Matrices, Vectors & Complex Numbers

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Text : Engineering Mathematics, K.A. Stroud, 5th Ed., Palgrave

Mathematics Prep 2007

MATRICES

A matrix is a set of numbers arranged in rows and columns to form a rectangular array.

A matrix having *m* **rows** and *n* **columns** is called an *m x n* matrix and is referred to as having **order** *m x n*.

A matrix is indicated by writing the array within brackets

e.g. $\begin{pmatrix} 5 & 7 & 2 \\ 6 & 3 & 8 \end{pmatrix}$ is a 2 x 3 matrix, where

5, 7, 2, 6, 3, 8 are known as the **elements** of the matrix.

A matrix is usually denoted by a boldfaced letter, e.g. **A**.

Double suffix notation: Each element is a matrix has its own particular 'address' or location which can be defined by a system of double suffixes, the first indicating the row, the second the column.

e.g. In the matrix,

$$\begin{pmatrix} 6 & -5 & 1 & -3 \\ 2 & -4 & 8 & 3 \\ 4 & -7 & -6 & 5 \\ -2 & 9 & 7 & -1 \end{pmatrix}$$

$a_{24} = 3$, $a_{44} = -1$, $a_{42} = 9$

Addition and subtraction of matrices

To be added or subtracted, two matrices must be of the *same order*. The sum or difference is then determined by adding or subtracting corresponding elements.

e.g. $$\begin{pmatrix} 4 & 2 & 3 \\ 5 & 7 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 8 & 9 \\ 3 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 4+1 & 2+8 & 3+9 \\ 5+3 & 7+5 & 6+4 \end{pmatrix} = \begin{pmatrix} 5 & 10 & 12 \\ 8 & 12 & 10 \end{pmatrix}$$

and $$\begin{pmatrix} 6 & 5 & 12 \\ 9 & 4 & 8 \end{pmatrix} - \begin{pmatrix} 3 & 7 & 1 \\ 2 & 10 & -5 \end{pmatrix} = \begin{pmatrix} 6-3 & 5-7 & 12-1 \\ 9-2 & 4-10 & 8+5 \end{pmatrix} = \begin{pmatrix} 3 & -2 & 11 \\ 7 & -6 & 13 \end{pmatrix}$$

Scalar multiplication of matrices

To multiply a matrix by a single number (i.e. a scalar), each individual element of the matrix is multiplied by that factor:

e.g. $$4 \times \begin{pmatrix} 3 & 2 & 5 \\ 6 & 1 & 7 \end{pmatrix} = \begin{pmatrix} 12 & 8 & 20 \\ 24 & 4 & 28 \end{pmatrix}$$

i.e. in general, $k(a_{ij}) = (ka_{ij})$

Multiplication of two matrices

Two matrices can be multiplied together only when the number of columns in the first is equal to the number of rows in the second.

e.g. if **A** = (a_{ij}) = $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ and **b** = (b_i) = $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

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then **A . b** = $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$$= \begin{pmatrix} a_{11}b_1 + a_{12}b_2 + a_{13}b_3 \\ a_{21}b_1 + a_{22}b_2 + a_{23}b_3 \end{pmatrix}$$

i.e. each element in the top row of **A** is multiplied by the corresponding element in the first column of **b** and the products added. Similarly, the second row of the product is found by multiplying each element in the second row of **A** by the corresponding element in the first column of **b**.

Example 1

$$\begin{pmatrix} 4 & 7 & 6 \\ 2 & 3 & 1 \end{pmatrix} \bullet \begin{pmatrix} 8 \\ 5 \\ 9 \end{pmatrix} = \begin{pmatrix} 4\times8+7\times5+6\times9 \\ 2\times8+3\times5+1\times9 \end{pmatrix} = \begin{pmatrix} 32+35+54 \\ 16+15+9 \end{pmatrix} = \begin{pmatrix} 121 \\ 40 \end{pmatrix}$$

Example 2

If **A** = $\begin{pmatrix} 1 & 5 \\ 2 & 7 \\ 3 & 4 \end{pmatrix}$ and **B** = $\begin{pmatrix} 8 & 4 & 3 & 1 \\ 2 & 5 & 8 & 6 \end{pmatrix}$

then **A . B** $= \begin{pmatrix} 1 & 5 \\ 2 & 7 \\ 3 & 4 \end{pmatrix} \bullet \begin{pmatrix} 8 & 4 & 3 & 1 \\ 2 & 5 & 8 & 6 \end{pmatrix}$

$$= \begin{pmatrix} 1\times8+5\times2 & 1\times4+5\times5 & 1\times3+5\times8 & 1\times1+5\times6 \\ 2\times8+7\times2 & 2\times4+7\times5 & 2\times3+7\times8 & 2\times1+7\times6 \\ 3\times8+4\times2 & 3\times4+4\times5 & 3\times3+4\times8 & 3\times1+4\times6 \end{pmatrix}$$

$$= \begin{pmatrix} 8+10 & 4+25 & 3+40 & 1+30 \\ 16+14 & 8+35 & 6+56 & 2+42 \\ 24+8 & 12+20 & 9+32 & 3+24 \end{pmatrix}$$

$$= \begin{pmatrix} 18 & 29 & 43 & 31 \\ 30 & 43 & 62 & 44 \\ 32 & 32 & 41 & 27 \end{pmatrix}$$

Note that multiplying a (3 x 2) matrix and a (2 x 4) matrix gives

a product matrix of order (3 x 4)

i.e. order (3 x 2) x order (2 x 4) $\longrightarrow$ order (3 x 4)

(same)

In general then, the product of an (*m x p*) matrix and an (*p x n*) matrix has order (*m x n*).

i.e. $$\underbrace{\mathbf{A}}_{m\times p}\ \underbrace{\mathbf{B}}_{p\times n} = \underbrace{\mathbf{C}}_{m\times n}$$

e.g. $\begin{pmatrix} 1 & 5 & 6 \\ 4 & 9 & 7 \end{pmatrix} \bullet \begin{pmatrix} 2 & 3 & 5 \\ 8 & 7 & 1 \end{pmatrix}$ has no meaning.

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It follows that a matrix can be squared only if it is itself a square matrix, i.e. the number of rows equals the number of columns.

If **A** = $\begin{pmatrix} 4 & 7 \\ 5 & 2 \end{pmatrix}$

$\mathbf{A}^2 = \begin{pmatrix} 4 & 7 \\ 5 & 2 \end{pmatrix} \bullet \begin{pmatrix} 4 & 7 \\ 5 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 16+35 & 28+14 \\ 20+10 & 35+4 \end{pmatrix} = \begin{pmatrix} 51 & 42 \\ 30 & 39 \end{pmatrix}$$

Note that, in matrix multiplication, **A.B** $\neq$ **B.A**, i.e. multiplication is not commutative. The order of the factors is important!

eg, if **A** = $\begin{pmatrix} 5 & 2 \\ 7 & 4 \\ 3 & 1 \end{pmatrix}$ and **B** = $\begin{pmatrix} 9 & 2 & 4 \\ -2 & 3 & 6 \end{pmatrix}$

then **A.B** =$\begin{pmatrix} 41 & 16 & 32 \\ 55 & 26 & 52 \\ 25 & 9 & 18 \end{pmatrix}$, and **B.A** = $\begin{pmatrix} 71 & 30 \\ 29 & 14 \end{pmatrix}$

Properties of matrix operations

$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$	$\alpha(\beta\mathbf{A})=(\alpha\beta)\mathbf{A}$
$(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$	$(\alpha+\beta)\mathbf{A}=\alpha\mathbf{A}+\beta\mathbf{A}$
$\mathbf{AB}\neq\mathbf{BA}$	$\alpha(\mathbf{A}+\mathbf{B})=\alpha\mathbf{A}+\alpha\mathbf{B}$
$(\mathbf{AB})\mathbf{C}=\mathbf{A}(\mathbf{BC})$	$\mathbf{A}+(-\mathbf{A})=\mathbf{0}$

Transpose of a matrix

If the rows and columns of a matrix are interchanged,

then the new matrix so formed is called the **transpose** of the original matrix. If **A** is the original matrix, its transpose is denoted by $\mathbf{A}^T$.

