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Calculus III - Notes

Absolute Minimums and Maximums

In this section we are going to extend the work from the previous section. In the previous section we were asked to find and classify all critical points as relative minimums, relative maximums and/or saddle points. In this section we want to optimize a function, that is identify the absolute minimum and/or the absolute maximum of the function, on a given region in $\mathbb{R}^2$. Note that when we say we are going to be working on a region in $\mathbb{R}^2$ we mean that we're going to be looking at some region in the xy-plane.

In order to optimize a function in a region we are going to need to get a couple of definitions out of the way and a fact. Let's first get the definitions out of the way.

Definitions

1. A region in $\mathbb{R}^2$ is called **closed** if it includes its boundary. A region is called **open** if it doesn't include any of its boundary points.
2. A region in $\mathbb{R}^2$ is called **bounded** if it can be completely contained in a disk. In other words, a region will be bounded if it is finite.

Let's think a little more about the definition of closed. We said a region is closed if it includes its boundary. Just what does this mean? Let's think of a rectangle. Below are two definitions of a rectangle, one is closed and the other is open.

Open	Closed
$-5 < x < 3$	$-5 \le x \le 3$
$1 < y < 6$	$1 \le y \le 6$

In this first case we don't allow the ranges to include the endpoints (*i.e.* we aren't including the edges of the rectangle) and so we aren't allowing the region to include any points on the edge of the rectangle. In other words, we aren't allowing the region to include its boundary and so it's open.

In the second case we are allowing the region to contain points on the edges and so will contain its entire boundary and hence will be closed.

This is an important idea because of the following fact.

Extreme Value Theorem

If $f(x,y)$ is continuous in some closed, bounded set D in $\mathbb{R}^2$ then there are points in D, (x_1, y_1) and (x_2, y_2) so that $f(x_1, y_1)$ is the absolute maximum and $f(x_2, y_2)$ is the absolute minimum of the function in D.

Note that this theorem does NOT tell us where the absolute minimum or absolute maximum will occur. It only tells us that they will exist. Note as well that the absolute minimum and/or absolute maximum may occur in the interior of the region or it may occur on the boundary of the region.

The basic process for finding absolute maximums is pretty much identical to the process that we used in Calculus I when we looked at finding absolute extrema of functions of single variables. There will however, be some procedural changes to account for the fact that we now are dealing with functions of two variables. Here is the process.

Finding Absolute Extrema

1. Find all the critical points of the function that lie in the region D and determine the function value at each of these points.

2. Find all extrema of the function on the boundary. This usually involves the Calculus I approach for this work.

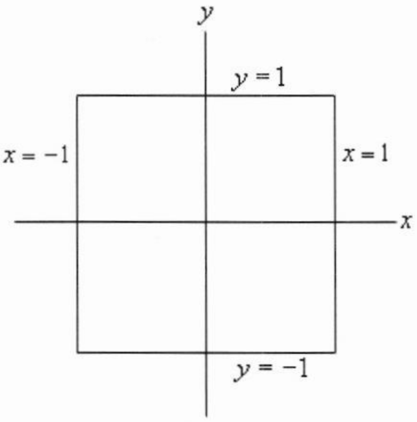
3. The largest and smallest values found in the first two steps are the absolute minimum and the absolute maximum of the function.

The main difference between this process and the process that we used in Calculus I is that the "boundary" in Calculus I was just two points and so there really wasn't a lot to do in the second step. For these problems the majority of the work is often in the second step as we will often end up doing a Calculus I absolute extrema problem one or more times.

Let's take a look at an example or two.

Example 1 Find the absolute minimum and absolute maximum of $f(x,y) = x^2 + 4y^2 - 2x^2y + 4$ on the rectangle given by $-1 \le x \le 1$ and $-1 \le y \le 1$.

Solution
Let's first get a quick picture of the rectangle for reference purposes.



The boundary of this rectangle is given by the following conditions.

$$\begin{aligned} &\text{right side}: && x = 1,\ -1 \le y \le 1 \\ &\text{left side}: && x = -1,\ -1 \le y \le 1 \\ &\text{upper side}: && y = 1,\ -1 \le x \le 1 \\ &\text{lower side}: && y = -1,\ -1 \le x \le 1 \end{aligned}$$

These will be important in the second step of our process.

