+\int \cos x \sin^2 x\; e^{2\int \cos x}\right) + c\right]$$

$$z = e^{-2\sin x}\left[\left(\int \cos x \sin^2 x\; e^{2\sin x}\right) + c\right] \qquad u = \sin x$$

$$z = e^{-2u}\left[\left(\int u^2 e^{2u}\, du\right) + c\right]$$

$$z = e^{-2u}\left[\left(\frac{u^2}{2} - \frac{2u}{4} + \frac{2}{8}\right) e^{2u} + c\right]$$

u^2	e^{2u}
$2u$	$\frac{1}{2} e^{2u}$
2	$\frac{1}{4} e^{2u}$
0	$\frac{1}{8} e^{2u}$

$$z = \frac{u^2}{2} - \frac{u}{2} + \frac{1}{4} + c\, e^{-2u}$$

$$\cos y = \frac{\sin^2 x}{2} - \frac{\sin x}{2} + \frac{1}{4} + c\, e^{-2\sin x}$$

ex 2 solv

$$y' + yx = y^3 e^{-x^2/2}$$

$$y^{-3}y' + y^{-2}x = 1 \qquad \text{let } z = y^{-2}$$

$$z' = y^{-3}y'$$

$$z' + zx = 1$$

$$z = e^{-\int x}\left[\left(\int e^{-x^2/2} \cdot e^{\int x} dx\right) + c\right]$$

$$= e^{-x^2/2}\left[\left(\int e^{x^2/2} \cdot e^{-x^2/2} dx\right) + c\right]$$

$$= e^{-x^2/2}\left[\left(x + c\right)\right]$$

$$= x e^{-x^2/2} + c e^{-x^2/2}$$

H.W.

① $y' + 2yx - e^{-x^2} y^3 x = 0$

② $(x^2 y^3 + xy)\, dy = dx$

Method to solve higgher order DE.

Undetermint coefficient methoded

General form of Linear D.E. of higher order with constant coefficient

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = f(x) \quad (2)$$

if $f(x) \neq 0$ (1) is called non-Homogeneous L.D.E.

if $f(x) = 0$

$$a_0 y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0 \quad (2)$$

is Homogeneous Linear D.E.

eq (2) has "n" solutions and its sum is called complementary ~~solution~~ (function) "$y_{c.f}$"

eq (1) has an particulal solution "$y_{P.I}$"

The General sol. of (1) "G.S"

$$G.S = y_{c.f} + y_{P.I}$$

How To find $y_{c.f}$ for (2)

I) make the coefficient of highest derivetiv "one"

II) write Axulary eq. by Transfer (A.E.) — (3)

$y = 1$
$y' = m$
$y'' = m^2$
$\vdots$
$y^{(n)} = m^n$

so, eq. (2) will be

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_2 m^2 + a_1 m + a_0 = 0 \quad \text{—— (3)}$$

solve (3) for m, so we find n-time of Roots m_i

Then

$$y_{c.f} = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

ex1

$$3y'' + 6y' - 9y = 0$$

$$y'' + 2y' - 3y = 0$$

$$m^2 + 2m - 3 = 0 \Longrightarrow (m-1)(m+3) = 0$$

$$\therefore m_1 = 1 \quad , \quad m_2 = -3$$

$$y_{c.f} = c_1 e^{1 \cdot x} + c_2 e^{-3x}$$

ex2 solve

$$y''' - 2y'' - 5y' + 6y = 0$$

$$m^3 - 2m^2 - 5m + 6 = 0$$

Factors of 6

$$(\mp 1, \mp 2, \mp 3, \mp 6)$$

since 1 is o.k satisfies The eq. $\Rightarrow (m-1)$ factor

$$\begin{array}{r|l} & m^2 - m - 6 \\ \hline m-1 & m^3 - 2m^2 - 5m + 6 \\ & m^3 - m^2 \\ \hline & -m^2 - 5m + 6 \\ & -m^2 + m \\ \hline & -6m + 6 \\ & -6m + 6 \\ \hline & 0 \quad 0 \end{array}$$

$$(m-1)(m^2 - m - 6) = 0$$

$$(m-1)(m+2)(m-3) = 0$$

$m_1 = 1$

$m_2 = -2$

$m_3 = 3$

$\therefore$

$$y_{c.f} = c_1 e^{x} + c_2 e^{-2x} + c_3 e^{3x}$$

note

There are Three case of Root for eq (3) H.E.

I) different Real Roots (see ex1 & ex2)

II) Repeated real Roots (see ex3 & ex4, ex5)

III) Complex Roots (conjugate as usual) see ex 6

ex3 $y'' + 2y' + y = 0$

repeated Two time

$m^2 + 2m + 1 = 0 \Rightarrow m_1 = m_2 = -1$

$\therefore$ $y_{c.f} = (c_1 + xc_2)e^{-x} = c_1 e^{-x} + x c_2 e^{-x}$

if m is repeated n-Times Root (m is a root of AE.)

$y_{c.f} = [c_1 + xc_2 + x^2 c_3 + \dots + x^{n-1} c_n]\, e^{mx}$

ex4 $y'''' + 4y''' + 6y'' + 4y' + y = 0$

$m^4 + 4m^3 + 6m^2 + 4m + 1 = 0$

$m_1 = m_2 = m_3 = m_4 = -1$ H.w.