So, if $\mathbf{A} = \begin{pmatrix} 4 & 6 \\ 7 & 9 \\ 2 & 5 \end{pmatrix}$, then $\mathbf{A}^T = \begin{pmatrix} 4 & 7 & 2 \\ 6 & 9 & 5 \end{pmatrix}$

Example : If

$\mathbf{A} = \begin{pmatrix} 2 & 7 & 6 \\ 3 & 1 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 0 \\ 3 & 7 \\ 1 & 5 \end{pmatrix}$

then $\mathbf{A.B}$ = $\begin{pmatrix} 35 & 79 \\ 20 & 32 \end{pmatrix}$ .... and $(\mathbf{A.B})^T$ = $\begin{pmatrix} 35 & 20 \\ 79 & 32 \end{pmatrix}$ ....

Special matrices

(a) ***Square matrix*** is a matrix of order *m x m*

e.g. $\begin{pmatrix} 1 & 2 & 5 \\ 6 & 8 & 9 \\ 1 & 7 & 4 \end{pmatrix}$ is a 3 x 3 matrix.

A square matrix (a_{ij}) is **symmetric** if $a_{ij}=a_{ji}$, eg. $\begin{pmatrix} 1 & 2 & 5 \\ 2 & 8 & 9 \\ 5 & 9 & 4 \end{pmatrix}$

i.e. it is symmetrical about the leading diagonal.

Note that **A** is symmetric $\Leftrightarrow$ $\mathbf{A} = \mathbf{A}^T$.

A square matrix (a_{ij}) is **skew-symmetric** if $a_{ij}=-a_{ji}$, eg. $\begin{pmatrix} 0 & 2 & 5 \\ -2 & 0 & 9 \\ -5 & -9 & 0 \end{pmatrix}$

In that case, $\mathbf{A} = -\mathbf{A}^T$.

(b) ***Unit matrix*** is a diagonal matrix in which the elements on the leading diagonal are all unity, i.e. $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

The unit matrix is denoted by **I**.

If **A** = $\begin{pmatrix}5 & 2 & 4\\ 1 & 3 & 8\\ 7 & 9 & 6\end{pmatrix}$ and **I** = $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ then **A.I** = $\begin{pmatrix}5 & 2 & 4\\ 1 & 3 & 8\\ 7 & 9 & 6\end{pmatrix}$ i.e. **A.I** = **A**

Similarly, if we can show that **I.A** = **A**

Therefore, the unit matrix **I** behaves very much like the unit factor in the ordinary algebra and arithmetic.

(c) ***Null matrix***: A null matrix is one whose elements are all zero.

i.e. $\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$ and is denoted by **0**.

If **A.B**=**0**. we cannot say that therefore **A**=**0** or **B**=**0**

for if **A** = $\begin{pmatrix}2 & 1 & -3\\ 6 & 3 & -9\end{pmatrix}$ and **B** = $\begin{pmatrix}1 & 9\\ 4 & -6\\ 2 & 4\end{pmatrix}$

then **A.B** $= \begin{pmatrix}2 & 1 & -3\\ 6 & 3 & -9\end{pmatrix} \bullet \begin{pmatrix}1 & 9\\ 4 & -6\\ 2 & 4\end{pmatrix}$

$$= \begin{pmatrix}2+4-6 & 18-6-12\\ 6+12-18 & 54-18-36\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

That is **A.B**=**0**, but clearly $\mathbf{A} \neq \mathbf{0}$ and $\mathbf{B} \neq \mathbf{0}$.

Inverse of a square matrix

The inverse of a matrix, **A**, is denoted by $\mathbf{A}^{-1}$.

The matrix product of a matrix and a matrix is the unit matrix.

i.e., $\boxed{\mathbf{A.A^{-1}=A^{-1}.A=I}}$

Example :

If $\mathbf{A}=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{pmatrix}$ and $\mathbf{B}=\frac{1}{28}\begin{pmatrix} 2 & -4 & 7 \\ 22 & -16 & 7 \\ -6 & 12 & -7 \end{pmatrix}$

then we get **A.B**=**B.A**=**I**

i.e., $\mathbf{B}=\mathbf{A}^{-1}$ or $\mathbf{A}=\mathbf{B}^{-1}$

For a *2x2* matrix,

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Its inverse is given by,

$$\boxed{\mathbf{A}^{-1}=\frac{\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}}{|\mathbf{A}|}}$$

where |**A**| is called the determinant of **A** given by

$$\boxed{|\mathbf{A}|=a_{11}a_{22}-a_{12}a_{21}}$$

Example : If

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Then $|\mathbf{A}| = (1 \times 4) - (2 \times 3) = -2$

and

$$\mathbf{A}^{-1} = \frac{\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}}{-2} = \begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix}$$

Check :

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The steps for inverting a square matrix are listed below and are explained through an example which follows

(a) Evaluate the **determinant** of **A**, i.e. |**A**|.

(b) Form a matrix **C** of the **cofactors** of the elements of |**A**|.

(c) Write the transpose of **C**, i.e. $\mathbf{C}^T$, to obtain the **adjoint** of **A**.

(d) Divide each element of $\mathbf{C}^T$ by |**A**|.

(e) The resulting matrix is the inverse $\mathbf{A}^{-1}$ of the original matrix **A**.

Example : Find the inverse of

$$\begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{pmatrix}$$

(a) The determinant of $\begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{pmatrix}$ is

Note "-" sign here

$$\begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{vmatrix} = 2\begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} - 3\begin{vmatrix} 4 & 6 \\ 1 & 0 \end{vmatrix} + 5\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix}$$

$$= 2(1\times 0 - 6\times 4) - 3(4\times 0 - 6\times 1) + 5(4\times 4 - 1\times 1)$$

$$= -48 + 18 + 75 = 45$$

(b) The matrix of cofactors of $\mathbf{A} = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{pmatrix}$ is

$$\mathbf{C} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

where

$$A_{11} = +\begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} = +(0-24) = -24$$

$$A_{12} = -\begin{vmatrix} 4 & 6 \\ 1 & 0 \end{vmatrix} = -(0-6) = 6$$

$$A_{13} = +\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} = +(16-1) = 15$$

...cont'd

$$A_{21} = -\begin{vmatrix} 3 & 5 \\ 4 & 0 \end{vmatrix} = -(0-20) = 20 \qquad A_{31} = +\begin{vmatrix} 3 & 5 \\ 1 & 6 \end{vmatrix} = +(18-5) = 13$$

$$A_{22} = +\begin{vmatrix} 2 & 5 \\ 1 & 0 \end{vmatrix} = +(0-5) = -5 \qquad A_{32} = -\begin{vmatrix} 2 & 5 \\ 4 & 6 \end{vmatrix} = -(12-20) = 8$$

$$A_{23} = -\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = -(8-3) = -5 \qquad A_{33} = +\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = +(2-12) = -10$$

The signs in front of A_{ij} are determined from the sign matrix

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

...cont'd

Therefore, the matrix of cofactors of $\begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{pmatrix}$

is $\mathbf{C} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} -24 & 6 & 15 \\ 20 & -5 & -5 \\ 13 & 8 & -10 \end{pmatrix}$

(c) The adjoint of **A** is $\mathbf{C}^T$

$$\mathbf{C}^T = \begin{pmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{pmatrix}$$

(d) The inverse of **A** is $\mathbf{C}^T/|\mathbf{A}|$

$$\mathbf{A}^{-1} = \frac{1}{45}\begin{pmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{pmatrix}$$

Simultaneous equations in matrix form

One of the most common application of matrices in engineering is in the representation of a set of algebraic equations.