We'll start this off by finding all the critical points that lie inside the given rectangle. To do this we'll need the two first order derivatives.

$$f_x = 2x - 4xy \qquad\qquad f_y = 8y - 2x^2$$

Note that since we aren't going to be classifying the critical points we don't need the second order derivatives. To find the critical points we will need to solve the system,

$$2x - 4xy = 0$$
$$8y - 2x^2 = 0$$

We can solve the second equation for y to get,

$$y = \frac{x^2}{4}$$

Plugging this into the first equation gives us,

$$2x - 4x\left(\frac{x^2}{4}\right) = 2x - x^3 = x\left(2 - x^2\right) = 0$$

This tells us that we must have $x = 0$ or $x = \pm\sqrt{2} = \pm 1.414\ldots$. Now, recall that we only want critical points in the region that we're given. That means that we only want critical points for which $-1 \le x \le 1$. The only value of x that will satisfy this is the first one so we can ignore the last two for this problem. Note however that a simple change to the boundary would include these two so don't forget to always check if the critical points are in the region (or on the boundary since that can also happen).

Plugging $x = 0$ into the equation for y gives us,

$$y = \frac{0^2}{4} = 0$$

The single critical point, in the region (and again, that's important), is $(0, 0)$. We now need to get the value of the function at the critical point.

$$f(0,0) = 4$$

Eventually we will compare this to values of the function found in the next step and take the largest and smallest as the absolute extrema of the function in the rectangle.

Now we have reached the long part of this problem. We need to find the absolute extrema of the function along the boundary of the rectangle. What this means is that we're going to need to look at what the function is doing along each of the sides of the rectangle listed above.

Let's first take a look at the right side. As noted above the right side is defined by

$$x = 1, \quad -1 \le y \le 1$$

Notice that along the right side we know that $x = 1$. Let's take advantage of this by defining a new function as follows,

$$g(y) = f(1, y) = 1^2 + 4y^2 - 2\left(1^2\right)y + 4 = 5 + 4y^2 - 2y$$

Now, finding the absolute extrema of $f(x, y)$ along the right side will be equivalent to finding the absolute extrema of $g(y)$ in the range $-1 \le y \le 1$. Hopefully you recall how to do this from Calculus I. We find the critical points of $g(y)$ in the range $-1 \le y \le 1$ and then evaluate $g(y)$ at the critical points and the end points of the range of y's.

Let's do that for this problem.

$$g'(y) = 8y - 2 \qquad \Rightarrow \qquad y = \frac{1}{4}$$

This is in the range and so we will need the following function evaluations.

$$g(-1) = 11 \qquad g(1) = 7 \qquad g\left(\frac{1}{4}\right) = \frac{19}{4} = 4.75$$

Notice that, using the definition of $g(y)$ these are also function values for $f(x,y)$.

$$g(-1) = f(1,-1) = 11$$
$$g(1) = f(1,1) = 7$$
$$g\left(\frac{1}{4}\right) = f\left(1,\frac{1}{4}\right) = \frac{19}{4} = 4.75$$

We can now do the left side of the rectangle which is defined by,

$$x = -1, \; -1 \le y \le 1$$

Again, we'll define a new function as follows,

$$g(y) = f(-1,y) = (-1)^2 + 4y^2 - 2(-1)^2 y + 4 = 5 + 4y^2 - 2y$$

Notice however that, for this boundary, this is the same function as we looked at for the right side. This will not always happen, but since it has let's take advantage of the fact that we've already done the work for this function. We know that the critical point is $y = \frac{1}{4}$ and we know that the function value at the critical point and the end points are,

$$g(-1) = 11 \qquad g(1) = 7 \qquad g\left(\frac{1}{4}\right) = \frac{19}{4} = 4.75$$

The only real difference here is that these will correspond to values of $f(x,y)$ at different points than for the right side. In this case these will correspond to the following function values for $f(x,y)$.

$$g(-1) = f(-1,-1) = 11$$
$$g(1) = f(-1,1) = 7$$
$$g\left(\frac{1}{4}\right) = f\left(-1,\frac{1}{4}\right) = \frac{19}{4} = 4.75$$

We can now look at the upper side defined by,

$$y = 1, \; -1 \le x \le 1$$

We'll again define a new function except this time it will be a function of x.

$$h(x) = f(x,1) = x^2 + 4(1^2) - 2x^2(1) + 4 = 8 - x^2$$

We need to find the absolute extrema of $h(x)$ on the range $-1 \le x \le 1$. First find the critical point(s).

$$h'(x) = -2x \qquad \Rightarrow \qquad x = 0$$

The value of this function at the critical point and the end points is,

$$h(-1) = 7 \qquad h(1) = 7 \qquad h(0) = 8$$

and these in turn correspond to the following function values for $f(x,y)$

$$h(-1) = f(-1,1) = 7$$
$$h(1) = f(1,1) = 7$$
$$h(0) = f(0,1) = 8$$

Note that there are several "repeats" here. The first two function values have already been computed when we looked at the right and left side. This will often happen.