$y_{c.f} = [c_1 + x c_2 + x^2 c_3 + x^3 c_4]\, e^{-x}$

ex5 $y''' - 3y'' + 4y = 0$

$m^3 - 3m^2 + 4 = 0 \Rightarrow m_1 = -1$

$m_2 = m_3 = 2$

$y_{c.f} = c_1 e^{-x} + [c_2 + x c_3]\, e^{2x}$

~~ex6~~ Case III

if A.E_3 has complex Root $z = a \mp ib$

Then
$$y_{c.f} = e^{ax}\left[A\cos bx + B\sin bx\right]$$

ex6

sol. $y'' + 2y' + 2y = 0$

$$m_{1,2} = \frac{-2 \mp \sqrt{4-8}}{2} = -1 \mp \sqrt{-1} = -1 \mp i$$

$a = -1$
$b = 1$

$$y_{c.f} = e^{ax}\left[A\cos bx + B\sin bx\right]$$

$$y_{c.f} = e^{-x}\left[A\cos x + B\sin x\right]$$

How to find $y_{P.I}$

if The R.H.S of (1) ($f(x) =$) is one of The function e^{rx} or $\sin bx$ (or other Trig.) or polynomial we will use The following Table

&c ....

$f(x)$	Cases of Root AE.	$y_{P.I}$
e^{rx}	r is not root of AE.	$A e^{rx}$
	r is single root of AE.	$A x e^{rx}$
	r is root of order "n" for AE.	$A x^n e^{rx}$
$\sin Kx$ or $\cos Kx$	K is not root of AE	$A \cos Kx + B \sin Kx$
	K is single root of AE	$x(A\cos Kx + B \sin Kx)$
	K is root of order "n"	$x^n (A \cos Kx + B \sin Kx)$
$P_m(x)$	0 is not root of AE	$\bar{P}_m(x)$
	0 is a single Root of AE	$\bar{P}_{m+1}(x)$
	0 is root of order "n" for AE.	$\bar{P}_{m+n}(x)$

note

by $P_m(x)$ we mean polynomial of order m

$$a x^m + b x^{m-1} + c x^{m-2} + \cdots + z x + a_0 = 0$$

by $\bar{P}_m(x)$ we mean polynomial other Than $P_n(x)$ s.t.

$$A \cancel{x}^m + B x^{m-1} + C x^{m-2} + \cdots + Z x + A_0 = 0$$

ex1 find $y_{P.I}$ for $y'' + 2y' + y = 4e^{3x}$ —— ①

by ex3 page 84 $y_{c.f} = (C_1 + xC_2)\, e^{-x}$

$$\left.\begin{aligned} y_{P.I} &= A e^{3x} \\ y' &= 3A e^{3x} \\ y'' &= 9A e^{3x} \end{aligned}\right\} \text{ in } ①$$

$$9A e^{3x} + 6A e^{3x} + A e^{3x} = 4 e^{3x}$$

$$16A = 4 \Longrightarrow A = \frac{1}{4}$$

$$\therefore y_{P.I} = \frac{1}{4} e^{3x}$$

ex2 $y'' - 3y' + 2y = e^{2x}$ —— ①

$A.E:\ m^2 - 3m + 2 = 0 \Longrightarrow m_1 = 1 \quad m_2 = 2$

$$y_{c.f} = C_1 e^{x} + C_2 e^{2x}$$

$y_{P.I} = Ax e^{2x}$ (γ is single root of AE.)

$$\left.\begin{aligned} y'_{P.I} &= 2Ax e^{2x} + A e^{2x} \\ y''_{P.I} &= 4Ax e^{2x} + 2A e^{2x} + 2A e^{2x} \end{aligned}\right\} \text{ in } ①$$

$\vdots$

$$A e^{2x} = e^{2x} \longrightarrow A = 1$$

$$\therefore y_{P.I} = x e^{2x} \longrightarrow G.S. = C_1 e^{x} + (C_2 + x) e^{2x}$$

ex 3

$$y'' + 3y' + 2y = 8\sin 2x \quad \text{—— (1)}$$

A.E. $m^2 + 3m + 2 = 0 \Rightarrow m_1 = -1 \quad m_2 = -2$

$$y_{c.f} = c_1 e^{-x} + c_2 e^{-2x}$$

$$\left.\begin{aligned} y_{P.I} &= A\cos 2x + B\sin 2x \\ y'_{P.I} &= -2A\sin 2x + 2B\cos 2x \\ y''_{P.I} &= -4A\cos 2x - 4B\sin 2x \end{aligned}\right\} \text{ in (1)}$$

$$-4A\cos 2x - 4\sin 2x - 6A\sin 2x + 6B\cos 2x + 2A\cos 2x + 2B\sin 2x = 8\sin 2x$$

$-4A + 6B + 2A = 0$ ($\cos 2x$ معاملات)

$-4B - 6A + 2B = 8$ ($\sin 2x$ معاملات)

$$B = \frac{-2}{5}, \quad A = \frac{-6}{5}$$

$$\therefore y_{P.I} = \frac{-6}{5}\cos 2x - \frac{2}{5}\sin 2x$$

note

if $f(x) = f_1(x) + f_2(x)$

Then $y_{P.I} = y_{P.I}(f_1) + y_{P.I}(f_2)$ as in Table P 83

H.W. 5

1- $y'' + 3y' + 2y = \sin 2x + e^{x}$

2- $y'' + 3y' + 2y = \cos x + \sin 3x$

3- $y'' - y' = -8x + 3$ ("0" is a root of A.E.)

4- $y'' - y = e^{x} + x^{2}$

5- $y'' - y' - 6y = e^{-x} - 7\cos x$

6- $y'' - 5y' = x + \cos 5x$

next lecture is

"Laplace Transformations"