For example,

$$\begin{aligned} x + 2y &= 4 \\ 2x + 3y &= 6 \end{aligned} \quad \Leftrightarrow \quad \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

and

$$\begin{aligned} x + 2y + z &= 3 \\ 2x - 3y &= 6 \\ x + y + 2z &= 0 \end{aligned} \quad \Leftrightarrow \quad \begin{pmatrix} 1 & 2 & 1 \\ 2 & -3 & 0 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}$$

In general, *n* simultaneous equations involving *n* unknowns can be written as

$$\begin{matrix} a_{11}x_1+a_{12}x_2+a_{13}x_3+ \;\dots\; +a_{1n}x_n = b_1 \\ a_{21}x_1+a_{22}x_2+a_{23}x_3+ \;\dots\; +a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1+a_{n2}x_2+a_{n3}x_3+ \;\dots\; +a_{nn}x_n = b_n \end{matrix} \quad \Leftrightarrow \quad \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix}}_{n \times n \text{ square matrix}} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

n x 1 column matrices

i.e. $\boxed{\mathbf{Ax} = \mathbf{b}}$

Where **A** is a square matrix of order *n*, **x** is a column matrix of *n* unknowns and **b** is a column matrix of *n* constants.

If we multiply both sides of the matrix equation by the inverse of A, we have:

$$\mathbf{A}^{-1}.\mathbf{A}.\mathbf{x} = \mathbf{A}^{-1}.\mathbf{b}$$

but $\mathbf{A}^{-1}.\mathbf{A} = \mathbf{I}$ So $\mathbf{I}.\mathbf{x} = \mathbf{A}^{-1}.\mathbf{b}$ i.e. $\mathbf{x} = \mathbf{A}^{-1}.\mathbf{b}$

Therefore, if we form the inverse of the matrix of coefficients and pre-multiply matrix **b** by it, we shall determine the matrix of the solutions of **x**.

Example

To solve the set of equations:

$x_1 + 2x_2 + x_3 = 4$

$3x_1 - 4x_2 - 2x_3 = 2$

$5x_1 + 3x_2 + 5x_3 = -1$

First write the set of equations in the matrix form, which gives:

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & -4 & -2 \\ 5 & 3 & 5 \end{pmatrix} \bullet \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$$

i.e. **A.x** = **b** So **x** = $\mathbf{A}^{-1}$.**b**

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So the next step is to find the inverse of **A** where **A** is the matrix of the coefficients of **x**. We have already seen how to determine the inverse of a matrix. In this case $\mathbf{A}^{-1}$ = $-\frac{1}{35}\begin{pmatrix} -14 & -7 & 0 \\ -25 & 0 & 5 \\ 29 & 7 & -10 \end{pmatrix}$

$$\therefore \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = -\frac{1}{35}\begin{pmatrix} -14 & -7 & 0 \\ -25 & 0 & 5 \\ 29 & 7 & -10 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} = -\frac{1}{35}\begin{pmatrix} -70 \\ -105 \\ 140 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$$

So finally $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$ so x_1=2; x_2=3; x_3=-4.

As we have seen, a basic knowledge of matrices provides a neat and concise way of dealing with sets of linear equations. In practice, the numerical coefficients are not always simple numbers, neither is the number of equations in the set limited to three. In more extensive problems, recourse to computing facilities is a great help, but the underlying methods are still the same.

A final note…

The inverse of a matrix is given by $\mathbf{A}^{-1}=\mathrm{adj}(\mathbf{A})/|\mathbf{A}|$.

Therefore, when $|\mathbf{A}|=0$, $\mathbf{A}^{-1}$does not exist and **A** is said to be singular.

When the matrix **A** of a set of simultaneous equations is singular, this means that the equations cannot be solved, i.e. there is no unique solution.

VECTORS

A ***scalar*** *quantity* is one that has only magnitude,

e.g.

- mass
- time
- pressure
- temperature

A ***vector*** *quantity* is one which has magnitude and direction,

e.g.

- force
- velocity
- acceleration

A vector quantity can be represented graphically by a line, drawn so that:

(a) the *length* of the line denotes the magnitude of the quantity, according to some stated vector scale.

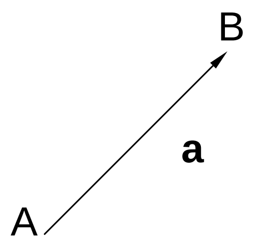
(b) the *direction* of the line denotes the direction in which the vector quantity acts. The sense of the direction is indicated by an arrowhead.

e.g. A horizontal force of 35 N acting to the right, would be indicated by a line $\longrightarrow$ and if the chosen vector scale were 1 cm = 10 N, the line would be 3.5 cm long.

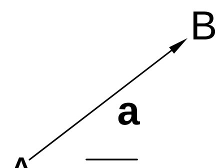
The vector quantity AB is referred to as $\overline{AB}$ or **a**.

The magnitude of the vector quantity is written

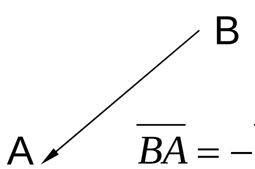
$|\overline{AB}|$, or |**a**|, or simply AB or *a*.



Note that $\overline{BA}$ would represent a vector quantity of the same magnitude but with opposite sense.



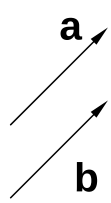
$\overline{AB} = \mathbf{a}$



$\overline{BA} = -\overline{AB} = -\mathbf{a}$

Two equal vectors

If two vectors, **a** and **b**, are said to be equal, they have the same magnitude and the same direction.



If **a** = **b**, then

(a) $a = b$ (magnitudes equal)

(b) the direction of **a** = direction of **b**, i.e. the two vectors are parallel and in the same sense.

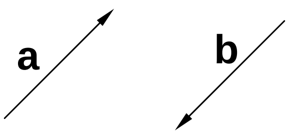
Similarly, if two vectors **a** and **b** are such that **b** = -**a**,

what can we say about:

(a) their magnitudes? ....Magnitudes are equal..................................

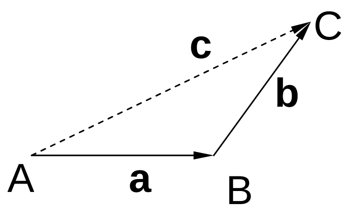
(b) their directions? The vectors are parallel but opposite in sense..

i.e. if **b** = -**a**, then



Addition of vectors

The sum of two vectors, $\overline{AB}$ and $\overline{BC}$, is defined as the single or equivalent or resultant vector $\overline{AC}$.



i.e. $\overline{AB} + \overline{BC} = \overline{AC}$

or **a** + **b** = **c**

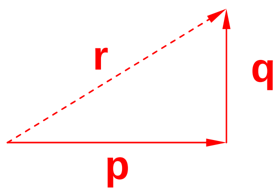
To find the sum of two vectors **a** and **b** then, we draw them as a chain, starting the second where the first ends: the sum **c** is given by the single vector joining the start of the first to the end of the second.

e.g. if **p** $\equiv$ a force of 40 N, acting in the direction due east

q $\equiv$ a force of 30 N, acting in the direction due north

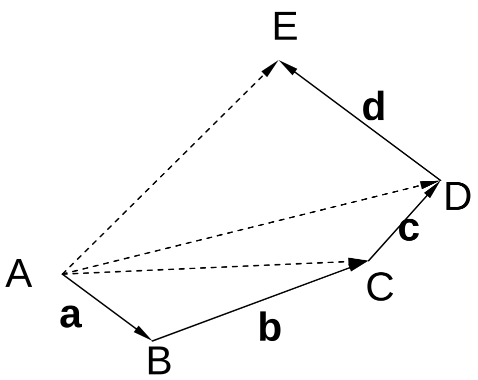
then the magnitude of the vector sum *r* of these will forces will be ....50 N..........