Finally, we need to take care of the lower side. This side is defined by,

$$y = -1, \ -1 \le x \le 1$$

The new function we'll define in this case is,

$$h(x) = f(x,-1) = x^2 + 4(-1)^2 - 2x^2(-1) + 4 = 8 + 3x^2$$

The critical point for this function is,

$$h'(x) = 6x \qquad \Rightarrow \qquad x = 0$$

The function values at the critical point and the endpoint are,

$$h(-1) = 11 \qquad h(1) = 11 \qquad h(0) = 8$$

and the corresponding values for $f(x,y)$ are,

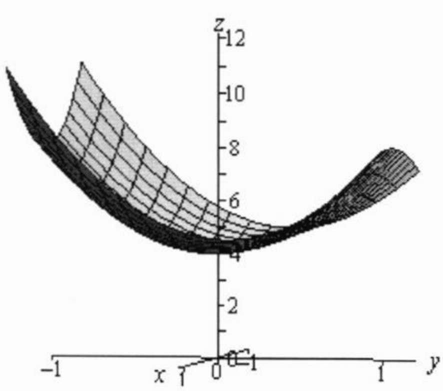
$$h(-1) = f(-1,-1) = 11$$
$$h(1) = f(1,-1) = 11$$
$$h(0) = f(0,-1) = 8$$

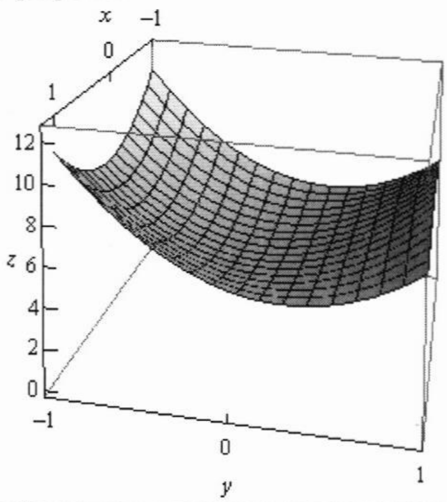
The final step to this (long…) process is to collect up all the function values for $f(x,y)$ that we've computed in this problem. Here they are,

$$f(0,0) = 4 \qquad f(1,-1) = 11 \qquad f(1,1) = 7$$
$$f\left(1,\frac{1}{4}\right) = 4.75 \qquad f(-1,1) = 7 \qquad f(-1,-1) = 11$$
$$f\left(-1,\frac{1}{4}\right) = 4.75 \qquad f(0,1) = 8 \qquad f(0,-1) = 8$$

The absolute minimum is at $(0,0)$ since gives the smallest function value and the absolute maximum occurs at $(1,-1)$ and $(-1,-1)$ since these two points give the largest function value.

Here is a sketch of the function on the rectangle for reference purposes.





Lagrange Multipliers

In the previous section we optimized (*i.e.* found the absolute extrema) a function on a region that contained its boundary. Finding potential optimal points in the interior of the region isn't too bad in general, all that we needed to do was find the critical points and plug them into the function. However, as we saw in the examples finding potential optimal points on the boundary was often a fairly long and messy process.

In this section we are going to take a look at another way of optimizing a function subject to given constraint(s). The constraint(s) may be the equation(s) that describe the boundary of a region although in this section we won't concentrate on those types of problems since this method just requires a general constraint and doesn't really care where the constraint came from.

So, let's get things set up. We want to optimize (*i.e.* find the minimum and maximum value of) a function, $f(x,y,z)$, subject to the constraint $g(x,y,z)=k$. Again, the constraint may be the equation that describes the boundary of a region or it may not be. The process is actually fairly simple, although the work can still be a little overwhelming at times.

Method of Lagrange Multipliers

1. Solve the following system of equations.

$$\nabla f(x,y,z) = \lambda \, \nabla g(x,y,z)$$
$$g(x,y,z) = k$$

2. Plug in all solutions, (x,y,z), from the first step into $f(x,y,z)$ and identify the minimum and maximum values, provided they exist.

The constant, λ, is called the **Lagrange Multiplier**.