Because



$$r^2 = p^2 + q^2 = 1600 + 900 = 2500$$
$$r = \sqrt{2500} = 50\,\mathrm{N}$$

The sum of a number of vectors **a** + **b** + **c** + **d** + ...



(a) Draw the vectors as a chain

(b) Then: $\mathbf{a}+\mathbf{b} = \overline{AC}$

$$\overline{AC} + \mathbf{c} = \overline{AD}$$

$$\therefore\ \mathbf{a}+\mathbf{b}+\mathbf{c} = \overline{AD}$$

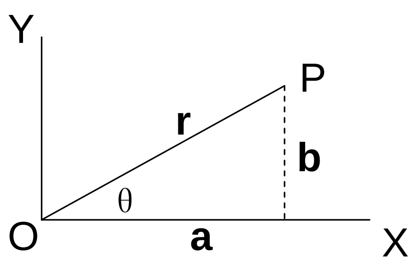
$$\overline{AD} + \mathbf{d} = \overline{AE}$$

$$\therefore\ \mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d} = \overline{AE}$$

i.e. the sum of all vectors, **a**, **b**, **c, d**, is given by the single vector joining the start of the first to the end of the last – in this case, $\overline{AE}$

This follows directly from our previous definition of the sum of two vectors.

Components of a vector in terms of unit vectors



The vector $\overline{OP}$ is defined by its magnitude (*r*) and its direction (θ).

It could also be defined by its two components in the OX and OY directions.

i.e. $\overline{OP}$ is equivalent to a vector **a** in the OX direction + a vector **b** in the OY direction.

i.e. $\overline{OP}$ = **a** (along OX) + **b** (along OY)

If we now define **i** to be a ***unit vector*** in the OX direction

then **a** = *a***i**

Similarly, if we now define **j** to be a *unit vector* in the OY direction

then **b** = *b***j**

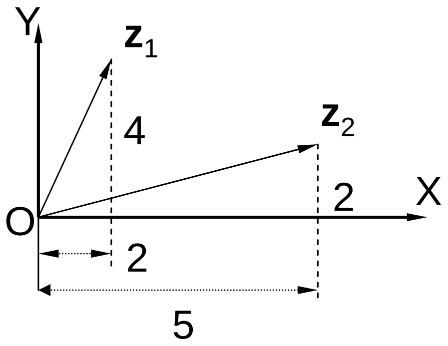
So that the vector OP can be written as:

$\mathbf{r} = a\mathbf{i} + b\mathbf{j}$

where **i** and **j** are unit vectors in the OX and OY directions.

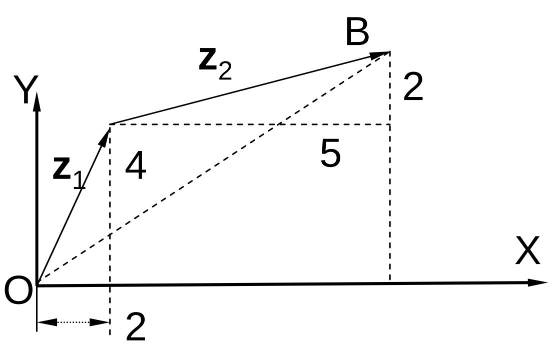
(**i** and **j** are also called the ***basis vectors***.)

Let $\mathbf{z}_1 = 2\mathbf{i} + 4\mathbf{j}$ and $\mathbf{z}_2 = 5\mathbf{i} + 2\mathbf{j}$



To find $\mathbf{z}_1 + \mathbf{z}_2$, draw the two vectors in a chain.

$$\mathbf{z}_1 + \mathbf{z}_2 = \overline{OB} = (2+5)\mathbf{i} + (4+2)\mathbf{j} = 7\mathbf{i} + 6\mathbf{j}$$



i.e. total up the vector components along OX, and total up the vector components along OY

Of course, we can do this without a diagram:

If $\mathbf{z}_1 = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{z}_2 = 4\mathbf{i} + 3\mathbf{j}$ So $\mathbf{z}_1 + \mathbf{z}_2 = 7\mathbf{i} + 5\mathbf{j}$

And in much the same way, $\mathbf{z}_2 - \mathbf{z}_1$ = ...$\mathbf{i} + \mathbf{j}$..........

Similarly, if $\mathbf{z}_1 = 5\mathbf{i} - 2\mathbf{j}$; $\mathbf{z}_2 = 3\mathbf{i} + 3\mathbf{j}$; $\mathbf{z}_3 = 4\mathbf{i} - 1\mathbf{j}$

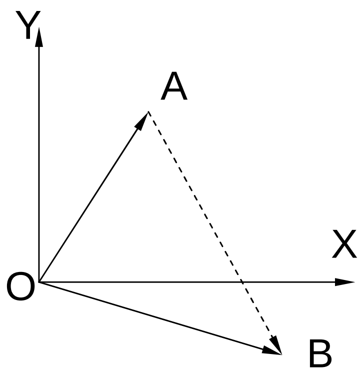
then (a) $\mathbf{z}_1 + \mathbf{z}_2 + \mathbf{z}_3$ = ...$12\mathbf{i}$..........

(b) $\mathbf{z}_1 - \mathbf{z}_2 - \mathbf{z}_3$ = ...$-2\mathbf{i} - 4\mathbf{j}$..........

Now this one

If $\overline{OA} = 3\mathbf{i} + 5\mathbf{j}$ and $\overline{OB} = 5\mathbf{i} - 2\mathbf{j}$, find $\overline{AB}$

As usually, a diagram will help. Here it is:



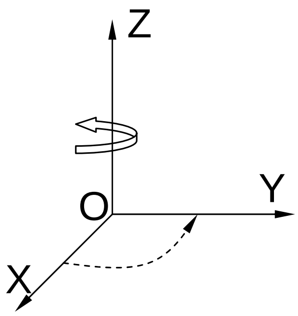
First of all, from the diagram, write down a relationship between the vector. Then express them in terms of the unit vectors

$\overline{AB} =$ ....$2\mathbf{i} - 7\mathbf{j}$..........

Because, from diagram:

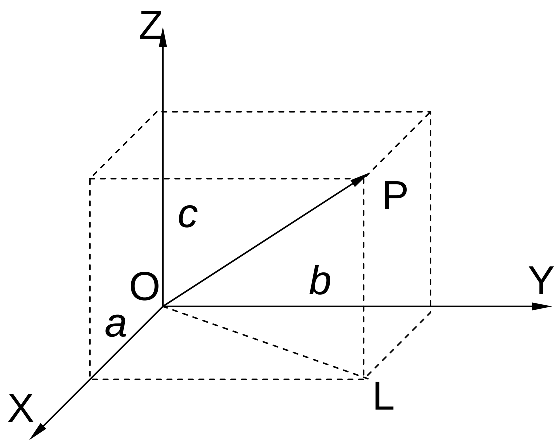
$$\overline{AB} = \overline{OB} - \overline{OA} = (5\mathrm{i} - 2\mathrm{j}) - (3\mathrm{i} + 5\mathrm{j}) = 2\mathrm{i} - 7\mathrm{j}$$

Vectors in space



The axes of reference are defined by the 'right-hand' rule.

OX, OY, OZ form a right-handed set if rotation from OX to OY takes a right-handed corkscrew action along the positive direction of OZ.



Vector $\overline{OP}$ is defined by its components

a along OX

b along OY

c along OZ

Let **i** = unit vector in OX direction

j = unit vector in OY direction

k = unit vector in OZ direction

Then $\overline{OP} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

Also $OL^2 = a^2 + b^2$ and $OP^2 = OL^2 + c^2$

So $OP^2 = a^2 + b^2 + c^2$

So, if $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then $r = \sqrt{a^2 + b^2 + c^2}$

This gives us an easy way of finding the magnitude of a vector expressed in terms of the basis vectors.

Example :

If $\overline{PQ} = 4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, then $\left|\overline{PQ}\right|$ = ...5.385.......

Because $\overline{PQ} = 4\mathrm{i} + 3\mathrm{j} + 2\mathrm{k}$

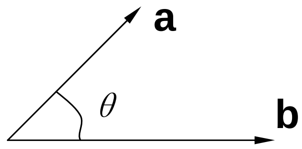
$$\left|\overline{PQ}\right| = \sqrt{16 + 9 + 4} = \sqrt{29} = 5.385$$

Hence, a unit vector parallel to $\overline{PQ}$ is

$$\frac{\overline{PQ}}{\left|\overline{PQ}\right|} = \frac{4\mathrm{i} + 3\mathrm{j} + 2\mathrm{k}}{5.385}$$

Scalar product of two vectors

The scalar product is denoted by **a.b** is also called the ‘dot product’.



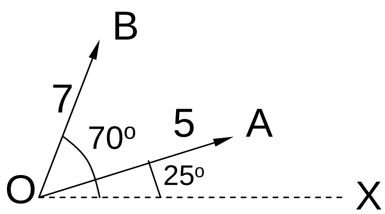
It is defined by

$$\boxed{\mathbf{a.b} = ab\cos\theta}$$

The result is a *scalar* quantity.

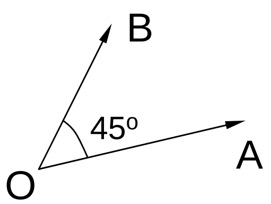
Note : **a.b** = **b.a**

For example:



$$\overline{OA}\bullet\overline{OB} = \ldots\ldots\frac{35\sqrt{2}}{2}\ldots\ldots$$

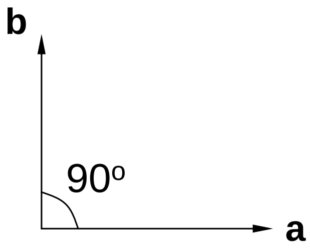
Because we have



$$\overline{OA}\bullet\overline{OB} = OA.OB.\cos\theta$$

$$= 5 \times 7 \times \cos 45^{o} = \frac{35\sqrt{2}}{2}$$

Now what about this case:

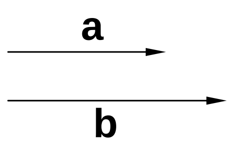


The scalar product **a** and **b** = **a.b** = ...**0**......

Because in the case cos90°=0. So the scalar product of any two vectors at right-angles to each other is *always zero*.

And in this case now, with two vectors in the same direction,

θ =0°.



So **a.b** = ...*ab*......

Because **a.b** = *ab*cos0° = *ab*

Now suppose our two vectors are expressed in terms of the unit vectors **i**, **j** and **k**.

Let $\quad \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

and $\quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

Then $\quad \mathbf{a}.\mathbf{b} = a_1b_1\mathbf{i}.\mathbf{i} + a_1b_2\mathbf{i}.\mathbf{j} + a_1b_3\mathbf{i}.\mathbf{k} + a_2b_1\mathbf{j}.\mathbf{i} + a_2b_2\mathbf{j}.\mathbf{j}$

$$+ a_2b_3\mathbf{j}.\mathbf{k} + a_3b_1\mathbf{k}.\mathbf{i} + a_3b_2\mathbf{k}.\mathbf{j} + a_3b_3\mathbf{k}.\mathbf{k}$$

This can now be simplified.

Because **i.i** = (1).(1)cos0° = 1 So **i.i** = **j.j** = **k.k** =1 (a)

Also **i.j** = (1).(1)cos90° = 0 So **i.j** = **i.k** = **j.k** = 0 (b)

So using the results (a) and (b), we get

$$\boxed{\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3}$$

Because $\mathbf{a.b} = a_1b_1.1 + a_1b_2.0 + a_1b_3.0 + a_2b_1.0 + a_2b_2.1$

$+ a_2b_3.0 + a_3b_1.0 + a_3b_2.0 + a_3b_3.1$

$= a_1b_1 + a_2b_2 + a_3b_3$

i.e. we just sum the products of the coefficients of the unit vectors along the corresponding axes.

For example:

If $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 1\mathbf{j} + 6\mathbf{k}$

then $\mathbf{a}.\mathbf{b} = 2 \times 4 + 3 \times 1 + 5 \times 6 = 41$

If $\mathbf{p} = 3\mathbf{i} - 2\mathbf{j} + 1\mathbf{k}$ and $\mathbf{q} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$

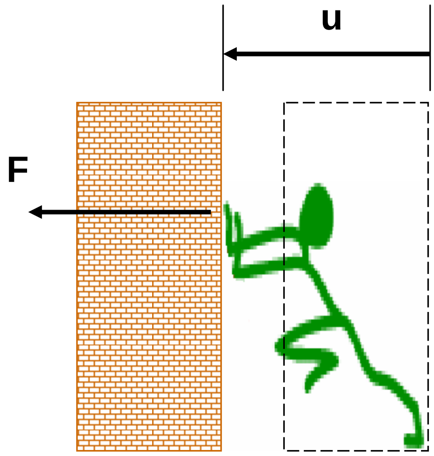
then $\mathbf{p}.\mathbf{q} =$ ....-4........................

Work – An application of dot product

Work is done by a force when it undergoes a displacement.

When a constant force of magnitude ***F*** displaces a distance ***u*** in the *same direction* as the force, the work done is

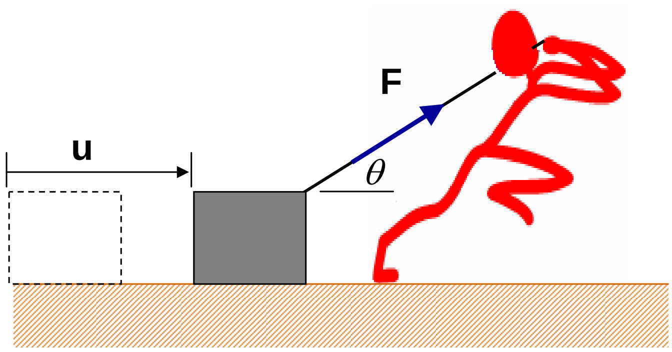
$$W=|\mathbf{F}||\mathbf{u}|$$



Work – An application of dot product

If **F** is constant by is not in the same direction as **u**, then work is done only by the component of **F** in the direction of **u**, i.e.,

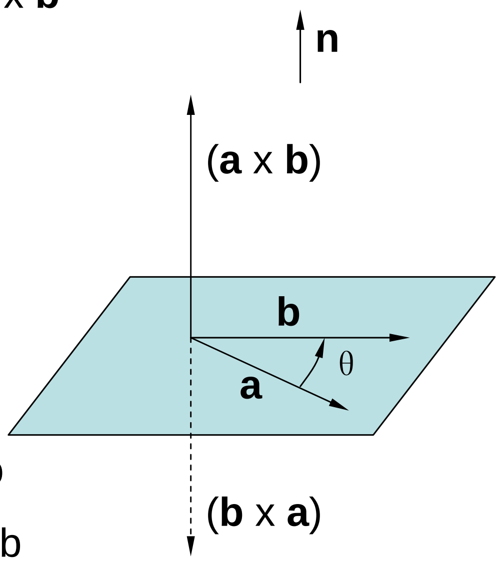
$$\begin{aligned} W &= |\mathbf{F}||\mathbf{u}|\cos\theta \\ &= \mathbf{F}\cdot\mathbf{u} \end{aligned}$$



Note that when **F** and **u** are in the same direction, $\theta=0$ i.e. $\cos\theta=1$

Vector product of two vectors

The vector product of **a** and **b** is written **a** x **b** (often called the 'cross product') and is defined as a <u>*vector*</u>



$$\boxed{\mathbf{a} \times \mathbf{b} = (ab\sin\theta)\,\mathbf{n}}$$

n is a unit vector perpendicular to **a** and **b** pointing in the direction of your right thumb when your fingers are curled from **a** to **b**.