Notice that the system of equations actually has four equations, we just wrote the system in a simpler form. To see this let's take the first equation and put in the definition of the gradient vector to see what we get.

$$\langle f_x, f_y, f_z \rangle = \lambda \langle g_x, g_y, g_z \rangle = \langle \lambda g_x, \lambda g_y, \lambda g_z \rangle$$

In order for these two vectors to be equal the individual components must also be equal. So, we actually have three equations here.

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z$$

These three equations along with the constraint, $g(x,y,z)=c$, give four equations with four unknowns x, y, z, and λ.

Note as well that if we only have functions of two variables then we won't have the third component of the gradient and so will only have three equations in three unknowns x, y, and λ.

As a final note we also need to be careful with the fact that in some cases minimums and maximums won't exist even though the method will seem to imply that they do. In every problem we'll need to go back and make sure that our answers make sense.

Let's work a couple of examples.

Example 1 Find the dimensions of the box with largest volume if the total surface area is 64 cm^2.

Solution

Before we start the process here note that we also saw a way to solve this kind of problem in Calculus I, except in those problems we required a condition that related one of the sides of the box to the other sides so that we could get down to a volume and surface area function that only involved two variables. We no longer need this condition for these problems.

Now, let's get on to solving the problem. We first need to identify the function that we're going to optimize as well as the constraint. Let's set the length of the box to be x, the width of the box to be y and the height of the box to be z. Let's also note that because we're dealing with the dimensions of a box it is safe to assume that x, y, and z are all positive quantities.

We want to find the largest volume and so the function that we want to optimize is given by,

$$f(x, y, z) = xyz$$

Next we know that the surface area of the box must be a constant 64. So this is the constraint. The surface area of a box is simply the sum of the areas of each of the sides so the constraint is given by,

$$2xy + 2xz + 2yz = 64 \qquad \Rightarrow \qquad xy + xz + yz = 32$$

Note that we divided the constraint by 2 to simplify the equation a little. Also, we get the function $g(x, y, z)$ from this.

$$g(x, y, z) = xy + xz + yz$$

Here are the four equations that we need to solve.

$$\begin{aligned} yz &= \lambda(y+z) & (f_x &= \lambda g_x) & (1)\\ xz &= \lambda(x+z) & (f_y &= \lambda g_y) & (2)\\ xy &= \lambda(x+y) & (f_z &= \lambda g_z) & (3)\\ xy + xz + yz &= 32 & (g(x,y,z) &= 32) & (4) \end{aligned}$$

There are many ways to solve this system. We'll solve it in the following way. Let's multiply equation (1) by x, equation (2) by y and equation (3) by z. This gives,

$$\begin{aligned} xyz &= \lambda x(y+z) & (5)\\ xyz &= \lambda y(x+z) & (6)\\ xyz &= \lambda z(x+y) & (7) \end{aligned}$$

Now notice that we can set equations (5) and (6) equal. Doing this gives,

$$\begin{aligned} \lambda x(y+z) &= \lambda y(x+z)\\ \lambda(xy+xz) - \lambda(yx+yz) &= 0\\ \lambda(xz - yz) &= 0 \qquad \Rightarrow \qquad \lambda = 0 \quad \text{or} \quad xz = yz \end{aligned}$$

This gave two possibilities. The first, $\lambda = 0$ is not possible since if this was the case equation (1) would reduce to

$$yz = 0 \qquad \Rightarrow \qquad y = 0 \text{ or } z = 0$$

Since we are talking about the dimensions of a box neither of these are possible so we can discount $\lambda = 0$. This leaves the second possibility.

$$xz = yz$$

Since we know that $z \neq 0$ (again since we are talking about the dimensions of a box) we can cancel the z from both sides. This gives,

$$x = y \qquad (8)$$

Next, let's set equations (6) and (7) equal. Doing this gives,

$$\lambda y(x+z) = \lambda z(x+y)$$
$$\lambda(yx + yz - zx - zy) = 0$$
$$\lambda(yx - zx) = 0 \qquad \Rightarrow \qquad \lambda = 0 \text{ or } yx = zx$$

As already discussed we know that $\lambda = 0$ won't work and so this leaves,

$$yx = zx$$

We can also say that $x \neq 0$ since we are dealing with the dimensions of a box so we must have,

$$z = y \qquad (9)$$

Plugging equations (8) and (9) into equation (4) we get,

$$y^2 + y^2 + y^2 = 3y^2 = 32 \qquad\qquad y = \pm\sqrt{\frac{32}{3}} = \pm 3.266$$

However, we know that y must be positive since we are talking about the dimensions of a box. Therefore the only solution that makes physical sense here is

$$x = y = z = 3.266$$

So, it looks like we've got a cube here.