Note : $\mathbf{a} \times \mathbf{b} = -(\mathbf{a} \times \mathbf{b})$

Note that **b** x **a** reverses the direction of rotation and the product vector would now act downwards, i.e.

$\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$

If $\theta = 0^o$, then $|\mathbf{a} \times \mathbf{b}|$ = ...0...................

and if $\theta = 90^o$, then $|\mathbf{a} \times \mathbf{b}|$ = ...*ab*..........

If **a** and **b** are given in terms of the unit vectors **i**, **j** and **k**:

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \text{and} \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

Then

$$\mathbf{a} \times \mathbf{b} = a_1b_1\mathbf{i} \times \mathbf{i} + a_1b_2\mathbf{i} \times \mathbf{j} + a_1b_3\mathbf{i} \times \mathbf{k} + a_2b_1\mathbf{j} \times \mathbf{i} + a_2b_2\mathbf{j} \times \mathbf{j}$$
$$+ a_2b_3\mathbf{j} \times \mathbf{k} + a_3b_1\mathbf{k} \times \mathbf{i} + a_3b_2\mathbf{k} \times \mathbf{j} + a_3b_3\mathbf{k} \times \mathbf{k}$$

Because $|\mathbf{i} \times \mathbf{i}| = (1).(1)\sin 0^o = 0$ So $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$ (a)

Also $|\mathbf{i} \times \mathbf{j}| = (1).(1)\sin 90^o = 1$ and $\mathbf{i} \times \mathbf{j}$ is in the direction of $\mathbf{k}$,

i.e. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ (same magnitude and same direction). Therefore:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}\,; \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i}\,; \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j} \qquad \text{(b)}$$

And remember too that therefore: $\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i}$; $\mathbf{j} \times \mathbf{k} = -\mathbf{k} \times \mathbf{j}$;

$\mathbf{k} \times \mathbf{i} = -\mathbf{i} \times \mathbf{k}$ since the sense of rotation is reversed.

Now with the results of (a) and (b), and this last reminder, you can simplify the expression for $\mathbf{a} \times \mathbf{b}$.

Remove the zero terms and tidy up what is left:

$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$

Because $\mathbf{a} \times \mathbf{b} = a_1b_1.0 + a_1b_2.\mathbf{k} + a_1b_3.(-\mathbf{j}) + a_2b_1.(-\mathbf{k}) + a_2b_2.0$

$$+ a_2b_3.\mathbf{i} + a_3b_1.\mathbf{j} + a_3b_2.(-\mathbf{i}) + a_3b_3.0$$

$$= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

and you may recognize this as the pattern of a determinant where the first row is made up of the vectors **i**, **j** and **k**.

So now we have that:

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

For example, if **p** = 2**i** + 4**j** +3**k** and **q** = **i** + 5**j** – 2**k**, then

$$\mathbf{p}\times\mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & 3 \\ 1 & 5 & -2 \end{vmatrix} \begin{array}{l} \text{Unit vectors} \\ \text{Coefficients of } \mathbf{p} \\ \text{Coefficients of } \mathbf{q} \end{array}$$

Expanding the determinant, we get:

p x **q** = ...-23**i** + 7**j** +6**k**........

Because

$$\mathrm{p}\times\mathrm{q} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ 2 & 4 & 3 \\ 1 & 5 & -2 \end{vmatrix} = \mathrm{i}\begin{vmatrix} 4 & 3 \\ 5 & -2 \end{vmatrix} - \mathrm{j}\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} + \mathrm{k}\begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix}$$

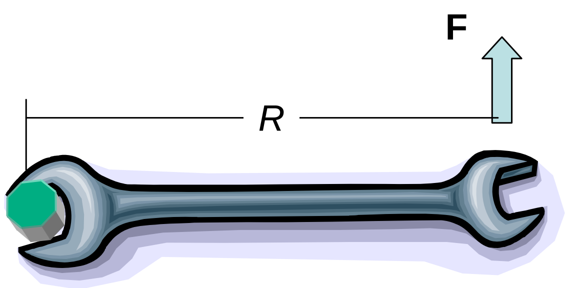
$$= \mathrm{i}(-8-15) - \mathrm{j}(-4-3) + \mathrm{k}(10-4)$$

$$= -23\,\mathrm{i} + 7\,\mathrm{j} + 6\,\mathrm{k}$$

Moment – An application of cross product

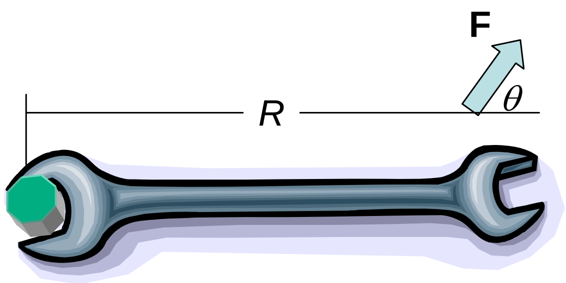
When a force, **F**, is applied <u>perpendicularly</u> to a moment arm of length *R*, the resultant moment is

M=|**F**|*R*



When a force, **F**, is not applied perpendicularly to a moment arm, the resultant moment is due to the component of **F** perpendicular to the moment arm

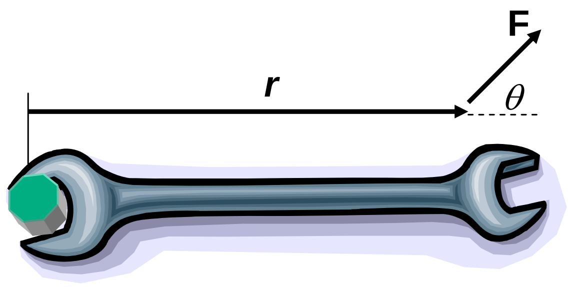
M=|**F**|*R*sinθ



Moment – An application of cross product

In general, moments are given by

M=**r**x**F**



Note that **M** is a vector, i.e. it has a direction and a magnitude.

Since **M**=**r**x**F**=|**r**||**F**|(sinθ)**n**, the direction of **M** is **n**. The directional sense of the moment is given by curling the fingers of the right hand with the thumb pointing in the direction of **n**.

Mathematics Prep 2007

COMPLEX NUMBERS

Imaginary Numbers

Recall $ax^2+bx+c=0 \quad \Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$b^2-4ac>0 \Rightarrow$ real and unequal roots

$b^2-4ac=0 \Rightarrow$ real and equal root

$b^2-4ac<0 \Rightarrow$ complex and unequal roots

E.g. $x^2+2x+2=0 \quad \Rightarrow \quad x=\frac{-2\pm\sqrt{-4}}{2}=\frac{-2\pm\sqrt{4\times(-1)}}{2}=\frac{-2\pm2\sqrt{-1}}{2}$

$$=-1\pm\sqrt{-1}$$

Define $\boxed{i=\sqrt{-1}}$

i.e. $x^2+2x+2=0 \quad \Rightarrow \quad x_1=-1+i, \quad x_2=-1-i$

Imaginary Numbers

$i=\sqrt{-1}$ is an *imaginary number*

Similarly, 2i, -2i, 2.5i, 3.141i, are imaginary numbers.

while, 2, -2, 2.5, 3.141, are *real numbers*.