We should be a little careful here. Since we've only got one solution we might be tempted to assume that these are the dimensions that will give the largest volume. The method of Lagrange Multipliers will give a set of points that will either maximize or minimize a given function subject to the constraint, provided there actually are minimums or maximums.

The function itself, $f(x, y, z) = xyz$ will clearly have neither minimums or maximums unless we put some restrictions on the variables. The only real restriction that we've got is that all the variables must be positive. This, of course, instantly means that the function does have a minimum, zero.

The function will not have a maximum if all the variables are allowed to increase without bound. That however, can't happen because of the constraint,

$$xy + xz + yz = 32$$

Here we've got the sum of three positive numbers (because x, y, and z are positive) and the sum must equal 32. So, if one of the variables gets very large, say x, then because each of the products must be less than 32 both y and z must be very small to make sure the first two terms are less than 32. So, there is no way for all the variables to increase without bound and so it should make some sense that the function, $f(x, y, z) = xyz$, will have a maximum.

This isn't a rigorous proof that the function will have a maximum, but it should help to visualize that in fact it should have a maximum and so we can say that we will get a maximum volume if the dimensions are : $x = y = z = 3.266$.

Notice that we never actually found values for λ in the above example. This is fairly standard for these kinds of problems. The value of λ isn't really important to determining if the point is a maximum or a minimum so often we will not bother with finding a value for it. On occasion we will need its value to help solve the system, but even in those cases we won't use it past finding the point.

Example 2 Find the maximum and minimum of $f(x, y) = 5x - 3y$ subject to the constraint $x^2 + y^2 = 136$.

Solution

This one is going to be a little easier than the previous one since it only has two variables. Also, note that it's clear from the constraint that region of possible solutions lies on a disk of radius $\sqrt{136}$ which is a closed and bounded region and hence by the Extreme Value Theorem we know that a minimum and maximum value must exist.

Here is the system that we need to solve.

$$\begin{aligned} 5 &= 2\lambda x \\ -3 &= 2\lambda y \\ x^2 + y^2 &= 136 \end{aligned}$$

Notice that, as with the last example, we can't have $\lambda = 0$ since that would not satisfy the first two equations. So, since we know that $\lambda \neq 0$ we can solve the first two equations for x and y respectively. This gives,

$$x = \frac{5}{2\lambda} \qquad\qquad y = -\frac{3}{2\lambda}$$

Plugging these into the constraint gives,

$$\frac{25}{4\lambda^2} + \frac{9}{4\lambda^2} = \frac{17}{2\lambda^2} = 136$$

We can solve this for λ.

$$\lambda^2 = \frac{1}{16} \qquad \Rightarrow \qquad \lambda = \pm\frac{1}{4}$$

Now, that we know λ we can find the points that will be potential maximums and/or minimums.

If $\lambda = -\frac{1}{4}$ we get,

$$x = -10 \qquad\qquad y = 6$$

and if $\lambda = \frac{1}{4}$ we get,

$$x = 10 \qquad\qquad y = -6$$

To determine if we have maximums or minimums we just need to plug these into the function. Also recall from the discussion at the start of this solution that we know these will be the minimum and maximums because the Extreme Value Theorem tells us that minimums and maximums will exist for this problem.

Here are the minimum and maximum values of the function.

$$\begin{aligned} f(-10,6) &= -68 \qquad \text{Minimum at } (-10,6) \\ f(10,-6) &= 68 \qquad \text{Maximum at } (10,-6) \end{aligned}$$

In the first two examples we've excluded $\lambda = 0$ either for physical reasons or because it wouldn't solve one or more of the equations. Do not always expect this to happen. Sometimes we will be able to automatically exclude a value of λ and sometimes we won't.

Let's take a look at another example.

Example 3 Find the maximum and minimum values of $f(x, y, z) = xyz$ subject to the constraint $x + y + z = 1$. Assume that $x, y, z \geq 0$.

Solution

First note that our constraint is a sum of three positive or zero number and it must be 1. Therefore it is clear that our solution will fall in the range $0 \le x, y, z \le 1$. Therefore the solution must lie in a closed and bounded region and so by the Extreme Value Theorem we know that a minimum and maximum value must exist.