In

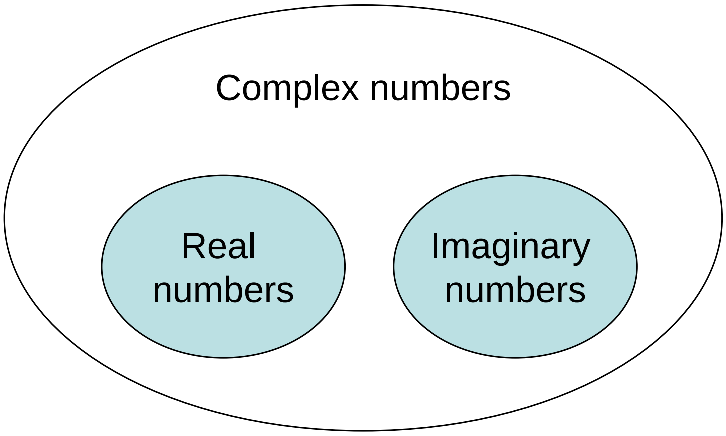
$x^2+2x+2=0 \quad \Rightarrow \quad x_1=-1+i, \quad x_2=-1-i$

x_1 and x_2 are also numbers.

They are called ***complex numbers*** because they have a *real part* and an *imaginary part*.

$\boxed{a+bi \text{ is a Complex Number}}$

Imaginary Numbers



Complex numbers are used in many areas of Mechanical and Electrical Engineering, e.g. vibration analysis, feedback control, circuits, phase diagrams.

Addition and Subtraction

When complex numbers are added or subtracted, the operations are performed on the real and imaginary parts of the numbers independently.

i.e.

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i$$

$$(a_1 + b_1 i) - (a_2 + b_2 i) = (a_1 - a_2) + (b_1 - b_2)i$$

e.g.

$$(3 + 5i) + (6 + i) = 9 + 6i$$

$$(3 + 5i) - (6 + i) = -3 + 4i$$

Multiplication

Complex numbers are multiplied together in the same manner sums of numbers are multiplied together.

Just remember that : $i^2 = -1$.

i.e. $$\begin{aligned}(a_1 + b_1 i)(a_2 + b_2 i) &= a_1(a_2 + b_2 i) + b_1 i(a_2 + b_2 i) \\ &= a_1 a_2 + a_1 b_2 i + a_2 b_1 i + b_1 b_2 \underbrace{i^2}_{-1} \\ &= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i\end{aligned}$$

e.g.

$$\begin{aligned}(3+5i)(6+i) &= 3\times 6 + 3i + 5i \times 6 + 5i^2 \\ &= 18 - 5 + 3i + 30i \\ &= 13 + 33i\end{aligned}$$

Multiplication

Given a complex number, $z = a + bi$

The ***conjugate*** to z is, $\bar{z} = a - bi$

Note that $z\bar{z}$ Is a real number since

$$\boxed{z\bar{z} = (a+bi)(a-bi) = a^2 + b^2}$$

e.g.

$$\begin{aligned}(3+5i)(3-5i) &= 3\times 3 - 3\times 5i + 5i \times 3 - 5\times 5 \times i^2 \\ &= 3^2 + 5^2 \\ &= 34\end{aligned}$$

Division

To perform division of complex numbers, we multiply the numerator and denominator by the conjugate of the denominator

i.e.

$$\frac{a_1+b_1i}{a_2+b_2i}=\frac{a_1+b_1i}{a_2+b_2i}\times\frac{a_2-b_2i}{a_2-b_2i}$$
$$=\frac{(a_1a_2+b_1b_2)+(a_2b_1-a_1b_2)i}{a_2^{\;2}+b_2^{\;2}}$$

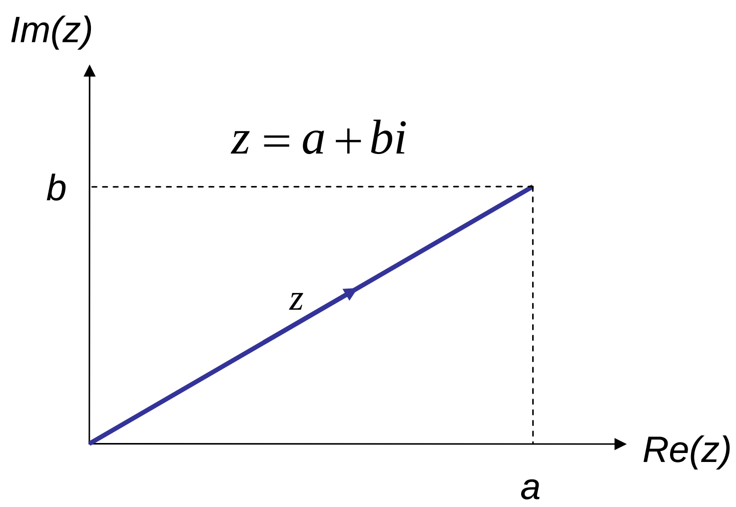
e.g.

$$\frac{3+5i}{6+i}=\frac{3+5i}{6+i}\times\frac{6-i}{6-i}$$
$$=\frac{3\times6-3\times i+5i\times6-5i\times i}{6^2+1^2}$$
$$=\frac{18-3i+30i+5}{37}=\frac{23}{37}+\frac{27}{37}i$$

Argand Diagram

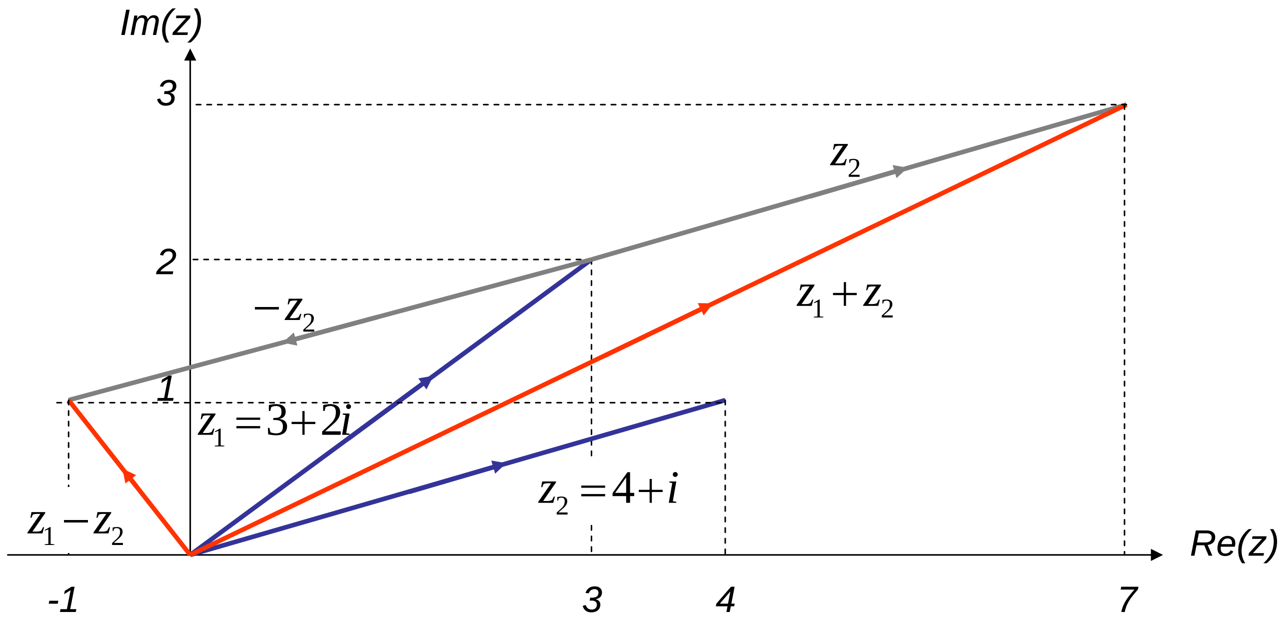
Since complex numbers are represented by two real numbers (the real and imaginary parts) just like the 2 components of vectors in 2D, they can also be graphically represented on an

Argand Diagram



Argand Diagram

As with vectors, complex numbers can also be added and subtracted graphically on the Argand Diagram



Argand Diagram

The equivalent to the magnitude and direction of vectors for complex numbers are

- modulus, $|z| = \sqrt{a^2 + b^2}$
- argument, $\arg(z) = \tan^{-1}\left(\dfrac{b}{a}\right)$

Note : The argument is measured anticlockwise from the x-axis

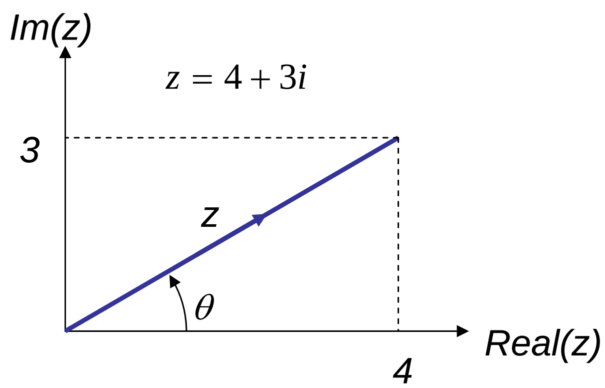
e.g.

$$z = 4 + 3i$$

$$\Rightarrow$$

$$|z| = \sqrt{4^2 + 3^2} = 5$$

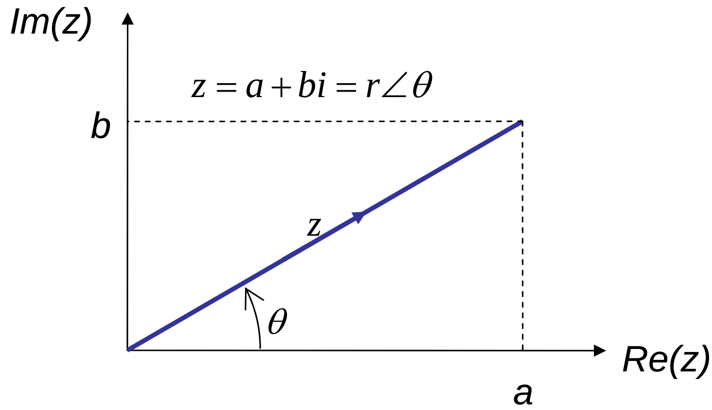
$$\arg(z) = \tan^{-1}\left(\frac{3}{4}\right)$$



The Modulus-Argument Form

If $|z|$=r and arg(z)=θ, then the z can be expressed as

$$\boxed{z = r\angle\theta}$$



Exercise : Covert the following to the modulus-argument form.
Hint : Sketch the Argand Diagrams

$$z = \frac{7-i}{3-4i}, \quad z = 2+2i, \quad z = 2-2i, \quad z = -2-2i$$

Ans :
$$\sqrt{2}\angle\pi/4, \quad 2\sqrt{2}\angle\pi/4, \quad 2\sqrt{2}\angle-\pi/4, \quad 2\sqrt{2}\angle 5\pi/4$$

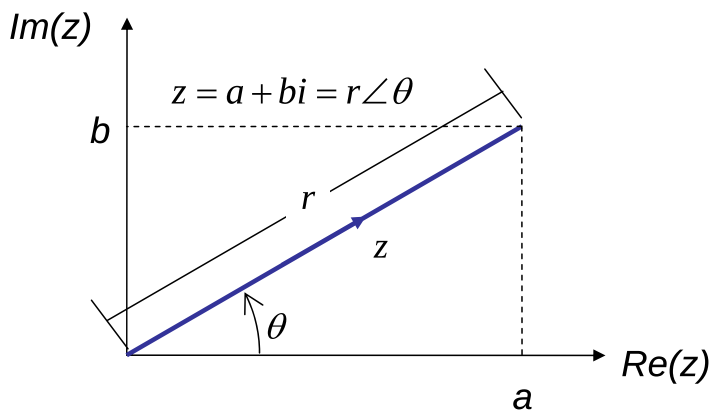
The Polar Form

If $z = r\angle\theta$, it can be seen from the Argand Diagram below that

$$a = r\cos\theta, \quad b = r\sin\theta$$

i.e.

$$\boxed{\begin{aligned} z &= r\angle\theta = a + bi \\ &= r(\cos\theta + i\sin\theta) \end{aligned}}$$



The Polar Form

Consider the *multiplication of 2 complex numbers* written in polar form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1), \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2 [\underbrace{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)}_{\cos(A+B)=\cos A\cos B - \sin A\sin B} + \underbrace{(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)}_{\sin(A+B)=\sin A\cos B + \sin B\cos A} i] \end{aligned}$$

$$\boxed{\therefore \quad z_1 z_2 = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]}$$

i.e.

$$\boxed{|z_1 z_2| = |z_1||z_2|, \qquad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)}$$

The Polar Form

Similarly, consider the *division of 2 complex numbers* written in polar form

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{r_2(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1 r_2[\overbrace{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)}^{\cos(A-B)=\cos A\cos B + \sin A\sin B} + \overbrace{(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}^{\sin(A-B)=\sin A\cos B - \sin B\cos A} i]}{r_2^{\,2}\underbrace{(\cos^2\theta_2 + \sin^2\theta_2)}_{=1}}$$

$$\boxed{\therefore \quad \frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]}$$

i.e.

$$\boxed{\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \qquad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)}$$

Exercises

Given, $z_1 = 2 + 2i, \quad z_2 = 1\angle(\pi/2)$

Determine the modulus and argument of,

$$z_1 z_2, \quad z_1 (z_2)^2$$

Sketch the Argand Diagram for the 2 products.

Ans: $2\sqrt{2}\angle(3\pi/4), \quad 2\sqrt{2}\angle(5\pi/4),$

The Exponential Form

From the series expansion of e^x, $\cos x$ and $\sin x$ below,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$$

We can show that

$$\boxed{z = r(\cos\theta + i\sin\theta) = re^{i\theta}}$$

The Exponential Form

Recall that

$$a^x a^y = a^{x+y} \qquad \text{and} \qquad \frac{a^x}{a^y} = a^{x-y}$$

Therefore, given 2 complex numbers

$$z_1 = r_1 e^{i\theta_1} \qquad \text{and} \qquad z_2 = r_2 e^{i\theta_2}$$

We get

$$\boxed{z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} \qquad \text{and} \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}}$$

This again gives what we saw with the polar form

$$|z_1 z_2| = |z_1||z_2|, \quad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \qquad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

The Exponential Form

With the exponential form, we can perform the logarithm of complex numbers.

$$z = re^{i\theta}$$

Then

$$\ln z = \ln\left(re^{i\theta}\right) = \ln r + \ln\left(e^{i\theta}\right) = \ln r + \theta i \ln(e)$$
$$= \ln r + \theta i$$

θ must be in radians

e.g. Find ln$(1+i)$.

$$\theta = \arg(1+i) = \tan^{-1}(1) = \frac{\pi}{4} \qquad r = |1+i| = \sqrt{1+1} = \sqrt{2}$$

$$\therefore \ln(1+i) = \ln\left(\sqrt{2}e^{i\pi/4}\right) = \ln\sqrt{2} + \frac{\pi}{4}i$$

Summary

- Complex numbers can be expressed in various forms.
- It is easy to convert from one form to another.
- Some operations can be performed more readily with certain forms.

Good for addition and subtraction

1. Simple form : $z = a + bi$
2. Argand diagram

Good for division, multiplication and powers.

3. Modulus-Argument form : $z = r\angle\theta$
4. Polar form : $z = r(\cos\theta + i\sin\theta)$
5. Exponential form : $z = re^{i\theta}$

Mathematics Prep 2007

END