Here is the system of equation that we need to solve.

$$yz = \lambda \qquad (10)$$
$$xz = \lambda \qquad (11)$$
$$xy = \lambda \qquad (12)$$
$$x + y + z = 1 \qquad (13)$$

Let's start this solution process off by noticing that since the first three equations all have λ they are all equal. So, let's start off by setting equations (10) and (11) equal.

$$yz = xz \qquad \Rightarrow \qquad z(y-x) = 0 \qquad \Rightarrow \qquad z = 0 \text{ or } y = x$$

So, we've got two possibilities here. Let's start off with by assuming that $z = 0$. In this case we can see from either equation (10) or (11) that we must then have $\lambda = 0$. From equation (12) we see that this means that $xy = 0$. This in turn means that either $x = 0$ or $y = 0$.

So, we've got two possible cases to deal with there. In each case two of the variables must be zero. Once we know this we can plug into the constraint, equation (13), to find the remaining value.

$$z = 0,\ x = 0: \qquad \Rightarrow \qquad y = 1$$
$$z = 0,\ y = 0: \qquad \Rightarrow \qquad x = 1$$

So, we've got two possible solutions $(0,1,0)$ and $(1,0,0)$.

Now let's go back and take a look at the other possibility, $y = x$. We also have two possible cases to look at here as well.

This first case is $x = y = 0$. In this case we can see from the constraint that we must have $z = 1$ and so we now have a third solution $(0,0,1)$.

The second case is $x = y \ne 0$. Let's set equations (11) and (12) equal.

$$xz = xy \qquad \Rightarrow \qquad x(z-y) = 0 \qquad \Rightarrow \qquad x = 0 \text{ or } z = y$$

Now, we've already assumed that $x \ne 0$ and so the only possibility is that $z = y$. However, this also means that,

$$x = y = z$$

Using this in the constraint gives,

$$3x = 1 \qquad \Rightarrow \qquad x = \frac{1}{3}$$

So, the next solution is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

We got four solutions by setting the first two equations equal.

To completely finish this problem out we should probably set equations (10) and (12) equal as well as setting equations (11) and (12) equal to see what we get. Doing this gives,

$$yz = xy \quad \Rightarrow \quad y(z-x)=0 \quad \Rightarrow \quad y=0 \text{ or } z=x$$
$$xz = xy \quad \Rightarrow \quad x(z-y)=0 \quad \Rightarrow \quad x=0 \text{ or } z=y$$

Both of these are very similar to the first situation that we looked at and we'll leave it up to you to show that in each of these cases we arrive back at the four solutions that we already found.

So, we have four solutions that we need to check in the function to see whether we have minimums or maximums.

$$f(0,0,1)=0 \qquad f(0,1,0)=0 \qquad f(1,0,0)=0 \qquad \text{All Minimums}$$
$$f\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)=\frac{1}{27} \qquad \text{Maximum}$$

So, in this case the maximum occurs only once while the minimum occurs three times.

Note as well that we never really used the assumption that $x, y, z \geq 0$ in this problem. This assumption is here mostly to make sure that we really do have a maximum and a minimum of the function. Without this assumption it wouldn't be too difficult to find points that give both larger and smaller values of the functions. For example.

$$x=-100,\ y=100,\ z=1: \ -100+100+1=1 \qquad f(-100,100,1)=-10000$$
$$x=-50,\ y=-50,\ z=101: \ -50-50+101=1 \qquad f(-50,-50,101)=252500$$

With these examples you can clearly see that it's not too hard to find points that will give larger and smaller function values. However, all of these examples required negative values of x, y and/or z to make sure we satisfy the constraint. By eliminating these we will know that we've got minimum and maximum values by the Extreme Value Theorem.

Calculus III - Practice Problems

Lagrange Multipliers

1. Find the maximum and minimum values of $f(x,y)=81x^2+y^2$ subject to the constraint $4x^2+y^2=9$.

2. Find the maximum and minimum values of $f(x,y)=8x^2-2y$ subject to the constraint $x^2+y^2=1$.

3. Find the maximum and minimum values of $f(x,y,z)=y^2-10z$ subject to the constraint $x^2+y^2+z^2=36$.

4. Find the maximum and minimum values of $f(x,y,z)=xyz$ subject to the constraint $x+9y^2+z^2=4$. Assume that $x\geq 0$ for this problem. Why is this assumption needed?

5. Find the maximum and minimum values of $f(x,y,z)=3x^2+y$ subject to the constraints $4x-3y=9$ and $x^2+z^2=9$